Singularity of type D_4 arising from four qubit systems

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An intriguing correspondence between four-qubit systems and simple singularity of type D_4 is established. We first consider the algebraic variety X of separable states within the projective Hilbert space $\mathbb{P}(\mathcal{H}) = \mathbb{P}^{15}$. Then, cutting X with a specific hyperplane H, we prove that the X-hypersurface, defined from the section $X \cap H \subset X$, has an isolated singularity of type D_4; it is also shown that this is the "worst-possible" isolated singularity one can obtain by this construction. Moreover, it is demonstrated that this correspondence admits a dual version by proving that the equation of the dual variety of X, which is nothing but the Cayley hyperdeterminant of type $2 \times 2 \times 2 \times 2$, can be expressed in terms of the SLOCC invariant polynomials as the discriminant of the miniversal deformation of the D_4-singularity.

Keywords: Quantum Information Theory, Entangled states, Simple singularities of hypersurfaces, Dynkin diagrams, Hyperdeterminant.

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I. INTRODUCTION

Several branches of geometry and algebra tend to play an increasing role in quantum information theory. We have in mind algebraic geometry for describing entanglement classes of multiple qubits[6,11,12,21], representation theory and Jordan algebras for entanglement and the black-hole/qubit correspondence[3–5], and geometries over finite fields/rings for deriving point-line configurations of observables relevant to quantum contextuality[17,22,23]. The topology of hypersurface singularitites, and the related Coxeter-Dynkin diagrams, represent another field worthwhile to be investigated in quantum information, as shown in this paper.

Dynkin diagrams are well known for classifying simple Lie algebras, Weyl groups, subgroups of $SU(2)$ and simple singularities, *i.e.* isolated singularities of complex hypersurfaces that are stable under small perturbations. More precisely, if we consider simple-laced Dynkin diagrams, *i.e.* diagrams of type $A-D-E$, we find objects of different nature classified by the same diagrams:

Type	Lie algebra	Subgroup of $SU(2)$	Hypersurface with simple singularity
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	cyclic group	$x_1^{n+1}+x_2^2+\cdots+x_k^2=0$
D_n	$\mathfrak{so}_{2n}(\mathbb{C})$	binary dihedral group	$x_1^{n-1}+x_1x_2^2+x_3^2+\cdots+x_k^2=0$
E_6	$\mathfrak{e}_6$	binary tetrahedral	$x_1^4+x_2^3+x_3^2+\cdots+x_k^2=0$
E_7	$\mathfrak{e}_7$	binary octahedral	$x_1^3x_2+x_2^3+x_3^2+\cdots+x_k^2=0$
E_8	$\mathfrak{e}_8$	binary icosahedral	$x_1^5+x_2^3+x_3^2+\cdots+x_k^2=0$

A challenging question in mathematics is to understand these ADE-correspondences by establishing a direct construction from one class of objects to the other. For instance, the construction of surfaces with simple singularities from the corresponding subgroup of $SU(2)$ is called the McKay correspondence. A construction due to Grothendieck allows us to recover the simple singularities of a given type from the nullcone (the set of nilpotent elements) of the corresponding simple Lie algebra. For an overview of such ADE correspondences, see Ref[24,25] and references therein.

Another construction connecting simple Lie algebras and simple singularities is due to Knop[14]. In his construction, Knop considers a unique smooth orbit, X, for the adjoint action of Lie group G on the projectivization of its Lie algebra $\mathbb{P}(\mathfrak{g})$ and cuts this variety by a specific hyperplane. The resulting X-hypersurface has a unique singular point of the same type as $\mathfrak{g}$.

Looking at ADE-correspondences in the context of QIT is a way to understand the role played by those diagrams in this field. In different classification schemes of four-qubit systems, the Dynkin diagram D_4 has already appeared thanks to the role played by the Lie algebra $\mathfrak{so}(8)$ (that is the type D_4). For instance, Verstraete *et al*'s classification[26] is based on the classification of the $SO(4) \times SO(4) \subset SO(8)$ orbits on $\mathcal{M}_4(\mathbb{C})$. Chterental and Djokovic[7] use the same group action and refer to (Remark 5.3 of Ref[7]) the Hilbert space of four qubits as a subspace of $\mathfrak{so}(8)$ whose SLOCC orbits arise from the trace of the adjoint $SO(8)$ orbits. In their study of the four-qubit classification from the string theory point of view, Borsten *et al*[2] employ a correspondence between nilpotent orbits of $\mathfrak{so}(4,4)$ (the real form of $\mathfrak{so}(8)$ with signature $(4,4)$) and nilpotent orbits of four-qubit systems. Last but not least, the relation between $\mathfrak{so}(8)$ and four-qubit systems has been pointed out by Lévay[15] in his paper on the black-hole/qubit correspondence. In this paper Lévay describes the Hilbert space of four qubits as the tangent space of $SO(4,4)/(SO(4) \times SO(4))$.

