Extreme values of the Dedekind Ψ function

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Abstract

Let $\Psi(n) := n\prod_{p|n}(1+\frac{1}{p})$ denote the Dedekind Ψ function. Define, for $n \geq 3$, the ratio $R(n) := \frac{\Psi(n)}{n \log\log n}$. We prove unconditionally that $R(n) < e^\gamma$ for $n \geq 31$. Let $N_n = 2 \cdots p_n$ be the primorial of order n. We prove that the statement $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ for $n \geq 3$ is equivalent to the Riemann Hypothesis.

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I. Introduction

The Dedekind Ψ function is an arithmetic multiplicative function defined for every integer $n > 0$ by

$$\Psi(n) := n\prod_{p|n}(1+\frac{1}{p}).$$

It occurs naturally in questions pertaining to dimension of spaces of modular forms [2] and to the commutation of operators in quantum physics [6]. It is related to the sum of divisor function

$$\sigma(n) = \sum_{d|n} d$$

by the inequalities

$$\Psi(n) \leq \sigma(n),$$

and the fact that they coincide for n squarefree. It is also related to Euler φ function by the inequalities

$$n^2 > \varphi(n)\Psi(n) > \frac{n^2}{\zeta(2)}$$

derived in Proposition 5 below.

In view of the studies of large values of σ [7] and of low values of φ [4], it is natural to study both the large and low values of Ψ. To that end, we define the ratio $R(n) := \frac{\Psi(n)}{n \log\log n}$. The motivation for this strange quantity is the asymptotics of Proposition 3. We prove unconditionally that $R(n) < e^\gamma$, for $n \geq 31$ in Corollary 2. Note that this bound would follow also from the Robin inequality

$$\sigma(n) \leq e^\gamma n \log\log n$$

for $n \geq 5041$ under Riemann Hypothesis (RH) [7], since $\Psi(n) \leq \sigma(n)$.
In the direction of lower bounds, we prove that the statement $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ for $n \geq 3$ is equivalent to RH, where $N_n = 2 \cdots p_n$ is the primorial of order n. The proof relies on Nicolas's work on the Euler totient function [4].

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II. Reduction to primorial numbers

Define the primorial number N_n of index n as the product of the first n primes

$$N_n = \prod_{k=1}^{n} p_k,$$

so that $N_1 = 2,\ N_2 = 6, \cdots$ and so on. As in [4], the primorial numbers play the role here of superabundant numbers in [7]. They are champion numbers (ie left to right maxima) of the function $x \mapsto \Psi(x)/x$:

$$\frac{\Psi(m)}{m} < \frac{\Psi(n)}{n} \text{ for any } m < n, \tag{1}$$

We give a proof of this fact, which was observed in [5].

Proposition 1: The primorial numbers N_n are exactly the champion numbers of the function $x \mapsto \Psi(x)/x$.

Proof: The proof is by induction on n. The induction hypothesis H_n is that the statement is true up to N_n. It is clear that H_2 is true. Let $N_n \leq m < N_{n+1}$ be a generic integer. The number m has at most n distinct prime factors. This, in combination with the observation that $1 + 1/x$ is monotonically decreasing as a function of x, shows that $\Psi(m)/m \leq \Psi(N_n)/N_n$. Further $\Psi(N_n)/N_n < \Psi(N_{n+1})/N_{n+1}$. The proof of H_{n+1} follows. ■

In this section we reduce the maximization of $R(n)$ over all integers n to the maximization over primorials.

Proposition 2: Let n be an integer ≥ 2. For any m in the range $N_n \leq m < N_{n+1}$ one has $R(m) \leq R(N_n)$.

Proof: Like in the preceding proof we have

$$\Psi(m)/m \leq \Psi(N_n)/N_n$$

Since $0 < \log\log 6 < \log\log N_n \leq \log\log m,$ the result follows. ■

III. Ψ at primorial numbers

We begin with an easy application of Mertens formula [3, Th. 429].

Proposition 3: We have, as $n \to \infty$

$$\lim R(N_n) = \frac{e^\gamma}{\zeta(2)} \approx 1.08.$$

Proof: Writing $1+1/p = (1-1/p^2)/(1-1/p)$ in the definition of $\Psi(n)$ we can combine the Eulerian product for $\zeta(2)$ with Mertens formula

$$\prod_{p \leq x} (1 - 1/p)^{-1} \sim e^\gamma \log(x)$$

to obtain

$$\frac{\Psi(N_n)}{N_n} \sim \frac{e^\gamma}{\zeta(2)} \log(p_n),$$

Now the Prime Number Theorem [3, Th. 6, Th. 420] states that $x \sim \theta(x)$ for x large. where $\theta(x)$ stands for Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \leq x} \log p.$$

This shows that, taking $x = p_n$ we have

$$p_n \sim \theta(p_n) = \log(N_n).$$

The result follows. ■

This motivates the search for explicit upper bounds on $R(N_n)$ of the form $\frac{e^\gamma}{\zeta(2)}(1+o(1))$. In that direction we have the following bound.

Proposition 4: For n large enough to have $p_n \geq 20000$, that is $n \geq 2263$, we have

$$\frac{\Psi(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(2)}(\log\log N_n + \frac{1.125}{\log p_n})$$

So, armed with this bound, we derive a bound of the form $R(N_n) < e^\gamma$ for $n \geq A$, with A a constant.

