Reduced models identification from experimental modal analysis of non-self adjoint systems: rotordynamics, active control and vibroacoustics applications

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Abstract

In the context of experimental modal analysis of non-self adjoint problems, like rotordynamics, vibroacoustics or active control applications, the nonsymmetry of the system induces specificities that must be considered for proper use of identification techniques. In this paper, the particularities of this kind of problem are addressed in order to be able to efficiently identify the dynamic behaviour. There are many aspects induced by the non-self adjoint character of the behaviour, some of them have been studied in the past years, while some remain unclear. This paper addresses some of these points. The first matter which is detailed is related to the ability of the technique to identify both right and left eigenvectors. The classical LSCF technique is adapted in order to reach this objective. A few adaptations are required, but the extension is quite natural and the efficiency of the technique is not degraded in the context of non self-adjoint problems. The second point is associated to the regularization of inverse problem for matrices identification using the complex eigenvectors. This inverse procedure, which is one of the ways allowing the damping matrix identification, is known to be very sensitive to noise. The technique of properness enforcement, already available in the context of structural dynamics, has been extended to non-self adjoint in order to regularize the problem. The basic idea is to slightly modify the identified complex vectors, in order that they verify the so-called properness condition. A low-cost specific iterative procedure is proposed to reach this objective. A numerical test-case is then presented on a rotordynamics application, and some experimental results are presented on a structural active control application, in order to show the ability of the approach to identify reduced models from experimental modal analysis. In particular, some considerations about proper damping identification are exhibited. Finally, some issues about vibroacoustics applications are considered: in this case, a direct relationship between left and right eigenvectors is available. The technique has then to be slightly adapted in order to take into account the dependency of the variables used in a constrained optimization equation. This leads to a non linear matrix problem, which can be difficult to solve. Some approximate techniques are exhibited and applied on an experimental test-case in order to illustrate their efficiency.

1 Introduction

Experimental modal analysis is a very common tool in structural dynamics. Using only experimental data, the dynamic behavior of a structure can be represented in a frequency range of interest using identified modal parameters (eigen frequencies, eigen shapes, modal damping ratios and modal masses). Using the modal basis, one can evaluate the response of the structure at measured points to an arbitrary excitation. A more evolved objective can be the determination of a matrix-based model, which can be seen as an experimental reduced model. For some specific applications, one can be interested to ask this model to be physical, in other words to have a topology of the matrices which is the same as that of a physical system. This is for example of first interest in the context of experimental identification of damping matrices.

This paper focuses on this particular point. It has been shown [1] that for structural dynamics problems with symmetric matrices, the existence of a physical experimental reduced model is equivalent to the so-called properness condition of complex vectors. In the same paper, an efficient methodology has been proposed to enforce that property when identified modes do not verify the properness condition. This procedure can be seen as an optimal correction of complex vectors for reduced model identification.

Some particular problems lead to second-order non symmetric formulations, like vibroacoustics [2] or rotordynamics [3]. For those systems, the quadratic eigenvalue problem [4] must be solved to obtain a coherent modal description of the problem, using left and right complex modes. Section 2 recalls classical modal properties of non symmetric second-order problems. Section 3 proposes extension of the properness condition to non-self adjoint problems and associated inverse relations for matrices reconstruction. These new results are almost trivial to obtain but constitute important results for matrices identification from experimental data. Section 4 presents an original technique for optimal correction of left and right eigen shapes in order that the properness condition is verified. A non-classical Riccati equation is derived from a constrained minimization problem, and a numerical procedure is proposed to solve it. The corrected eigenvectors can then be used for optimal reconstruction of matrices. A physical application is then considered in section 5 in the context of rotordynamics. An experimental test-case is presented in section 6 using a structure with an active control feedback, an extension to vibroacoustics is finally considered in section 7.

2 Problem description and modal decomposition

This part exhibits the typical problem which is considered in this work, and recalls classical modal properties of non-self adjoint problems that will be used in the paper.

2.1 Second-order typical problem

The typical second-order problem which is considered in this paper is:

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{f(t)\}, \tag{1}$$

in which $\{q(t)\}$ is the vector of the unknown discretized field, $[M]$ is the mass matrix, $[K]$ is the stiffness matrix, $[C]$ is the damping matrix, $\{f(t)\}$ is the force vector. All matrices are also supposed to be real, but not necessarily symmetric. The notations used here are in accordance with those proposed in [5]. In particular, brackets are used for matrices and curly braces for vectors.

To this time-domain equation are associated the following direct quadratic eigenvalue problem:

$$\left([M]\lambda_j^2 + [C]\lambda_j + [K]\right)\{\phi_{Rj}\} = 0, \tag{2}$$

and the corresponding adjoint eigenvalue problem:

$$\left([M]^T\lambda_j^2 + [C]^T\lambda_j + [K]^T\right)\{\phi_{Lj}\} = 0, \tag{3}$$

in which λ_j is the j-th eigenvalue associated to the j-th right eigenvector $\{\phi_{Rj}\}$ and j-th left eigenvector $\{\phi_{Lj}\}$.

Since the matrices are not necessarily symmetric, the eigenvalues of the problem are real or come in pairs (λ_j, λ_j^*). If $\{\phi_j\}$ is a (right or left) eigenvector associated to λ_j, then $\left\{\phi_j^*\right\}$ is a (right or left) eigenvector associated to λ_j^*. A very complete review of this kind of problem is addressed in [4].

