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Asymptotic Stabilisation of Distributed Port-Hamiltonian Systems by Boundary Energy-Shaping Control

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Abstract: This paper illustrates a general synthesis methodology of asymptotic stabilising, energy-based, boundary control laws, that is applicable to a large class of distributed port-Hamiltonian systems. Similarly to the finite dimensional case, the idea is to design a state feedback law able to perform the energy-shaping task, i.e. able to map the open-loop port-Hamiltonian system into a new one in the same form, but characterised by a new Hamiltonian with a unique and isolated minimum at the equilibrium. Asymptotic stability is then obtained via damping injection on the boundary, and is a consequence of the La Salle's Invariance Principle in infinite dimensions. The general theory is illustrated with the help of a simple concluding example, i.e. the boundary stabilisation of a transmission line with distributed dissipation.

Keywords: distributed port-Hamiltonian systems, boundary control, energy-shaping control, stability of PDEs

1. INTRODUCTION

Port-Hamiltonian systems have been introduced about twenty years ago as the mathematical formalisation of bond-graphs to describe lumped parameter physical systems in an unified manner. Further information can be found e.g. in Maschke and van der Schaft [1992], van der Schaft [2000], in Duindam et al. [2009], van der Schaft and Jeltsema [2014], and also in Macchelli [2014a] as far as an extension to macro-economic systems is concerned. For this class of systems, the dynamic results from the power conserving interconnection of a limited set of components, each characterised by a particular "energetic behaviour," i.e. storage, dissipation, generation and conversion. The generalisation to the infinite dimensional scenario leads to the definition of distributed port-Hamiltonian systems (see e.g., van der Schaft and Maschke [2002], Macchelli and Maschke [2009]) that have been introduced about one decade ago, and that have proved to represent a powerful framework for modelling, simulation and control physical systems described by PDEs. Distributed port-Hamiltonian systems share analogous geometric properties with their finite dimensional counterpart, and also the development of stabilising control laws follows the same rationale of the lumped parameter case. Since in most of the cases the Hamiltonian is the total energy of the system, stabilisation could be obtained by driving the Hamiltonian to zero. As a consequence, having such a physical quantity at our disposal simplifies the controller design considerably.

Most of the current research on the stability and stabilisation of distributed port-Hamiltonian systems deals with the development of boundary controllers. For example, in Rodriguez et al. [2001], Macchelli and Melchiorri [2004, 2005], Pasumarthy and van der Schaft [2007], Siuka et al. [2011], Schöberl and Siuka [2013], this task has been accomplished by looking at, or generating, a set of Casimir functions in closed-loop that robustly (i.e., independently from the Hamiltonian functions) relates the state of the infinite dimensional port-Hamiltonian system with the state of the controller, which is a finite dimensional port-Hamiltonian system interconnected to the boundary of the distributed parameter one. The shape of the closed-loop energy function is changed by acting on the Hamiltonian of the controller e.g. to introduce a minimum in a desired configuration. As discussed in van der Schaft [2000], Ortega et al. [2001], this procedure is the generalisation of the control by interconnection via Casimir generation (energy-Casimir method) developed for finite dimensional systems. The result is an energy-balancing passivity-based controller that is not able to deal with equilibria that require an infinite amount of supplied energy in steady state, i.e. with the so-called "dissipation obstacle."

In this paper, it is shown how to enlarge the class of boundary energy-shaping controllers beyond the dissipation obstacle by focusing on the trajectories that correspond to a particular Hamiltonian, rather than on the geometric structure (i.e., the Dirac structure), of the system only (see e.g., Macchelli [2014b,c]). Since the state dependent control action obtained thanks to the energy-Casimir method is able to shape the Hamiltonian function, the idea is to procede in a more direct manner, i.e. by determining a feedback law that maps the open-loop trajectories into the trajectories of a target system with the same port-Hamiltonian structure (i.e., Dirac and resistive structures are not modified), but characterised by a *shaped* Hamiltonian with the desired stability properties. This is the same concept adopted in finite dimensions in case of stabilisation with state-modulated sources discussed in Ortega et al. [2001], or with the more general IDA-PBC control technique presented in Ortega et al. [2002].