In the present paper, we will establish a correspondence between four-qubit systems and D_4-singularities by using a construction inspired by Knop's paper. In other words, we will establish an ADE-type correspondence between $SO(4,4)$ and singularities of type D_4 using the Hilbert space of four qubits.

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ be the Hilbert space of four-qubit systems. Up to scalar multiplication, a four-qubit $|\Psi\rangle \in \mathcal{H}$ can be considered as a point of the projective space $\mathbb{P}^{15} = \mathbb{P}(\mathcal{H})$. The set of separable states in $\mathcal{H}$ corresponds to tensors of rank one, *i.e.* tensors which can be factorized as $|\Psi\rangle = v_1 \otimes v_2 \otimes v_3 \otimes v_4$ with $v_i \in \mathbb{C}^2$. Adopting the notation $\{|0\rangle, |1\rangle\}$ for the single-qubit computational basis and $|ijkl\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle$ for the four-qubit basis, a general four-qubit state can be expressed as

$$|\Psi\rangle = \sum_{0 \leq i,j,k,l \leq 1} a_{ijkl} |ijkl\rangle \text{ with } a_{ijkl} \in \mathbb{C}.$$

Let G be the group of Stochastic Local Operation and Classical Communication (SLOCC) of four qubits [acting on $\mathbb{P}(\mathcal{H})$], *i.e.* $G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. It is well known that G acts transitively on the set of separable states. The projectivization of the corresponding orbit – also called the highest weight orbit – is the unique smooth orbit X for the action of G on $\mathbb{P}(\mathcal{H})$, that is

$$X = \mathbb{P}(G.|0000\rangle) = \{\text{The set of separable states}\} \subset \mathbb{P}^{15}.$$

A parametrization of X is given by the Segre embedding of four projective lines[11,12]

$$\phi : \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \to & \mathbb{P}^{15} \\ ([w_0 : w_1], [x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) & \mapsto & [w_0x_0y_0z_0 : \cdots : W_J : \cdots : w_1x_1y_1z_1] \end{cases}$$

where $W_J = w_i x_j y_k z_l$ for $J = \{i, j, k, l\} \in \{0, 1\}^4$ and the monomial order is such that $W_{J_1} \prec W_{J_2}$ if $8i_1 + 4j_1 + 2k_1 + l_1 \leq 8i_2 + 4j_2 + 2k_2 + l_2$.

A hyperplane $H \subset \mathbb{P}(\mathcal{H})$ is the set of states $|\Phi\rangle \in \mathbb{P}(\mathcal{H})$ on which a linear form $L_H \in \mathcal{H}^*$ vanishes. Given $H \subset \mathbb{P}(\mathcal{H})$, the hyperplane section $X \cap H \subset X$ is the hypersurface of X defined by the restriction of L_H to X. Due to the duality of Hilbert spaces, for any $H \subset \mathbb{P}(\mathcal{H})$ there exists a state $|\Psi\rangle \in \mathbb{P}(\mathcal{H})$ such that H is defined by the linear form $\langle\Psi|$. In what follows, we will often identify the hyperplane H and the linear form defining it, and write $H = \langle\Psi| = \sum_{0\leq i,j,k,l\leq 1} h_{ijkl}\langle ijkl|$ with $h_{ijkl} \in \mathbb{C}$. The hyperplane section $X \cap H$, or, equivalently, $X \cap \langle\Psi|$, will be the hypersurface of X given by

$$\langle\Psi|\phi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)\rangle = \sum_{0\leq i,j,k\leq 1} h_{ijkl} w_i x_j y_k z_l = 0. \tag{1}$$

To state our main Theorem, let us recall that the ring of polynomials invariant under G is generated by 4 invariants[18]. Let us denote by $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4$ a choice of four generators of the ring of invariants (that choice will be explained in Section III B), *i.e.* $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4]$. The quotient map $\Phi : \mathcal{H} \to \mathbb{C}^4$ is defined by $\Phi(x) = (\tilde{I}_1(x), \tilde{I}_2(x), \tilde{I}_3(x), \tilde{I}_4(x))$. The main result of this article is the following theorem:

Theorem 1. *Let $H = \langle\Psi|$ be a hyperplane of $\mathbb{P}(\mathcal{H})$ tangent to X and such that $X \cap H$ has only isolated singular points. Then the singularities are either of types A_1, A_2, A_3, A_4, or of type D_4, and there exist hyperplanes realizing each type of singularity. Moreover, if we denote by $\widehat{X}^* \subset \mathcal{H}$ the cone over the dual variety of X,* i.e. *the zero locus of the Cayley hyperdeterminant of format $2 \times 2 \times 2 \times 2$, then the quotient map $\Phi : \mathcal{H} \to \mathbb{C}^4$ is such that $\Phi(\widehat{X}^*) = \Sigma_{D_4}$, where Σ_{D_4} is the discriminant of the miniversal deformation of the D_4-singularity.*

The paper is organized as follows. In Section II, we will give the definition of a simple singularity and the invariants that follow from the Arnol'd classification[1] (Section II A). Then we will compute the singularity type of any hyperplane section of the set of separable states featuring only isolated singularities (see Section II B Proposition II.1). In Section

III, we will establish a dual version of Proposition II.1. We will first define the notion of discriminant of a singularity (see Section III A) and then show how it allows us to give a new expression for the Cayley hyperdeterminant Δ_4 (Section III B) and prove Proposition III.1 about the relation between Δ_4 and Σ_{D_4}. Propositions II.1 and III.1 lead to the proof of Theorem 1.

II. SIMPLE SINGULARITIES AND HYPERPLANE SECTIONS OF SEPARABLE STATES

A. Simple singularities following Arnol'd classification

Simple singularities have been studied from an algebraic geometrical viewpoint as rational double points of algebraic surfaces, Du Val singularities, and from a complex analytic perspective as critical points of holomorphic functions in several variables. These approaches lead to many equivalent characterizations of what a simple singularity is[9]. Here, we select the complex analytic approach introduced by Vladimir Arnol'd. We first recall the ingredients of Arnol'd classification of simple singularities[1].

Let us denote by $(f, 0)$ the germ of a holomorphic function, $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ at 0, and by $\mathcal{O}_k$ the set of all those germs. We consider the group $\mathcal{D}_k$ of biholomorphic maps $g : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ acting of $\mathcal{O}_k$ such that $g.f = f \circ g^{-1}$. A *singularity* is an equivalence class of a germ $(f, 0)$ such that $\frac{\partial f}{\partial x_i}(0) = 0$ for $i = 1, \dots, k$. In other words, a singularity is an orbit in $\mathcal{O}_k$ and we will write $[(f, 0)]$ for the orbit of the representative $(f, 0)$. We denote by $\mathcal{S}_k \subset \mathcal{O}_k$ the set of all singular germs. Let f be a representative of a singularity and let us denote by $A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)_{i,j}$ the corresponding Hessian matrix. The corank of the germ $(f, 0)$ is the dimension of the kernel of A. From the definition of the action of $\mathcal{D}_k$ it follows that equivalent germs will have the same corank, which means that the corank is an invariant of a singularity.

Definition II.1. *A singularity is said to be non-degenerate, or quadratic, or of the Morse type, if, and only if, its corank is zero.*

The Morse Lemma[19] ensures that if $(f, 0)$ is a non-degenerate singular germ, then $f \sim x_1^2 + \cdots + x_k^2$. The non-degenerate singularity is a dense orbit in $\mathcal{S}_k$. Assume that $[(f, 0)]$ is a singularity of corank l, a generalization of Morse's Lemma[1] tells us that $f \sim h(x_1, \dots, x_l) +$

$x_{l+1}^2 + \cdots + x_k^2$ and leads to an equivalence relation between germs of distinct number of variables.

Definition II.2. *Two function germs $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$, with $k < m$, are said to be stably equivalent if, and only if, $f(x_1, \ldots, x_k) + x_{k+1}^2 + \cdots + x_m^2 \sim g(x_1, \ldots, x_m)$.*

Remark II.1. In terms of the last definition we can compare singularities of functions which do not have the same number of variables. Adding quadratic terms of full rank in new variables do not affect the classification of the singular type.