2.2 Modal decomposition of the permanent harmonic response

The eigenmodes of the system can be efficiently used in particular for the modal decomposition of the permanent harmonic response. This can be done considering the state-space representation of the system:

$$[U]\left\{\dot{Q}(t)\right\} - [A]\{Q(t)\} = \{F(t)\}, \tag{4}$$

in which:

$$[U] = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \ [A] = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \ \{Q(t)\} = \begin{Bmatrix} q(t) \\ \dot{q}(t) \end{Bmatrix}, \ \{F(t)\} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}. \tag{5}$$

The eigenvalues of this problem can be stored in the spectral matrix Λ:

$$[\Lambda] = \left[{}^{\setminus}\lambda_{j\setminus}\right]. \tag{6}$$

The $j-th$ eigenvalue is associated to:

- a right eigenvector $\{\theta_{Rj}\}$ such as $(U\lambda_j - A)\{\theta_{Rj}\} = 0$, in which $\{\theta_{Rj}\} = \begin{Bmatrix} \phi_{Rj} \\ \phi_{Rj}\lambda_j \end{Bmatrix}$. Storing the eigenvectors (in the same order as the eigenvalues) in the modal matrix $[\theta_R] = \begin{bmatrix} \phi_R \\ \phi_R\Lambda \end{bmatrix}$, the following relationship is verified:
$$[U][\theta_R][\Lambda] = [A][\theta_R]. \tag{7}$$
- a left eigenvector $\{\theta_{Lj}\}$ such as $\{\theta_{Lj}\}^T(U\lambda_j - A) = 0$, in which $\{\theta_{Lj}\} = \begin{Bmatrix} \phi_{Lj} \\ \phi_{Lj}\lambda_j \end{Bmatrix}$. Storing the eigenvectors (in the same order as the eigenvalues) in the modal matrix $[\theta_L] = \begin{bmatrix} \phi_L \\ \phi_L\Lambda \end{bmatrix}$, the following relationships are verified:
$$[U]^T[\theta_L][\Lambda] = [A]^T[\theta_L] \quad \text{or} \quad [\Lambda][\theta_L]^T[U] = [\theta_L]^T[A]. \tag{8}$$

The orthogonality relationships can be written using $2n$ arbitrary values to build the diagonal matrix $[\xi] = \left[{}^{\setminus}\xi_{j\setminus}\right]$:

$$[\theta_L]^T[U][\theta_R] = [\xi] \ \text{ or } \ [\theta_L]^T[A][\theta_R] = [\xi][\Lambda]. \tag{9}$$

The modal decomposition of the permanent harmonic response at frequency ω is finally:

$$\{Q(t)\} = [\theta_R]\left([\xi]\left(j\omega[E_{2n}] - [\Lambda]\right)\right)^{-1}[\theta_L]^T\{F(\omega)\}e^{i\omega t}, \tag{10}$$

in which E_{2n} is the $2n \times 2n$ identity matrix and $F(\omega)$ is the complex amplitude of the harmonic excitation. This relationship can also be written using the n degrees of freedom notations in the frequency domain:

$$\{q(\omega)\} = [\phi_R]\left[{}^{\setminus}\frac{1}{\xi_j(i\omega - \lambda_j)}{}_{\setminus}\right][\phi_L]^T\{f(\omega)\}, \tag{11}$$

A classical way to use these modes is to perform the calculation of the harmonic response according to previous equations, using a limited number of modes, depending on the maximum frequency value that should be obtained. In the field of experimental modal analysis, an experimental reduced model is built from complex modes, which are identified using FRFs (Frequency Response Functions) using techniques like the ones described in [6, 7] for symmetric problems.

In the following, for practical reasons, without loss of generality, one will assume that the eigenshapes are normalized such as $[\xi] = [E_{2n}]$.

3 Properness condition

The properness condition for complex modes is well detailed in reference [1], in the context of symmetrical systems. Starting from a given set of $2n$ identified complex modes, this condition is related to the fact that the system can be exactly represented in the frequency range of interest, by a n degrees of freedom equivalent physical model, built from identified modes. If a set of vectors does not verify the properness condition, it means either that the system can not be represented by this reduced set of modes, or that the experimental identification has introduced some errors in the eigenshapes, in particular because the properness condition is very sensitive to noise. In this case, Balmès has proposed a methodology to enforce the properness condition in structural dynamics [1]. In the same paper, it is also shown that the properness condition is equivalent to the completeness of the basis, which is discussed for example in references [8] and [9]. In this section, the properness condition and associated matrices reconstruction relationships are extended to the case of non symmetric systems.

The properness condition is associated to the inverse problem. The orthogonality relationships (9) can be inverted:

$$[U]^{-1} = [\theta_R]\,[\theta_L]^T \tag{12}$$

or

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}CM^{-1} \end{bmatrix} = \begin{bmatrix} \phi_R\phi_L^T & \phi_R\Lambda\phi_L^T \\ \phi_R\Lambda\phi_L^T & \phi_R\Lambda^2\phi_L^T \end{bmatrix}, \tag{13}$$

and

$$[A]^{-1} = [\theta_R]\,[\Lambda]\,[\theta_L]^T \tag{14}$$

or

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}^{-1} = \begin{bmatrix} -K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} = \begin{bmatrix} \phi_R\Lambda^{-1}\phi_L^T & \phi_R\phi_L^T \\ \phi_R\phi_L^T & \phi_R\Lambda\phi_L^T \end{bmatrix}. \tag{15}$$

It is then clear that the properness condition for nonsymmetric second order systems can be written as:

$$\left[\phi_R\phi_L^T\right] = 0. \tag{16}$$

Once this relationship is verified, the matrices can be found using the inverse relations:

$$M = \left[\phi_R\Lambda\phi_L^T\right]^{-1},\ \ K = -\left[\phi_R\Lambda^{-1}\phi_L^T\right]^{-1},\ \ C = -\left[M\phi_R\Lambda^2\phi_L^T M\right]. \tag{17}$$

In the context of symmetric systems, this is one of the most popular ways to identify damping matrices from experimental measurements. An important remark is that these relationships require the knowledge of n modes to build a n-degrees of freedom physical equivalent system.

4 Enforcement of the properness condition

It is assumed in this part that a modal identification has been done: the eigenvalues, the left and right eigenvectors have been identified from the measured FRFs (a discussion about that point will be presented in the part related to the experimental test-case). The objective is to perform an optimal correction of eigenvectors in order that the properness condition (16) is verified. In a practical point of view, the experimental eigenshapes identification is generally quite sensitive to measurement errors, and a small shift on vectors can induce some large changes not only in the properness relation but also on the values of matrices after the inversion process. This will be illustrated later on the test-cases.