In this paper, then, the boundary control via energy-shaping is developed for the class of linear, distributed, port-Hamiltonian systems presented in Le Gorrec et al. [2005], Jacob and Zwart [2012]. By transforming the original system via state feedback into a new one with an Hamiltonian function that has an isolated minimum at the equilibrium, simple stability is obtained. To have asymptotic stability, it is necessary to add damping by means of a further control loop. In this respect, another important contribution of this paper is to show that, if it is possible to impose full boundary dissipation to the port-Hamiltonian system resulting from the energy-shaping procedure, then the desired equilibrium can be proved to be asymptotically stable. It is worth noting that the proposed techniques can be easily extended to the nonlinear case: the difficult part is then to prove existence of solutions for the set of coupled PDEs and ODEs associated to the closed-loop system, and the invariance properties of the steady state trajectories determined by the damping injection loop.

The paper is organised as follows. In Section 2, the class of linear, distributed, port-Hamiltonian systems under investigation is briefly presented, together with some fundamental properties. The energy-shaping boundary control technique is presented in Section 3, while asymptotic stability in case of full boundary dissipation (damping injection) is discussed in Section 4. Then, in Section 5, the general methodology is illustrated with the help of an example, namely a trasmission line with distributed dissipation. Conclusions and a discussion about possible future research activities are reported in Section 6.

2. BACKGROUND

In this paper, we refer to the class of linear distributed port-Hamiltonian systems that have been studied in Le Gorrec et al. [2005], Villegas et al. [2009], Jacob and Zwart [2012], Ramírez et al. [2014], i.e. to systems described by the following PDE:

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z) \quad (1)$$

with $x \in \mathbb{R}^n$, and $z \in [a, b]$. Moreover, $P_1 = {P_1}^{\mathrm{T}}$, $P_0 = -{P_0}^{\mathrm{T}}$, $G_0 = G_0^{\mathrm{T}} \geq 0$, and $\mathcal{L}(\cdot)$ is a bounded and continuously differentiable matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z)$ and $\mathcal{L}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a, b]$. For the sake of clearness, $(\mathcal{L}x)(t,z) := \mathcal{L}(z)x(t,z)$. The state space is $X = L_2(a, b; \mathbb{R}^n)$, and is endowed with the inner product $\langle x_1 \mid x_2\rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2\rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1\rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot\rangle$ denotes the natural L_2-inner product. The selection of this space for the state variable is motivated by the fact that $\|\cdot\|^2_{\mathcal{L}}$ is proportional to the energy function. As a consequence, X is also called the space of energy variables, and $\mathcal{L}x$ are the co-energy variables. This class is quite general and includes models of flexible structures, traveling waves, heat exchangers, and bioreactors among others. The PDE (1) can be also written as $\dot{x} = \mathcal{J}x$, where $\mathcal{J}$ is the linear operator defined as

$$\mathcal{J}x := P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$$

with domain

$$D(\mathcal{J}) = \big\{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n)\big\}$$

To have a distributed port-Hamiltonian system, the PDE (1) has to be "completed" by proper boundary port. More precisely, given $\mathcal{L}x \in H^1(a, b; \mathbb{R}^n)$, the boundary port variables associated to (1) are the vectors $f_\partial, e_\partial \in \mathbb{R}^n$ defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \quad (2)$$

The boundary port variables are just a linear combination of the restriction of the boundary variables, and simple integration by parts shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} = e^{\mathrm{T}}_\partial(t) f_\partial(t)$$

The problem of determining the boundary inputs and outputs for (1) to have a so-called boundary control system on X, see e.g. Curtain and Zwart [1995], has been addressed in Le Gorrec et al. [2005], Villegas et al. [2009].

Theorem 1. Let W be a $n \times 2n$ real matrix. If W has full rank and satisfies $W\Sigma W^{\mathrm{T}} \geq 0$, being

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

then the system (1) with input

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: \mathcal{B}x \quad (3)$$

is a boundary control system on X, with $\mathcal{B} : H^1(a, b; \mathbb{R}^n) \to \mathbb{R}^n$. Furthermore, the operator $\bar{\mathcal{J}}x := P_1(\partial/\partial z)(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain

$$\begin{aligned} D(\bar{\mathcal{J}}) &= \left\{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \;\middle|\; \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \operatorname{Ker} W\right\} \\ &= \left\{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \mid \mathcal{B}x = 0\right\} \end{aligned}$$

generates a contraction semigroup on X. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible and let P be given by

$$P = \begin{pmatrix} W\Sigma W^{\mathrm{T}} & W\Sigma\tilde{W}^{\mathrm{T}} \\ \tilde{W}\Sigma W^{\mathrm{T}} & \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} \end{pmatrix}^{-1}$$

Define the output as

$$y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: \mathcal{C}x \quad (4)$$

with $\mathcal{C} : H^1(a, b; \mathbb{R}^n) \to \mathbb{R}^n$. Then, for $u \in C^2(0, \infty; \mathbb{R}^n)$ and $(\mathcal{L}x)(0) \in H^1(a, b; \mathbb{R}^n)$, the following energy balance equation is satisfied:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} P \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \quad (5)$$

Proof. See Le Gorrec et al. [2005].