Another important invariant of singular germs is the famous Milnor number[19]. Let $(f, 0)$ be a singular germ and consider $I_{\nabla f} = \mathcal{O}_k < \dfrac{\partial f}{\partial x_1}(0), \ldots, \dfrac{\partial f}{\partial x_k}(0) >$ the gradient ideal.

Definition II.3. *The Milnor number μ of a singular germ $(f, 0)$ is equal to the dimension of the local algebra of $(f, 0)$,* i.e. *the quotient of the algebra $\mathcal{O}_k$ by $I_{\nabla f}$,*

$$\mu = dim_{\mathbb{C}} \left(\mathcal{O}_k \big/ I_{\nabla f} \right).$$

The critical point 0 of the function f will be isolated if, and only if, its Milnor number is finite.

Let us now state what, in the sense of Vladimir Arnol'd, a simple singularity is .

Definition II.4. *The orbit $[(f, 0)]$ is a simple singularity if, and only if, a sufficiently small neighborhood of $(f, 0)$ intersects $\mathcal{S}_k$ with a finite number of non-equivalent orbits.*

Remark II.2. If we consider a representative of a non-degenerate singularity $f \sim x_1^2 + \cdots + x_k^2$, a small perturbation of f in $\mathcal{S}_k$, *i.e.* $f + \varepsilon h$ with $h \in \mathcal{S}_k$, will still have a Hessian of full rank for ϵ small. Thus $f \sim f + \varepsilon h$, which means that non-degenerate singularity is the most stable type of singularity. We can rephrase Definition II.4 by saying that $[(f, 0)]$ is a simple singularity if, and only if, a small perturbation of a representative f will only lead to a finite number of non-equivalent singularities.

In his classification of simple singularities[1], Arnol'd proved that being simple is equivalent to the following conditions:

- $\mu < +\infty$,

- corank $\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)\right) \leq 2,$

- if corank $\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(0)\right) = 2$ the cubic term in the degenerate direction of the Hessian is non-zero,

- if corank $= 2$ and the cubic term is a cube then $\mu < 9$.

With these conditions Arnol'd obtained the classification of simple singularities into five different types (Table I).

Type	A_n	D_n	E_6	E_7	E_8
Normal forms	x^{n+1}	$x^{n-1} + xy^2$	$x^3 + y^4$	$x^3 + xy^3$	$x^3 + y^5$
Milnor number	n	n	6	7	8

Table I. Simple singularities.

Remark II.3. The functions given in Table I are stably equivalent to the hypersurfaces given in the introduction. They are also clearly equivalent to the rational double points of algebraic surfaces.

The classification given by Arnol'd furnishes an algorithm to test if a singularity is simple or not.

Algorithm II.1. *Let $(f, 0)$ be a singularity.*

- *Compute μ; if $\mu = \infty$ the singularity is not isolated (and not simple),*

- *If not, compute $r = corank(Hess(f, 0))$.*

  - *if $r \geq 3$, the singularity is not simple,*

  - *if $r = 1$, the singularity is of type A_μ,*

  - *if $r = 2$, then*

    * *if the cubic term in the degenerate directions is non-zero and is not a cube, then the singularity is of type D_μ,*

    * *if the cubic term in the degenerate directions is a cube and $\mu < 9$, then the singularity is of type E_μ,*

* *if not, the singularity is not simple.*

In the next section we will follow this algorithm to compute the singular type of a given hyperplane section.

B. Computing singularities of hyperplane sections

Before we prove the first proposition, let us consider two examples in order to explain how we calculate the singular type of a hyperplane section.

Example II.1. Let $H \in \mathbb{P}(\mathcal{H}^*)$ be a hyperplane, or a linear form, given by $H = \langle \Psi_1| = \langle 0011| + \langle 1100|$. The corresponding hyperplane section $X \cap H$ is tangent to $|1111\rangle$. Indeed, a tangent vector to X at $|1111\rangle$ will be of the form $|v\rangle = \alpha|0111\rangle + \beta|1011\rangle + \gamma|1101\rangle + \delta|1110\rangle$ and it is clear that $\langle \Psi_1|v\rangle = 0$. The homogeneous form of the linear section $X \cap H$ corresponds to its restriction to (the cone over) X, that is to

$$f(w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1) = w_0x_0y_1z_1 + w_1x_1y_0z_0.$$