One then wants to find the matrices $[\tilde{\phi}_R]$ and $[\tilde{\phi}_L]$, that verify the properness relation (16) and that minimize the norm of $[\tilde{\phi}_R - \phi_R]$ and $[\tilde{\phi}_L - \phi_L]$. This problem can be written:

$$\text{Find } [\tilde{\phi}_R] \text{ and } [\tilde{\phi}_L] \text{ minimizing } \left\|\tilde{\phi}_R - \phi_R\right\| \text{ and } \left\|\tilde{\phi}_L - \phi_L\right\| \text{ with } [\tilde{\phi}_R\tilde{\phi}_L^T] = 0, \tag{18}$$

in which $[\phi_R]$ and $[\phi_L]$ are two given matrices $n \times 2n$ and $\|\cdot\|$ is a matrix norm.

A solution of this constrained minimization problem can be found using a Lagrange cost function H to be minimized, including a Lagrange multiplier matrix $[\delta]$. In a symbolic point of view, the cost function can be written as:

$$H = \frac{1}{2} \left\| \tilde{\phi}_R - \phi_R \right\| + \left\| \tilde{\phi}_L - \phi_L \right\| + [\delta] \otimes [\tilde{\phi}_R \tilde{\phi}_L^T], \tag{19}$$

where $\otimes$ can be seen as a tensor product. The minimization of this cost function leads to:

$$\begin{cases} [\tilde{\phi}_R] = [E_n - \delta\delta^T]^{-1}[\phi_R - \delta\phi_L^*], \\ [\tilde{\phi}_L] = [E_n - \delta^T\delta]^{-1}[\phi_L - \delta^T\phi_R^*]. \end{cases} \tag{20}$$

This gives the expression of the modified complex vectors when $[\delta]$ is known. To obtain the value of the Lagrange multipliers, one has to solve a Riccati equation:

$$0 = [\phi_R \phi_L^T] - [\phi_R \phi_R^{*T}]\,[\delta] - [\delta]\,[\phi_L^* \phi_L^T] + [\delta]\,[\phi_L^* \phi_R^{*T}]\,[\delta]. \tag{21}$$

This equation is the generalization of the one proposed in reference [1]. This Riccati equation is not the one which is classicaly used in active control theory, since both matrices in factor of $[\delta]$ are not transposed one from another. Nevertheless, a quite efficient method can be used to find a solution of this equation using a Newton technique, which requires the resolution of a Sylvester equation that can be solved using several techniques among which the one proposed in reference [10]. One could note that the proposed methodology to solve the Riccati equation could be inefficient in the context of active control, since in this case the "closest" solution to the initial value of $[\delta]$ which is found could be another solution than the optimal one in the sense of active control (real-valued positive-definite solution corresponding to the steady-state optimal configuration of the control). In our case, the optimal solution is supposed to be "close" to the initial state (i.e. the experimental identification is not too far from the "exact" solution), and the Newton algorithm is likely to converge to the solution of interest.

5 Rotordynamics application

A wide class of non self-adjoint problems is constituted of rotordynamics applications. A whirling beam example, which has been described in [11], is considered here to illustrate the application of the method in this context. This high speed gyroscopic system includes a lumped mass at the center of the beam. The system is discretized using 10 degrees of freedom, and the corresponding matrices (including gyroscopic and circulatory terms) are:

$$[M] = \begin{bmatrix} \mathcal{M} & 0 \\ 0 & \mathcal{M} \end{bmatrix}, \; [C] = \begin{bmatrix} \mathcal{C} & \mathcal{G} \\ -\mathcal{G} & \mathcal{C} \end{bmatrix}, \; [K] = \begin{bmatrix} \mathcal{K}^a & \mathcal{H} \\ -\mathcal{H} & \mathcal{K}^b \end{bmatrix}, \tag{22}$$

including 5×5 matrices such as:

$$\begin{aligned} \mathcal{M}_{pq} &= mL\delta_{pq} + 2M\sin(p\pi/2)\sin(q\pi/2), \\ \mathcal{C}_{pq} &= (c+h)L\delta_{pq}, \\ \mathcal{G}_{pq} &= -2\Omega\mathcal{M}_{pq}, \\ \mathcal{K}^a_{pq} &= 2\left(k_1 + (-1)^{p+q}k_2\right)pq\pi^2/L^2 + EI_x p^2q^2\pi^4/L^3\delta_{pq} - \Omega^2\mathcal{M}_{pq}, \\ \mathcal{K}^b_{pq} &= 2\left(k_1 + (-1)^{p+q}k_2\right)pq\pi^2/L^2 + EI_y p^2q^2\pi^4/L^3\delta_{pq} - \Omega^2\mathcal{M}_{pq}, \\ \mathcal{H}_{pq} &= -h\Omega L\delta_{pq}, \end{aligned} \tag{23}$$

where δ_{pq} is the Kronecker symbol, and the system properties are $m = 10$ kg.m^{-1}, $M = 10$ kg, $L = 5$ m, $EI_y = 9L^3/5\pi^2$ N.m^2, $EI_x = 4L^3/5\pi^2$ N.m^2, $k_1 = k_2 = L^2/20$ N.m, $\Omega = \sqrt{21.6}\pi$ rad.s^{-1}, $c = h = 0.25$ N.S.m^{-1}.

The complex eigenvalues and left and right eigenvectors are evaluated using the state-space form, and an artificial noise is added to the calculated values (random noise on real and imaginary parts with a relative magnitude of 1% for eigenvalues and 3% for eigenvectors).

The efficiency of the methodology can be illustrated using FRFs. The figure 1 exhibits the following curves:

- The *Reference* FRF, calculated from the initial given matrices. It is supposed to represent the behavior of the system to be identified. For illustration purpose, only one of the many FRFs is presented here (corresponding to the collocated input and output on the first degree of freedom).
- The *Modal* FRF re-built after noise introduction on eigenvectors. This is associated to ”experimentally” identified eigenvalues and eigenvectors. An important point is that this curve is visually coincident to the reference one. In an experimental procedure, this would indicate that the identification is correct.
- The *Direct* FRF reconstructed using the matrices obtained after solving the inverse problem from the eigenvalues and eigenvectors affected by noise. This procedure clearly fails in this case, because equations (13) and (15) are not verified strictly since the properness condition is not verified. Some small differences in the eigendata can induce large discrepancies on the identified system.
- The *Proper* FRF obtained using data with enforcement of the properness condition, to be discussed in the next paragraph. This curve is also visually coincident with the reference one.