In this paper, the matrices W and $\tilde{W}$ are selected in such a way that (1) is in impedance form, which means that

$$W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0 \qquad W\Sigma\tilde{W}^{\mathrm{T}} = I \tag{6}$$

and consequently the energy-balance relation (5) reduces to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|x(t)\right\|^2_{\mathcal{L}} \leq y^{\mathrm{T}}(t)u(t)$$

3. BOUNDARY CONTROL BY ENERGY-SHAPING

In this section, it is shown how to design a boundary state-feedback control action in the form

$$u(t) = \beta(x(t,\cdot)) + u'(t) \tag{7}$$

that is able to map the open-loop dynamic (1) into the target one

$$\begin{aligned}\frac{\partial x}{\partial t}(t,z) &= \\ &= P_1\frac{\partial}{\partial z}\frac{\delta H_d}{\delta x}(x(t,z)) + (P_0 - G_0)\frac{\delta H_d}{\delta x}(x(t,z)) \\ u'(t) &= WR\begin{pmatrix}\frac{\delta H_d}{\delta x}(t,b)\\ \frac{\delta H_d}{\delta x}(t,a)\end{pmatrix}\end{aligned} \tag{8}$$

in which $H_d(x) = H(x) + H_a(x)$. The target system has the same internal structure of the original one, i.e. the matrices P_1, P_0 and G_0 are not changed, but a different Hamiltonian H_d and boundary input u'.

The idea here is to overcome the intrinsic limitations of the energy-Casimir method that are associated with the admissible (internal) dissipation, and to realise an "explicit" energy-shaping procedure, as in the lumped parameter case. The energy-Casimir method, in fact, is constructive and it is based on the definition of a boundary controller in port-Hamiltonian form whose Dirac structure is chosen in order to have a proper set of Casimir functions in closed-loop. Such invariants show how it is possible to shape the total energy of the system by acting on the controller Hamiltonian: the stabilising control law depends then on this choice and on the relation between states of the plant and of the controller specified by the Casimir functions. The associated control action can be easily written in terms of a state-feedback law $\beta(x(t,\cdot))$ as in (7), but unfortunately such control action suffers of all the intrinsic limitations of the energy-Casimir method, i.e. it is not able to stabilise equilibria that require an infinite amount of supplied energy in steady state, van der Schaft [2000], Ortega et al. [2001].

Proposition 2. (Energy-shaping). Consider the boundary control system of Theorem 1, and denote by $H(x) = \frac{1}{2}\left\|x\right\|^2_{\mathcal{L}}$ its Hamiltonian function. Then, the boundary state-feedback law $u = \beta(x) + u'$, being u' an auxiliary boundary input, maps (1) into the target dynamical system (8), with $H_d(x) = H(x) + H_a(x)$ if

$$P_1\frac{\partial}{\partial z}\frac{\delta H_a}{\delta x}(x) + (P_0 - G_0)\frac{\delta H_a}{\delta x}(x) = 0 \tag{9}$$

$$\beta(x) + WR\begin{pmatrix}\frac{\delta H_a}{\delta x}(b)\\ \frac{\delta H_a}{\delta x}(a)\end{pmatrix} = 0 \tag{10}$$

Proof. The proof is immediate by comparison of initial and target dynamics. For a geometric interpretation of this result in the distributed parameter scenario, refer to Macchelli [2014c].