Corollary 1: For $n \geq 4$, we have $R(N_n) < e^\gamma = 1.78\cdots$

Proof:

For $p_n \geq 20000$, we use the preceding proposition. We need to check that

$$\exp(2/p_n)(1 + \frac{1.125}{\log(p_n)\log\log(N_n)}) \leq \zeta(2).$$

Since the LHS is a decreasing function of n it is enough to check this inequality for the first n such that $p_n \geq 20000$.
For $5 \leq p_n \leq 20000$, that is $3 \leq n \leq 2262$ we simply compute $R(N_n)$, and check that it is $< e^\gamma$. ■

We can extend this Corollary to all integers > 30 by using the reduction of preceding section, combined with some numerical calculations for $30 < n \leq N_4$.

Corollary 2: For $n > 30$, we have $R(n) < e^\gamma$.

We prepare for the proof of the preceding Proposition by a pair of Lemmas. First an upper bound on a partial Eulerian product from [8, (3.30) p.70].

Lemma 1: For $x \geq 2$, we have

$$\prod_{p \leq x}(1 - 1/p)^{-1} \leq e^\gamma(\log x + \frac{1}{\log x})$$

Next an upper bound on the tail of the Eulerian product for $\zeta(2)$.

Lemma 2: For $n \geq 2$ we have

$$\prod_{p > p_n}(1 - 1/p^2)^{-1} \leq \exp(2/p_n)$$

Proof: Use Lemma 6.4 in [1] with $x = p_n$ and $t = 2$. ■

We are now ready for the proof of Proposition 4.

Proof:

Write

$$\frac{\Psi(N_n)}{N_n} = \prod_{k=1}^{n} \frac{1 - 1/{p_k}^2}{1 - 1/p_k}$$

and use both lemmas to derive

$$\frac{\Psi(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(2)}(\log p_n + \frac{1}{\log p_n}).$$

Now we get rid of the first $\log$ in the RHS by the bound of [7, p.206]

$$\log(p_n) < \log\log N_n + \frac{0.125}{\log p_n}.$$

The result follows.

■

IV. Lower bounds

We reduce first to Euler's φ function.

Proposition 5: For $n \geq 2$ we have

$$n^2 > \varphi(n)\Psi(n) > \frac{n^2}{\zeta(2)}$$

Proof: The first inequality follows at once upon writing

$$\frac{\varphi(n)\Psi(n)}{n^2} = \prod_{p|n}(1 - 1/p^2),$$

a product of finitely many terms < 1. Notice for the second inequality that

$$\frac{\varphi(n)\Psi(n)}{n^2} = \prod_{p|n}(1 - 1/p^2) > \prod_{p}(1 - 1/p^2),$$

an infinite product that is the inverse of the Eulerian product for $\zeta(2)$.

■

Theorem 1: Under RH the ratio $R(N_n)$ is $> \frac{e^\gamma}{\zeta(2)}$ for $n \geq 3$. If RH is false, this is still true for infinitely many n.

Proof:

Follows by Proposition 5, combined with [4, Theorem 2]. ■

In view of this result and of numerical experiments the natural conjecture is

Conjecture 1: For all $n \geq 3$ we have $R(N_n) > \frac{e^\gamma}{\zeta(2)}$.

The main result of this note is the following.

Theorem 2: Conjecture 1 is equivalent to RH.

Proof: If RH is true we refer to the first statement of Theorem 1. If RH is false we consider the function

$$g(x) := \frac{e^\gamma}{\zeta(2)} \log\theta(x) \prod_{p\leq x}(1 + 1/p)^{-1},$$

Observing that $\log\theta(p_n) = \log\log N_n$, we see that $g(p_n) < 1$ is equivalent to $R(N_n) > \frac{e^\gamma}{\zeta(2)}$. We need to show that there exists an $x_0 \geq 3$ such that $g(x_0) > 1$ or equivalently $\log g(x_0) > 0$. Using once again the identity $1 - 1/p^2 = (1 - 1/p)(1 + 1/p)$, and [1, Lemma 6.4], we obtain, upon writing

$$-\log\zeta(2) = \sum_{p\leq x}\log(1 - 1/p^2) + \sum_{p>x}\log(1 - 1/p^2),$$

the bound

$$\log g(x) \geq \log f(x) - 2/x,$$

where f is the function introduced in [4, Theorem 3], that is

$$f(x) := e^\gamma \log\theta(x) \prod_{p\leq x}(1 - 1/p).$$

We know by [4, Theorem 3 (c)] that, if RH is false, there is a $0 < b < 1$ such that $\limsup x^{-b} f(x) > 0$ and hence $\limsup \log f(x) >> \log x$. Since $2/x = o(\log x)$, the result follows.

■

V. Conclusion

In this note we have derived upper and lower bounds on the Dedekind Ψ function. We show unconditionally that the function $\Psi(n)$ satisfies the Robin inequality. Since $\psi(n) \leq \sigma(n)$ this could be proved under RH [7] or by referring to [1]. Of special interest is Conjecture 1 which is shown here to be equivalent to RH. We hope this new RH criterion will stimulate research on the Dedekind Ψ function.

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