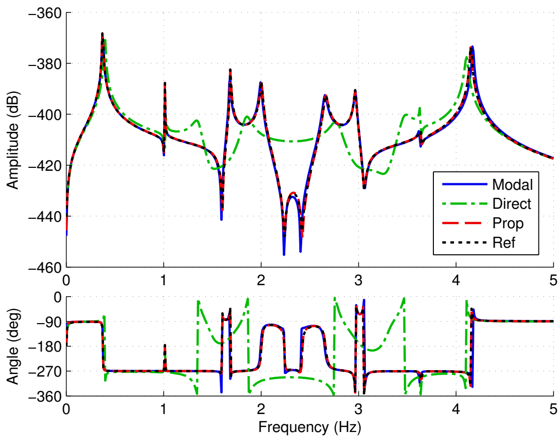

Figure 1: Frequency Response Functions between dofs 5 and 7 - Original system (Ref), modal response of disturbed system (Modal), direct reconstruction (Direct), properness enforced reconstruction (Prop)

One important point is that while the properness condition is not verified, there is no equivalent system that is able to represent the behavior of the system with 5 degrees of freedom, and this is the reason why the direct reconstruction fails.

The figure 1 exhibits the FRFs for the disturbed configuration and both reconstructions. One can observe first that including some noise in the eigenvectors does not affect much the FRFs, which once again indicates that even with good values of criteria based on FRFs errors, the matrix reconstruction can be poor. The second observation concerns the enforcement of the properness condition that leads to errors which are very low compared with those obtained by direct reconstruction. One interesting trend is that the lowest values of error are obtained for frequencies corresponding to resonances of the system, while the maximum values are related to anti resonances. Concerning FRFs obtained from matrices identified with inverse procedure, one can observe in figure 1 that the properness enforcement allows a correct reconstruction, while the direct inverse technique leads to significant errors. The figure 2 shows the efficiency of the properness enforcement using a graphical representation of system matrices, since the direct reconstruction introduces extra terms

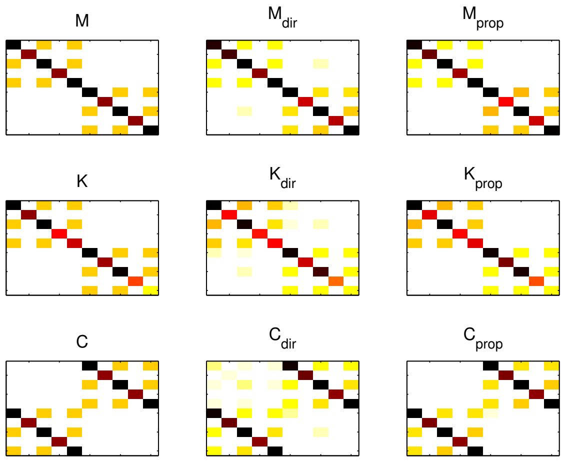

Figure 2: Graphical representation of matrices - Original system, direct reconstruction and reconstruction with properness enforcement (white = zero value, black = maximum amplitude)

in matrices that lead to large errors in FRFs. The matrix which is the most affected by the use of unproper eigenvectors is the damping matrix. This will be also reported in the next illustrations. One can conclude that damping matrices identification without properness identification has many chances to fail, and that the properness enforcement methodology can be seen as a physical regularization procedure.

6 Experimental illustration

6.1 Description of the experimental set-up

In this section an experimental illustration of the methodology is presented. The figure 3 shows the experimental set-up which has been used. It is constituted with two bending beams which are coupled through their bases by a common ”clamping” device. The frequency range of interest concerns the two firsts modes of the coupled system, which could be represented by a 2-degrees of freedom equivalent model, using points 1 and 2 indicated in figure 3 as reference points. These points are equipped with accelerometers and some contactless force transducers are used to excite the structure, with force sensors. An electrical intensity probe has also been used to check the value of the force sensors and to verify that the moving masses do not perturb the measured information. The system itself is characterized by symmetric matrices, the unsymmetric parts are introduced using an active device.

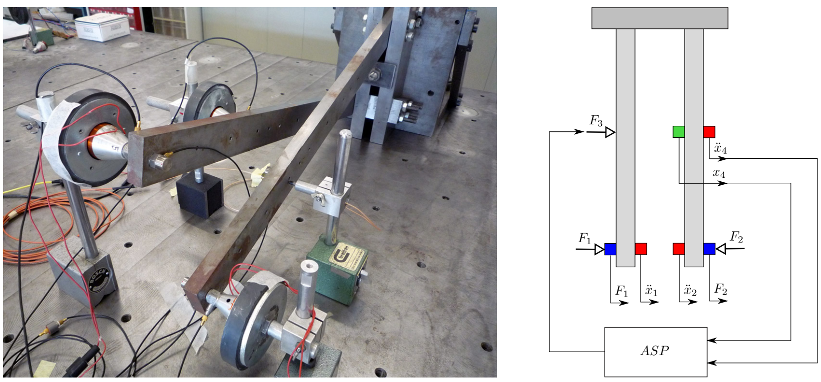

Figure 3: Experimental set-up with analogical loop

The active part includes an Analogic Signal Processing (ASP) circuit, which introduces a force at point 3 depending on the value of displacement or acceleration of point 4. The results which are presented here correspond to a force feedback which is directly proportional to the displacement: it can be seen as a non-symmetric term in the stiffness matrix. The gain of the feedback loop was chosen such that the system remains in the stable domain. All measurements have been performed using a sine-stepping approach to avoid undesirable effects due to broadband signal in feedback loop.

6.2 Left and right modes identification

The classical LSCF method [7] can be easily adapted to identify the left and right complex modes of the structure. As indicated in [12], the full identification of the left and right vectors requires the use of sensors and excitations at every point of interest (i.e. every degree of freedom of the model). It is emphasized that, as soon as relationships between left and right eigenvectors are available, this condition is no longer necessary and only a limited number of excitation points can be used. In the present case, in which there is no link between left and right vectors, there is no alternative to the excitation of all points of interest, and extension of LSCF technique for non-self adjoint problem has to be considered.