With $u' = 0$, energy is not increasing along the trajectories of (8), i.e. $\dot{H}_d(x(t)) \leq 0$. Asymptotic stability can be then obtained by damping injection, provided that a dual output to u' is properly defined. In this respect, with Theorem 1 in mind, the natural choice turns out to be

$$y'(t) = \tilde{W}R\begin{pmatrix}\frac{\delta H_d}{\delta x}(t,b)\\ \frac{\delta H_d}{\delta x}(t,a)\end{pmatrix} \tag{11}$$

which clearly implies that $\frac{\mathrm{d}}{\mathrm{d}t}H_d(x(t)) \leq y'^{\mathrm{T}}(t)u'(t)$. Such new boundary port (u', y') has now to be terminated over a dissipative element to obtain asymptotic stability of equilibria, or just to improve the convergence rate:

$$u'(t) = -\Xi y'(t), \qquad \Xi = \Xi^{\mathrm{T}} > 0 \tag{12}$$

Conditions for checking asymptotic stability in closed-loop are discussed in the next section, while an application of this technique on a particular example is presented in Section 5.

4. ASYMPTOTIC STABILITY ANALYSIS

The aim of this section is to show that the energy-shaping control law defined in (9) and (10), combined with the damping injection relation (12) is able to asymptotically stabilise (1) in an equilibrium $(\mathcal{L}x)_\star \in H^1(a,b;\mathbb{R}^n)$ solution of

$$P_1\frac{\partial}{\partial z}(\mathcal{L}z)_\star(z) + (P_0 - G_0)(\mathcal{L}z)_\star(z) = 0 \tag{13}$$

The main result is an application to the La Salle's Invariance Principle in infinite dimensions, see e.g. Luo et al. [1999]. The first step is to determine how to chose H_a so that (9) holds.

Lemma 3. The functions H_a solutions of (9) are in the form $H_a(x) = \hat{H}_a(\xi(x))$, with

$$\xi(x(t,\cdot)) = \int_a^b \hat{\Phi}^{\mathrm{T}}(z)x(t,z)\,\mathrm{d}z \tag{14}$$

where $\hat{\Phi}(z) = \big(\Phi_1(z),\, \dots,\, \Phi_{n_\xi}(z)\big)$. Here, the functions $\Phi_i \in H^1(a,b;\mathbb{R}^n)$, $i = 1,\dots,n_\xi \leq n$, are independent solutions of

$$P_1\frac{\partial}{\partial z}\Phi_i(z) + (P_0 - G_0)\Phi_i(z) = 0 \tag{15}$$

Since in this paper we have restricted ourselves to the linear case, let us assume that H_a is quadratic in ξ. Furthermore, denote by $\phi_\star \in \mathbb{R}^{n_\xi}$ a vector such that $(\mathcal{L}x)_\star(z) = \hat{\Phi}(z)\phi_\star$. Then, $H_d = H + H_a$ has a global minimum in $(\mathcal{L}x)_\star$ if

$$\begin{aligned}H_a(x) = \frac{1}{2}&\left\{\int_a^b \hat{\Phi}^{\mathrm{T}}\Big[x - \mathcal{L}^{-1}\hat{\Phi}\phi_\star\Big]\,\mathrm{d}z\right\}^{\mathrm{T}} \times \\ &\times Q_a\left\{\int_a^b \hat{\Phi}^{\mathrm{T}}\Big[x - \mathcal{L}^{-1}\hat{\Phi}\phi_\star\Big]\,\mathrm{d}z\right\} - \\ &- \phi_\star^{\mathrm{T}}\left(\int_a^b \hat{\Phi}^{\mathrm{T}}x\,\mathrm{d}z\right) + \kappa\end{aligned} \tag{16}$$

where $Q_a = {Q_a}^{\mathrm{T}} \geq 0$ and $\kappa \in \mathbb{R}$ is some constant.

Remark 4. If in (16) it is assumed that $Q_a = 0$, then the energy-shaping state-feedback law β defined in (10) reduces to a constant, namely

$$\beta(x) = WR \begin{pmatrix} \hat{\Phi}(b)\phi_\star \\ \hat{\Phi}(a)\phi_\star \end{pmatrix}$$

which are the boundary conditions associated to the equilibria $(\mathcal{L}x)_\star$. Then, the effect of the damping injection contribution (12) is to dissipate the total energy until the new minimum is reached. A simple application of Villegas et al. [2009] shows that the equilibrium is exponentially stable.

Theorem 5. (Asymptotic stability). Let us consider the linear, infinite dimensional, port-Hamiltonian system (1) and the equilibrium $(\mathcal{L}x)_\star$ satisfying (13). Then, the control action $u = \beta(x) + u'$ with β defined in (10), being H_a chosen as in (16), and with u' defined in (12), makes $(\mathcal{L}x)_\star$ asymptotically stable.