In a non-homogeneous form f can be written in the chart corresponding to $w_1, x_1, y_1, z_1 = 1$ as $f(w_0, x_0, y_1, z_1) = w_0x_0 + y_0z_0$. In this chart the point $|1111\rangle$ has coordinates $(0, 0, 0, 0)$ and (we can forget about the subscripts) the hyperplane section is a hypersurface of X defined (locally) by the equation

$$f(w, x, y, z) = wx + yz = 0.$$

This hypersurface has a unique singularity $\left(\frac{\partial f}{\partial w}(a), \frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a), \frac{\partial f}{\partial z}(a)\right) = (0, 0, 0, 0) \Leftrightarrow a = (0, 0, 0, 0)$, which corresponds to $|1111\rangle$, and the Hessian matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is of the full rank. One concludes that $(X \cap H, |1111\rangle)$ is an isolated singularity of type A_1 and we denote it by $(X \cap H, |1111\rangle) \sim A_1$, or, equivalently, by $(X \cap \langle \Psi_1|, |1111\rangle) \sim A_1$.

Example II.2. Let us consider the hyperplane section defined by $H = \langle\Psi_2| = \langle 0000| + \langle 1011| + \langle 1101| + \langle 1110| \in \mathcal{H}^*$. This section $X \cap H$ is tangent to $|0111\rangle$. It is clear that a tangent vector to X at $|0111\rangle$ will be of the form $|v\rangle = \alpha|1111\rangle + \beta|0011\rangle + \gamma|0101\rangle + |0110\rangle$ and $H|v\rangle = 0$. The homogeneous linear form corresponding to $X \cap H$ is $f(w_0, w_1, x_0, x_1, y_0, y_1, z_0, z_1) = w_0x_0y_0z_0 + w_1x_0y_1z_1 + w_1x_1y_0z_1 + w_1x_1y_1z_0$. In the chart $w_0 = x_1 = y_1 = z_1 = 1$ the form becomes a hypersurface defined by

$$xyz + wx + wy + wz = 0$$

and $(0, 0, 0, 0)$ is the only singularity of this hypersurface. Using the software SINGULAR[8] one can check that $\mu_{x=(0,0,0,0)}(f) = 4$ and the rank of the Hessian

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is 2. Thus, we conclude that $(X \cap H, |0111\rangle) \sim D_4$, or, equivalently, $(X \cap \langle\Psi_2|, |0111\rangle) \sim D_4$ (*i.e.* the unique isolated singularity where the corank equals 2 and $\mu = 4$).

We can now prove our first proposition.

Proposition II.1. *Let $X \cap H$ be a singular hyperplane section of the variety of separable states for four-qubit systems,* i.e. $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*, with an isolated singularity $x \in X \cap H$. Then the singularity $(X \cap H, x)$ will be of type A_1, A_2, A_3 or D_4 and each type can be obtained by such a linear section of X.*

Proof. To prove Proposition II.1, we compute the singular type of all possible hyperplane sections of X. As the variety X is G-homogeneous, the singular type of $X \cap H$ will be identical for any representative of the G orbit of H. By the duality of the Hilbert space, a hyperplane H corresponds to a point $h \in \mathbb{P}(\mathcal{H})$. But the G orbits of $\mathbb{P}(\mathcal{H})$ have been classified by Verstraete *et al.*[26] (with a corrected version provided by Chterental and Djokovic[7]). According to Verstraete *et al.*'s classification, the G-orbits of the four-qubit Hilbert space consist of 9 families (3 families are parameter free and 6 of them depend on parameters) and normal forms for each family are known[7,26]. From each of Verstraete *et al.*'s normal forms $|\Psi\rangle$ we compute the corresponding hyperplane section $X \cap \langle\Psi|$. Then we look at isolated singular points of each hyperplane section and we calculate the corresponding singular type with a formal algebra system following the procedure described in examples II.1, II.2 and Algorithm II.1. For the normal forms depending on parameters, the singular type of the hyperplane sections will depend on values of the parameters. The results of our calculations are given in Tables II and III and provide a proof of the proposition.□

Verstraete *et al.*'s notation	Hyperplane	Singular type of the hyperplane section
$L_{0_{7\oplus\bar{1}}}$	$\langle 0000\| + \langle 1011\| + \langle 1101\| + \langle 1110\|$	D_4 (a unique singularity)
$L_{0_{5\oplus\bar{3}}}$	$\langle 0000\| + \langle 0101\| + \langle 1000\| + \langle 1110\|$	non-isolated
$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$	$\langle 0000\| + \langle 0111\|$	non-isolated

Table II. Hyperplanes and the corresponding sections which do not depend on parameters.