6.3 Results and discussion

A similar analysis can be performed with the closed loop. In this case the system is not symmetric. The results presented here correspond to a gain value of 6 in the feedback loop, corresponding to a strong feedback that remains in the stable domain. The figure 4 shows two measured FRFs and the corresponding synthetized ones using the identified complex modes, the matrices obtained by direct inversion and those corresponding to properness enforcement. It is then clear that once again the properness enforcement gives very good results compared with the direct procedure.

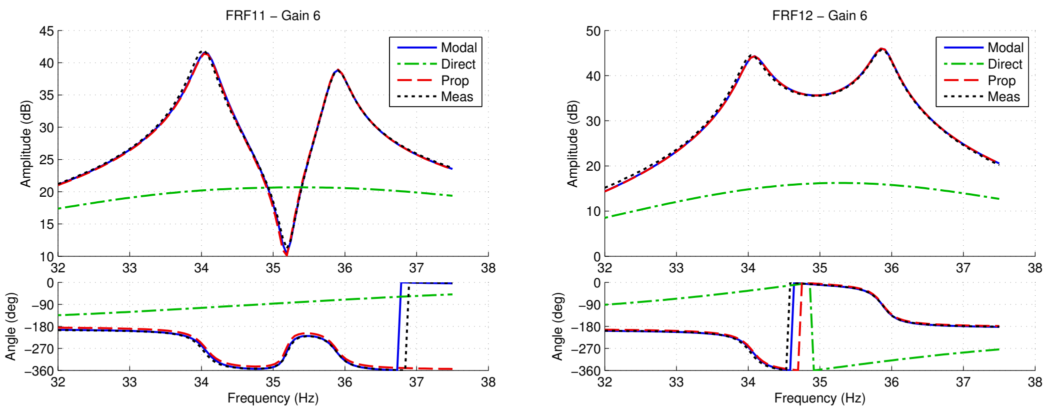

Figure 4: Examples of FRFs for closed loop with gain in 6th position - Original system (Ref), modal synthesis from identified modes (Modal), direct reconstruction (Direct), properness enforced reconstruction (Prop)

The modified LSCF technique leads to the identification of right and left eigenvectors:

$$[\phi_R^6] = \begin{bmatrix} 0.0256 - 0.0322i & 0.0188 - 0.0201i \\ 0.0368 - 0.042i & -0.0381 + 0.0505i \end{bmatrix}, [\phi_L^6] = \begin{bmatrix} 0.0406 - 0.0511i & 0.0399 - 0.0425i \\ 0.0165 - 0.0164i & -0.0246 + 0.0278i \end{bmatrix}. \quad (24)$$

The non symmetry of the system is clear. Once again, the properness enforcement induces small shifts in the identified complex modes:

$$[\tilde{\phi}_R^6] = \begin{bmatrix} 0.0287 - 0.0287i & 0.0216 - 0.0173i \\ 0.0380 - 0.0411i & -0.0397 + 0.0491i \end{bmatrix}, [\tilde{\phi}_L^6] = \begin{bmatrix} 0.0427 - 0.0490i & 0.0413 - 0.0413i \\ 0.0190 - 0.0138i & -0.0269 + 0.0249i \end{bmatrix}. \quad (25)$$

The changes in complex shapes are illustrated in figure 5, in which it can be observed that the changes mainly occur on the phase values, while the amplitudes of the vectors remain almost unchanged. The direct inversion

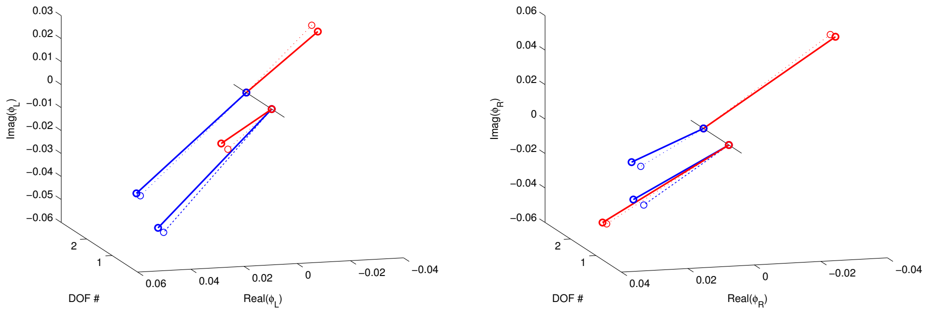

Figure 5: Right and left eigenmodes - original shapes (dashed lines), modified shapes (solid lines); mode 1 (blue), mode 2 (red)

of the problem to estimate the values of the matrices is clearly not the good way to obtain a physical value of the damping matrix:

$$[M^6] = \begin{bmatrix} 0.543 & 0.0173 \\ 0.0350 & 0.628 \end{bmatrix},\ [C^6] = \begin{bmatrix} -18.4 & -1.25 \\ -1.05 & -20.6 \end{bmatrix},\ [K^6] = \begin{bmatrix} 2.59\times10^4 & 98.9 \\ -1.01\times10^3 & 3.07\times10^4 \end{bmatrix}. \quad (26)$$

The reconstruction after properness enforcement is much better:

$$[\tilde{M}^6] = \begin{bmatrix} 0.541 & 0.0168 \\ 0.0356 & 0.627 \end{bmatrix},\ [\tilde{C}^6] = \begin{bmatrix} 1.60 & -0.451 \\ 1.48 & 0.608 \end{bmatrix},\ [\tilde{K}^6] = \begin{bmatrix} 2.57\times10^4 & 87.8 \\ -988 & 3.06\times10^4 \end{bmatrix}. \quad (27)$$

The values of these matrices are clearly in accordance with the experimental conditions: compared with identified values in open loop, the mass matrix is almost unchanged, the damping matrix is slightly changed, according to the phase delays in the feedback loop, while the extradiagonal term of the stiffness matrix is very affected by the force feedback.