Proof. Here, only a sketch of the proof is reported. At first, let us assume for simplicity and without loss of generality that $\hat{\Phi}(z) = (\mathcal{L}x)_\star(z)$, so that $\xi \in \mathbb{R}$ and $Q_a \in \mathbb{R}$. In spite of Remark 4, select $Q_a > 0$. Note at first that with a simple change of coordinates, studying the stability of $(\mathcal{L}x)_\star$ is equivalent to studying the stability of the origin. For the closed-loop system, the following energy-balancing relation holds true:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H_d(x) &= -\int_a^b \left(\frac{\delta H_d}{\delta x}\right)^{\mathrm{T}} G_0 \frac{\delta H_d}{\delta x}\,\mathrm{d}z - {y'}^{\mathrm{T}} \Xi y' \\ &\leq 0 \end{aligned}$$

with y' given by (11). Since Ξ is non-singular, and in spite of (12), it is easy to verify that energy is decreasing until a steady-state configuration $\bar{x}(t,z)$ is reached. Such configuration, possibly time-variant, satisfies

$$\begin{aligned} G_0\left[(\mathcal{L}\bar{x})(t,z) + Q_a\bar{\xi}(t)\hat{\Phi}(z)\right] &= 0 \\ \begin{pmatrix} \dfrac{\delta H_d}{\delta x}(\bar{x}(t,b)) \\ \dfrac{\delta H_d}{\delta x}(\bar{x}(t,a)) \end{pmatrix} &= \\ = \begin{pmatrix} (\mathcal{L}\bar{x})(t,b) \\ (\mathcal{L}\bar{x})(t,a) \end{pmatrix} + Q_a\bar{\xi}(t) \begin{pmatrix} \hat{\Phi}(b) \\ \hat{\Phi}(a) \end{pmatrix} &= 0 \end{aligned} \tag{17}$$

where $\bar{\xi}$ is the corresponding steay state evolution of ξ. The second relation in (17) is a consequence of Theorem 1. Under the assumption of pre-compactness of the orbits, asymptotic stability is a consequence of La Salles Invariance Principle, Luo et al. [1999]. More precisely, it is nessarily to verify that the only steady state solution $\bar{x}(t,z)$ which is invariant and compatible with $\dot{H}_d = 0$ is the origin. In this respect, it is easy to see that $\dot{\bar{\xi}}(t) = 0$, that means that $\bar{\xi}(t) = \xi_\star$ in steady state. With some further computations, it is possible to prove that

$$\phi(t,z) := \bar{x}(t,z) + \xi_\star Q_a \mathcal{L}^{-1}(z)\hat{\Phi}(z) = 0$$

for $t \geq \tau$, being τ sufficiently large, which implies that

$$\bar{x}(t,z) = -\xi_\star Q_a \mathcal{L}^{-1}(z)\hat{\Phi}(z)$$

when $t \geq \tau$. From (14) we have that

$$\begin{aligned} \xi_\star &= \int_a^b \hat{\Phi}^{\mathrm{T}}(z)\bar{x}(t,z)\,\mathrm{d}z \\ &= \xi_\star Q_a \int_a^b \hat{\Phi}^{\mathrm{T}}(z)\mathcal{L}^{-1}(z)\hat{\Phi}(z)\,\mathrm{d}z \end{aligned}$$

Since the integral term is greater than 0 since $\mathcal{L}(\cdot) > 0$, we have that $\xi_\star = 0$, and then that $\bar{x}(t,z) = 0$ for $t \geq \tau$. Then, the zero solution is the only invariant solution compatible with $\dot{H}_d = 0$, which turns out to be asymptotically stable based on La Salles Invariance Principle considerations.

5. EXAMPLE: BOUNDARY STABILISATION OF A TRANSMISSION LINE WITH DISSIPATION

The port-Hamiltonian formulation of the lossless transmission line equation is in the form (1) and given by (see van der Schaft and Maschke [2002]):