Remark II.4. Tables III, IV, V show that the classification of entangled states into 9 families can be refined according to the singular type of the corresponding section. The singular type of the linear section $X \cap \langle\Psi|$ is an invariant of the G-orbit of $|\Psi\rangle$ and may be used to distinguish two non-equivalent classes of entanglement. Thus, the values of the parameters which distinguish the sections indicate how we can decompose further the classification. However, to fully distinguish non-equivalent sections from their singular type, it would be necessary to investigate more precisely the non-isolated singular sections.

Remark II.5. It is worthwile to point out that the different isolated singular types we obtain by this construction (A_1, A_2, A_3 and D_4) are exactly the possible degenerations of the D_4-singularity. In particular, any small neighborhood of the singularity of type D_4 will meet, in $\mathcal{S}_k$, the orbits corresponding to the singular types A_1, A_2 and A_3 as shown in the adjacency diagrams of Arnold's classification (Corollary 8.7 in Ref[1]). The fact that D_4 is the "worst-possible" isolated singularity we get from the hyperplane sections of the set of separable states will be lighted with Proposition III.1.

Verstraete's notation	Hyperplane	parameters	Singular type
$L_{a_2 0_{3\oplus 1}}$	$a(\langle 0000\| + \langle 1111\|) + \langle 0011\| + \langle 0101\| + \langle 0110\|$	a generic $a = 0$	A_1 non-isolated
L_{a_4}	$a(\langle 0000\| + \langle 0101\| + \langle 1010\| + \langle 1111\|)$ $+i\langle 0001\| + \langle 0110\| - i\langle 1011\|$	a generic $a = 0$	A_3 (a unique singularity) non-isolated
L_{ab_3}	$a(\langle 0000\| + \langle 1111\|) + \frac{a+b}{2}(\langle 0101\| + \langle 1010\|)$ $+\frac{a-b}{2}(\langle 0110\| + \langle 1001\|)$ $+\frac{i}{\sqrt{2}}(\langle 0001\| + \langle 0010\| - \langle 0111\| - \langle 1011\|)$	a, b generic $a = b = 0$	A_2 (a unique singularity) non-isolated
$L_{a_2 b_2}$	$a(\|0000\rangle + \|1111\rangle) + b(\|0101\rangle + \|1010\rangle)$ $+\|0110\rangle + \|0011\rangle$	a, b generic $a = 0$ or $b = 0$ $a = b = 0$	smooth section non-isolated non-isolated
L_{abc_2}	$\frac{a+b}{2}(\langle 0000\| + \langle 1111\|) + \frac{a-b}{2}(\langle 0011\| + \langle 1100\|)$ $c(\langle 1010\| + \langle 0101\|) + \langle 0110\|$	a, b, c generic $a = \pm b$ $c = 0$ $a = \pm b = \pm c$ $a = c = 0$ or $b = c = 0$ $a = b = c = 0$	A_1 (a unique singularity) A_1 A_1 non-isolated non-isolated non-isolated
G_{abcd}	$\frac{a+d}{2}(\|0000\rangle + \|1111\rangle) + \frac{a-d}{2}(\|0011\rangle + \|1100\rangle)$ $+\frac{b+c}{2}(\|0101\rangle + \|1010\rangle) + \frac{b-c}{2}(\|0110\rangle + \|1001\rangle)$	a, b, c, d generic see Table IV see Table V	smooth section A_1 non-isolated

Table III. Hyperplanes and the corresponding sections which do depend on parameters.

III. THE CAYLEY $2 \times 2 \times 2 \times 2$ HYPERDETERMINANT AND THE D_4-DISCRIMINANT

Another fundamental concept associated with a simple singularity is its discriminant, *i.e.* the locus that parametrizes the deformation of the singular germs. In this section, we will show that the discriminant of the D_4-singularity is linked to the dual variety, in the sense of the projective duality, of the set of separable four-qubit states.

A. Discriminant of the miniversal deformation of the singularity

Consider a holomorphic germ $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ with a simple isolated singularity of Milnor number $\mu(f, 0) = n$. A *miniversal deformation*[1] of the germ f is given by

$$f + \sum \lambda_i g_i,$$

where $(g_1, \ldots, g_n)$ is a basis of $\mathcal{O}_k \diagup I_{\nabla f}$.