This analysis can be performed for several amplifier gain values in the feedback loop, in order to check the evolution of matrices terms when the gain changes. The figure 6 shows the results of the analysis. The mass matrix seems to be almost unsensitive to the gain value, which is in accordance with the fact that the feedback is supposed to be proportional to the displacement. The extradiagonal terms are varying, but they are much smaller than the diagonal ones, this is physically reasonable. The stiffness matrix extradiagonal term K_{12} exhibits the largest evolution when the amplifiers gain changes, this is in accordance with the nature of the feedback, while the diagonal terms remain almost constant. Finally, even the behavior of the damping matrix is coherent: there is a clear shift between the open loop configuration and the feedback loop configuration, which is due to the signal processing that induces phase delays which can be interpreted as damping effects. Nevertheless, once the feedback is considered, diagonal terms are almost constant, while an evolution on extradiagonal terms can be observed, due to the amplification of the feedback gain.

7 Extension of properness to vibroacoustics

7.1 Movement equations

Discretizing an internal vibro-acoustical problem using the natural fields for the description of the structure (those which can be directly measured), i.e. displacement for the structure and acoustic pressure for the cavity, leads to the following matrix system [13]:

$$\underbrace{\begin{bmatrix} M_s & 0 \\ L^T & M_a \end{bmatrix}}_{[M]} \underbrace{\begin{Bmatrix} \ddot{x} \\ \ddot{p} \end{Bmatrix}}_{\{\ddot{q}\}} + \underbrace{\begin{bmatrix} C_s & 0 \\ 0 & C_a \end{bmatrix}}_{[C]} \underbrace{\begin{Bmatrix} \dot{x} \\ \dot{p} \end{Bmatrix}}_{\{\dot{q}\}} + \underbrace{\begin{bmatrix} K_s & -L \\ 0 & K_f \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} x \\ p \end{Bmatrix}}_{\{q\}} = \underbrace{\begin{Bmatrix} F_s(t) \\ \dot{Q}_a(t) \end{Bmatrix}}_{\{f(t)\}}, \quad (28)$$

in which $\{x\}$ is the vector of generalized displacements of the structure, $\{p\}$ is the vector of acoustic pressures, $[M_s]$ is the mass matrix of the structure, $[M_a]$ is called "mass" matrix of acoustic fluid (its components

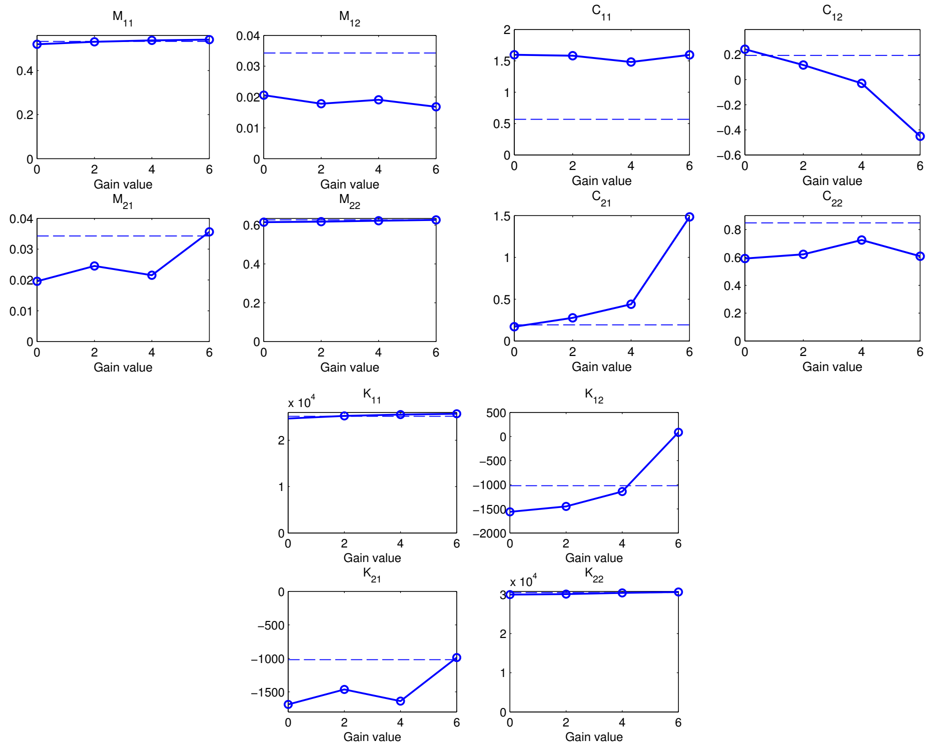

Figure 6: Evolution of mass, stiffness and damping matrices (SI units) terms versus gain value of amplifier (dashed line = open loop values)

are not homogeneous to masses, the name is chosen for analogy with structural denomination), $[K_s]$ is the stiffness matrix of the structure, $[K_a]$ is the "stiffness" matrix of fluid domain, $[L]$ is the vibro-acoustic coupling matrix, $[C_s]$ and $[C_a]$ respectively represent structural and acoustic losses. This formulation includes the hypothesis that there is no loss at the coupling between structural and acoustic parts, and that internal losses can be represented using equivalent viscous models. $\{F_s(t)\}$ is the vector representing the generalized forces on the structure, while $\{\dot{Q}_a(t)\}$ is associated to acoustic sources (volume acceleration) in the cavity.
The non self-adjoint character of the formulation induces difficulties for the resolution of this kind of problem using modal decomposition. Some research works have been done to find symmetric formulations dedicated to coupled vibroacoustic problems [13, 14], but up to now, these formulations are either not able to take into account dissipation in the fluid domain, or lead to full matrices which can not be efficiently used for large models. The technique which is widely used for model reduction in the field of numerical analysis is based on the use of two uncoupled bases (structural and fluid), and the solution of the coupled system is projected on these bases, even if some convergence problems can be found [15]. Being able to evaluate numerically the coupled modal basis in an efficient way is still a challenge, in particular for damped problems. On the other hand, starting from experimental data, it is possible to identify these modes [2], and one of the ways to build reduced models could be to follow the same methodology as the one used in structural dynamics, extended to vibroacoustics.