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} q(t,z) \\ p(t,z) \end{pmatrix} &= \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z} - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right\} \times \\ &\times \begin{pmatrix} \dfrac{\delta H}{\delta q}(q(t,z),p(t,z)) \\ \dfrac{\delta H}{\delta p}(q(t,z),p(t,z)) \end{pmatrix} \end{aligned} \tag{18}$$

where $z \in Z \equiv [0,\, \ell]$, q and p are the charge and magnetic flux densities along the line, and

$$H(q,p) = \frac{1}{2}\int_0^\ell \left(\frac{p^2}{L} + \frac{q^2}{C}\right)\mathrm{d}z \tag{19}$$

is the Hamiltonian (energy) function, with C and L the distributed capacitance and inductance. Moreover, in (18), $D \geq 0$ is the dissipation term associated to the presence of a distributed resistance along the line. The system exchanges power with the environment through a couple of ports defined in $z = 0$ and in $z = \ell$:

$$\begin{aligned} \big(I_0(t), V_0(t)\big) &= \left(\frac{\delta H}{\delta p}(0,t),\ \frac{\delta H}{\delta q}(0,t)\right) \\ \big(I_\ell(t), V_\ell(t)\big) &= \left(-\frac{\delta H}{\delta p}(0,t)(\ell,t),\ \frac{\delta H}{\delta q}(\ell,t)\right) \end{aligned} \tag{20}$$

that are the pair current/voltage at the extremities of the line itself. Finally, it is assumed that the controller is acting on the boundary ports (20) with the following causality:

$$u(t) = \begin{pmatrix} I_0(t) \\ V_\ell(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} V_0(t) \\ I_\ell(t) \end{pmatrix} \tag{21}$$

With the definition (21) of (boundary) inputs and outputs, a boundary control system in the sense of Theorem 1 is obtained. As discussed in Section 3, to stabilise (18), it is required to determine the state-feedback law $u = \beta(q,p) + u'$, such that the equilibrium $(q_\star, p_\star)$ is asymptotically stable. Such equilibrium is solution of

$$\begin{aligned} \frac{\partial}{\partial z}\frac{\delta H}{\delta p}(q_\star,p_\star) &= 0 \\ \frac{\partial}{\partial z}\frac{\delta H}{\delta q}(q_\star,p_\star) + D\frac{\delta H}{\delta p}(q_\star,p_\star) &= 0 \end{aligned} \tag{22}$$

where

$$\begin{aligned} \frac{\delta H}{\delta q}(q_\star(z),p_\star(z)) &= \frac{q_\star(z)}{C} \\ \frac{\delta H}{\delta p}(q_\star(z),p_\star(z)) &= \frac{p_\star(z)}{L} \end{aligned}$$

At first, let us assume that $D = 0$, i.e., there is no distributed resistance in the line. In this case, from Lemma 3, the class of function H_a that can be employed in the energy-shaping design procedure are in the form $H_a(q, p) = \hat{H}_a\big(\xi_1(q, p), \xi_2(q, p)\big)$, with

$$\begin{aligned} \xi_1(q(t,\cdot)) &= \int_0^\ell q(t,z)\,\mathrm{d}z \\ \xi_2(p(t,\cdot)) &= \int_0^\ell p(t,z)\,\mathrm{d}z \end{aligned} \tag{23}$$

and $\hat{H}_a$ that can be freely chosen. From (22), it is easy to find out that the equilibrium configuration is given by

$$q(t,z) = q_\star \qquad\qquad p(t,z) = p_\star \tag{24}$$

which means constant charge and flux densities along the line (or constant current and voltages since the Hamiltonian is quadratic). To have in closed-loop a port-Hamiltonian system with Hamiltonian $H_d = H + H_a$ with a minimum in (24), a possible choice of $\hat{H}_a$ is

$$\begin{aligned} \hat{H}_a(\xi_1,\xi_2) = \frac{1}{2}K_1(\xi_1-\xi_{1\star})^2 + \\ + \frac{1}{2}K_2(\xi_2-\xi_{2\star})^2 - q_\star\xi_1 - p_\star\xi_2 \end{aligned} \tag{25}$$

where $\xi_{1\star}$ and $\xi_{2\star}$ are the values of ξ_1 and ξ_2 at the equilibrium, i.e. $\xi_{1\star} = \ell q_\star$, and $\xi_{2\star} = \ell p_\star$, while K_1, K_2 are two positive gains. It is easy to check that the closed-loop system is lossless, so only simple stability has been achieved e.g. in the sense of Swaters [2000]. However, asymptotic stability can be obtained by damping injection at the boundary, as discussed in Section 3; then, asymptotic stability follows immediately from Theorem 5.