7.2 Complex modes for vibro-acoustics

The non-self adjoint character of vibroacoustic problem is particular since extradiagonal coupling terms that appear in mass and stiffness matrices are linked. It can be shown [2] that the left eigenvectors are related to the right ones by the following relationship:

$$\text{If } \{\phi_{Rj}\} = \left\{ \begin{array}{c} X_j \\ P_j \end{array} \right\} \text{ then } \{\phi_{Lj}\} = \left\{ \begin{array}{c} X_j \\ -P_j/\lambda_j^2 \end{array} \right\}. \tag{29}$$

This point is fundamental for modal analysis of coupled system, since only extraction of right eigenvectors is required to derive the left ones. The previous relation can also be written as:

$$\text{If } [\phi_R] = \begin{bmatrix} X \\ P \end{bmatrix} \text{ then } [\phi_L] = \begin{bmatrix} X \\ -P\Lambda^{-2} \end{bmatrix}. \tag{30}$$

7.3 Properness for vibro-acoustics

For the particular vibro-acoustic case, left eigenvectors are linked to right ones, and the properness condition can be written using only the right complex eigenvectors:

$$\begin{bmatrix} XX^T & -X\Lambda^{-2}P^T \\ PX^T & -P\Lambda^{-2}P^T \end{bmatrix} = 0. \tag{31}$$

7.4 Methodologies for properness enforcement

7.4.1 Structural dynamics based strategy

When the complex modes are available from experimental identification, one can use inverse relationships in order to find the reduced model which is supposed to have the same behavior as the measured one. The fact is that in general, the modes do not verify the properness condition (31). In the particular case of vibroacoustics, one can try to follow the same methodology as the one used in structural dynamics. The following constrained optimization problem should then be solved:

$$\begin{array}{c}\text{Find } [\tilde{X}] \text{ and } [\tilde{P}] \text{ minimizing } \left\|\tilde{X} - X\right\| \text{ and } \left\|\tilde{P} - P\right\| \\ \text{while } [\tilde{X}\tilde{X}^T] = 0,\ \left[\tilde{X}\tilde{P}^T\right] = 0,\ \left[\tilde{X}\Lambda^{-2}\tilde{P}^T\right] = 0,\ \left[\tilde{P}\Lambda^{-2}\tilde{P}^T\right] = 0,\end{array} \tag{32}$$

in which $[X]$ and $[P]$ are two given complex rectangular matrices and $[\Lambda]$ is a given diagonal complex matrix. This problem can be re-written using four Lagrange multipliers matrices $[\delta_j]$ (j=1 to 4):

$$\left\{\begin{array}{l} 0 = \left\{\begin{array}{c}\tilde{X} \\ \tilde{P}\end{array}\right\} - \left\{\begin{array}{c}X \\ P\end{array}\right\} + \frac{1}{2}\begin{bmatrix}\delta_1 + \delta_1^T & \delta_2 \\ \delta_2^T & 0\end{bmatrix}\left\{\begin{array}{c}\overline{\tilde{X}} \\ \overline{\tilde{P}}\end{array}\right\} \\ \qquad -\frac{1}{2}\begin{bmatrix}0 & \delta_3 \\ \delta_3^T & \delta_4 + \delta_4^T\end{bmatrix}\left\{\begin{array}{c}\overline{\tilde{X}\Lambda^{-2}} \\ \overline{\tilde{P}\Lambda^{-2}}\end{array}\right\} \\ 0 = [\tilde{X}\tilde{X}^T] \\ 0 = [\tilde{X}\tilde{P}^T] \\ 0 = [\tilde{X}\Lambda^{-2}\tilde{P}^T] \\ 0 = [\tilde{P}\Lambda^{-2}\tilde{P}^T]. \end{array}\right. \tag{33}$$

Solving this problem is clearly not easy because of the presence of the Λ matrices that makes impossible to find explicitly the expression of multipliers versus the unknown vectors. An iterative procedure could be investigated but this is not the best way to obtain quick results that can be used in real-time during modal analysis. Some simplified methods have been proposed [16], among which one is called over-properness: considering the fact that the method developed for structural dynamics [1] is valid for all matrix $[x]$ subjected to a properness condition $[xx^T = 0]$, one can use as $[x]$ matrix:

$$[x] = \begin{bmatrix} X \\ P \\ -P\Lambda^{-2} \end{bmatrix}, \tag{34}$$

then:

$$[xx^T] = \begin{bmatrix} XX^T & XP^T & -X\Lambda^{-2}P^T \\ PX^T & PP^T & -P\Lambda^{-2}P^T \\ -P\Lambda^{-2}X^T & -P\Lambda^{-2}P^T & P\Lambda^{-4}P^T \end{bmatrix}. \tag{35}$$

It can be observed that the four required terms of equation (31) are included in this matrix, while two of them are not theoretically required. Using this vector in the procedure detailed by equations (18) leads to a so-called over-proper solution which includes more constraints than those required, but that includes the required ones.

7.4.2 Alternative strategy

Another thinkable way for obtaining matrices of system (28) is to use a least-square approach. Being given a set of measured frequency responses $[X]$ corresponding to a set of measured excitations $[F]$, the matrices can be found by solving the following problem:

$$\min_{([M],[C],[K])\in\mathbb{M}\times\mathbb{C}\times\mathbb{K}} \varepsilon(M,C,K) = ||[-\omega^2 M + i\omega C + K][X] = [F]||, \tag{36}$$

where $\mathbb{M}\times\mathbb{C}\times\mathbb{K}$ is the space of admissible matrices (whose topology correspond to a vibroacoustic problem). The function to minimize can be written using a linear system:

$$\varepsilon(M,C,K) = ||[D]\{\alpha\} - \{G\}||, \tag{37}$$

where $\{\alpha\} = \{M_{11}M_{12}...K_{nn}\}^T$, while $[D]$ includes terms coming from $[X]$ and ω, and $\{G\}$ includes terms coming from $\{F\}$. The matrices components can finally be found using pseudo-inverse for minimization of least-square error:

$$\{\alpha\} = [D]^{\dagger}\{G\}. \tag{38}$$

This strategy can then be used to directly find the matrices without using the complex eigenvectors, which can be found in post processing stage by solving the eigenvalue problem. This approach implies undoubtedly a higher calculation cost than the previous strategies, in particular for systems with numerous degrees of freedom, while in the case of low order reduced models, this strategy could be appropriate.