It is possible to verify that the same control action can be determined by applying the energy-Casimir method. In fact, let us consider the following linear control system

$$\begin{cases} \dot{\xi}_C(t) = J_C\dfrac{\partial H_C}{\partial \xi_C}(\xi_C(t)) + u_C(t) \\ y_C(t) = \dfrac{\partial H_C}{\partial \xi_C}(\xi_C(t)) \end{cases}$$

in which $\xi_C = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the state variable, while $J_C = -J_C^{\mathrm{T}}$ and H_C are the interconnection matrices and Hamiltonian respectively, to be assigned later on. With simple computations, it can be checked that Casimir functions are not present in closed-loop if $J_C = 0$. From a physical point of view, this result is obvious. With this choice, in fact, such boundary controller consists of two separate systems, each required to provide a constant power flow in steady state: they are not energy-balancing controllers. So, it is necessary to couple these regulators and allow for an internal power flow at the controller side. This can be achieved by choosing

$$J_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

which implies that the closed loop system is characterised by the following Casimir functions:

$$\begin{aligned} C_1(\xi_1(t), q(t,\cdot)) &= \xi_1(t) - \int_0^\ell q(t,z)\,\mathrm{d}z \\ C_2(\xi_2(t), p(t,\cdot)) &= \xi_2(t) - \int_0^\ell p(t,z)\,\mathrm{d}z \end{aligned}$$

Note the similarities with (23), which lead to the same choice (25) as far as the controller Hamiltonian is concerned.

Due to internal dissipation, i.e. when $D \neq 0$, the energy-Casimir method cannot be applied, and it then preferable to rely on the energy-shaping methodology presented in Section 3. The PDE (9) provides the admissible functions H_a, and (10) the associated boundary control action. From (22), the equilibrium configuration takes the following form

$$q_\star(z) = -\frac{\bar{p}_\star}{\ell}Dz + \bar{q}_\star \qquad\qquad p_\star(z) = \bar{p}_\star$$

where $\bar{q}_\star$, $\bar{p}_\star$ are some real constants.

With Lemma 3 in mind, the admissible H_a takes the form $H_a(q, p) = \hat{H}_a(\xi(q, p))$ with

$$\xi(q(t,\cdot), p(t,\cdot)) = \int_0^\ell \begin{pmatrix} \dfrac{\bar{q}_\star}{\bar{p}_\star} - \dfrac{\bar{D}}{\ell}z \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} q(t,z) \\ p(t,z) \end{pmatrix} \mathrm{d}z$$

A possible choice for H_a is with

$$\hat{H}_a(\xi) = \frac{1}{2}K(\xi - \xi_\star)^2 - \bar{p}_\star\xi$$

where K is a positive gain, and $\xi_\star$ the value of ξ at the equilibrium. With this choice, the state feedback action β obtained thanks to (10) is able to shape the closed-loop Hamiltonian and to introduce a minimum in the desired equilibrium. Asymptotic stability is obtained via damping injection (12) on the *new* control port (u', y') defined in (8) and (11) in the general case, as discussed in Theorem 5.

6. CONCLUSIONS

The motivating idea of the paper has been the development of a general synthesis methodology of boundary control laws for linear, distributed port-Hamiltonian systems with one-dimensional spatial domain. As in the lumped parameter case, the feedback law is determined in such a way that its effect on the system is to shape the energy function, and to modify the dissipative structure. The first step is responsible for achieving simple stability of an equilibrium, while the second one for assuring asymptotic convergence of the trajectories. Usually, the first step has been usually accomplished thanks to the so-called energy-Casimir method, but due to the fact that the class of stabilising controller that such method can provide is quite limited because of the dissipation obstacle, the problem of determining a feedback law able to shape the Hamiltonian in a proper manner has been tackled here by directly focusing on the trajectories of open and closed-loop system, i.e. by determining the control action that maps the open-loop system into a new one, with the same (Dirac) structure but a different Hamiltonian. To achieve asymptotic stability, a further feedback loop is closed to implement a damping injection strategy: the first loop obtained by applying the energy-shaping procedure is responsible for having a new Hamiltonian with an isolated minimum at the equilibrium, while this second one for dissipating energy until such minimum is reached. The resulting control law is proved to asymptotically stabilise the system.

Even if the proposed methodology has been developed for linear systems, some of the techniques discussed here can be easily generalised to cope with the nonlinear case. This extension is the main future research topic, together with the stabilisation of distributed port-Hamiltonian systems with 2D or 3D spatial domain.

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