7.5 Experimental test-case

An experimental test-case based on measurements on a guitar given by F. Gautier from LAUM-Le Mans and J.-L. Le Carrou from LAM-Paris VI is now considered. In that case, only two degrees of freedom are considered, in order to represent the behavior of the guitar in the frequency range corresponding to the so-called A0 and T1 modes, which are of first interest in the design of the instrument [17, 18, 19]. These two modes have been identified experimentally by a curve fitting technique, and the FRFs built from these two modes is considered as the reference in the following. The direct approach leads once again to bad estimation of damping terms:

$$[M] = \begin{bmatrix} 3.10\times10^{-2} & 2.10\times10^{-9} \\ 3.88\times10^{-2} & 2.85\times10^{-7} \end{bmatrix}, \quad [C] = \begin{bmatrix} -2.23 & 2.19\times10^{-6} \\ -3.68 & -3.72\times10^{-5} \end{bmatrix}, \quad [K] = \begin{bmatrix} 2.30\times10^{4} & -3.59\times10^{-3} \\ 705 & 1.28\times10^{-5} \end{bmatrix}. \tag{39}$$

The properness enforcement allows the damping terms to become more physical:

$$[M] = \begin{bmatrix} 3.09\times10^{-2} & 1.88\times10^{-9} \\ 3.84\times10^{-2} & 2.83\times10^{-7} \end{bmatrix}, \quad [C] = \begin{bmatrix} 0.942 & -1.52\times10^{-6} \\ 0.315 & 7.55\times10^{-6} \end{bmatrix}, \quad [K] = \begin{bmatrix} 2.27\times10^{4} & -3.57\times10^{-3} \\ 632 & 1.26\times10^{-5} \end{bmatrix}. \tag{40}$$

Finally, the least-square error (LSE) approach leads to a correct topology of matrices, with physical damping term on structural part, while its value on the acoustic part is negative, but very small:

$$[M] = \begin{bmatrix} 2.91\times10^{-2} & 0 \\ 3.44\times10^{-2} & 2.57\times10^{-7} \end{bmatrix}, \quad [C] = \begin{bmatrix} 1.37 & 0 \\ 0 & -2.97\times10^{-6} \end{bmatrix}, \quad [K] = \begin{bmatrix} 2.15\times10^{4} & -3.45\times10^{-3} \\ 0 & 1.15\times10^{-5} \end{bmatrix}. \tag{41}$$

The comparison of FRFs, in figure 7, leads to the conclusion that the direct inversion is clearly not the right way to proceed. One can also observe that, even if the topology of the over-proper solution is not exactly the right one, the global error on FRFs reconstruction is of the same order as in the case of LSE technique. Indeed, depending on the objective, one should evaluate matrices from both formulations and choose the ones which are the most appropriate. One can also point out the fact that all methodologies lead to quite good estimation of mass and stiffness matrices, the critical point being the evaluation of damping matrix. The properness enforcement is not sufficient to obtain the correct topology, but the improvement is nevertheless clear, it can be seen as a regularization procedure for the inverse problem which is addressed here.

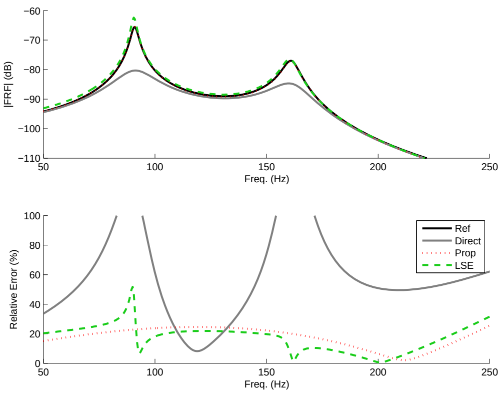

Figure 7: Methodologies for properness enforcement on guitar measurements

8 Conclusion

In this paper, the properness condition of complex modes is investigated and extended to non symmetric second order systems. An original methodology is proposed to correct the experimental complex mode shapes in an optimal way, in order to physically regularize the inverse procedure to identify the system matrices. Three illustrations are presented. The first one is a numerical test based on an arbitrary simulated system without any link to a particular physical problem. The second one is a numerical test coming from the discretization of a rotordynamic application. The third one is an experimental test based on two coupled bending beams with a non-colocated active feedback force. The trends observed on the three examples lead to the following conclusions:

- Enforcing the properness condition on complex modes produces very slight changes in their components, and these changes mainly occur on the phase values while the amplitudes remain almost unchanged.
- When reconstructing a physical model from complex eigenvalues and eigenvectors, very small differences in the complex modes lead to very large differences in system's matrices, especially in the damping matrix. As a consequence, even a low level of noise affecting the complex modes can lead to a totally wrong and non-physical estimation of the damping matrix.
- Modifying the complex modes by enforcing the properness condition is a good way to regularize the inverse problem. The reconstructed matrices and the derived FRFs show that the slight modifications applied to the eigenvectors drastically reduce the effect of uncertainties and allow the identification of physically meaningfull matrices.

Finally, the enforcement of the properness condition on experimentally identified complex modes should be considered each time the reconstruction of a physical system is envisaged.

The properness condition can be easily extended to vibroacoustics: this property must be verified by complex modes in order to be those of a physical system. Two techniques have been proposed to enforce the property on eigenshapes that do not verify it, leading to much better results than those corresponding to the use of initial identified vectors. The first technique is based on the structural dynamics procedure, leading to enforcement of more conditions than the theoretically required ones. The second one is based on a least square error minimization. None of the two methods exhibits perfect results, so it is clear that one of the next challenges in vibroacoustic reduced models identification based on experimental modal analysis will be the improvement of the properness enforcement methodology. Up to now, the two proposed methods can be applied for a given application and the user can choose between results depending of the efficiency of the identified reduced model.

Acknowledgements

The authors would like to thank Jean-Loïc Le Carrou from Laboratoire d'Acoustique Musicale (Paris VI) and François Gautier from the Laboratoire d'Acoustique de l'Université du Maine, for the fruitful discussions and for allowing us to use their measurements data, used in the last part of the chapter.

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