Definition III.1. *The discriminant $\Sigma \subset \mathbb{C}^n$ is the subset of values $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that the miniversal deformation $f + \sum \lambda_i g_i$ is singular,* i.e.

$$\Sigma = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n, \Delta(f + \sum_{i=1}^{n} \lambda_i g_i) = 0\},$$

where Δ is the usual notion of discriminant.

Remark III.1. The discriminant parametrizes all singular deformations of $(f, 0)$. It is known[28] that for hypersurfaces endowed with a simple singularity, the discriminant of the singularity characterizes its type.

Example III.1. Let $(f, 0)$ be a singularity of type A_n, *i.e.* $f \sim x^{n+1}$. Then $\mathcal{O}_1 \diagup I_{\nabla x^{n+1}} =< 1, x, \ldots, x^{n-1} >$. Thus, a miniversal deformation of f is

$$F(x, \lambda) = x^{n+1} + \lambda_1 x^{n-1} + \lambda_2 x^{n-2} + \cdots + \lambda_n.$$

The corresponding discriminant is the hypersurface $\Sigma_{A_n} \subset \mathbb{C}^n$ defined by

$$\Delta(x^{n+1} + \lambda_1 x^{n-1} + \lambda_2 x^{n-2} + \cdots + \lambda_n) = 0.$$

In the case where $n = 2$, *i.e.* when $f \sim x^3$ is a singularity of type A_2, then its discriminant is given by $\Delta(x^3 + \lambda_1 x + \lambda_2) = 0$, *i.e.* the discriminant is a cubic curve defined by $-4\lambda_1^3 - 27\lambda_2^2 = 0$.

The following example will be useful to prove the main result of the next section.

Example III.2. Consider now a singular germ $(f, 0)$ of type D_n; then $f \sim x^{n-1} + xy^2$. A basis of the local algebra $\mathcal{O}_2 \diagup I_{\nabla(x^{n-1}+xy^2)}$ is $(1, x, \ldots, x^{n-2}, y)$ and, hence, a miniversal deformation is

$$F(x, y, \lambda) = x^{n-1} + xy^2 + \lambda_1 x^{n-2} + \ldots \lambda_{n-2} x + \lambda_{n-1} + \lambda_n y.$$

Its discriminant is given by

$$\Delta(x^{n-1}+xy^2+\lambda_1 x^{n-2}+\dots\lambda_{n-2}x+\lambda_{n-1}+\lambda_n y)=0. \tag{2}$$

The following lemma proposes an alternative expression of the discriminant of the D_n singularities.

Lemma 1. *The discriminant of the miniversal deformation of $f\sim x^{n-1}+xy^2$ is the hypersurface $\Sigma_{D_n}\subset\mathbb{C}^n$ defined by*

$$\Delta(\lambda_1,\dots,\lambda_n)=\Delta(t^n+\lambda_1 t^{n-1}+\dots+\lambda_{n-1}-(\frac{1}{2}\lambda_n)^2)=0. \tag{3}$$

Proof. Let us denote by $\Sigma\subset\mathbb{C}^n$ the locus defined by eq. (3). To prove that equations (2) and (3) are equivalent, we will show that $\Sigma=\Sigma_{D_n}$.

To this end, let us characterize the hypersurfaces Σ and Σ_{D_n}. Given the definition of the discriminant, the expression $\Delta(F(t,\lambda))=0$ means there exists t_0 such that $F(t_0)=0$ and $\frac{\partial F}{\partial t}(t_0)=0$. In other words, $(\lambda_1,\dots,\lambda_n)\in\Sigma$ if, and only if, there exists t_0 such that

$$\left\{\begin{array}{rcl} t_0^n+\lambda_1 t_0^{n-1}+\dots+\lambda_{n-1}t_0-(\frac{1}{2}\lambda_n)^2 &=& 0,\\ nt_0^{n-1}+(n-1)\lambda_1 t_0^{n-2}+\dots+\lambda_{n-1} &=& 0.\end{array}\right\} \tag{4}$$

Similarly, $(\lambda_1,\dots,\lambda_n)\in\Sigma_{D_n}$ if, and only if, there exists (x_0,y_0) such that $F(x_0,y_0,\lambda)=\frac{\partial F}{\partial x}(x_0,y_0,\lambda)=\frac{\partial F}{\partial y}(x_0,y_0,\lambda)=0$, *i.e.*

$$\left\{\begin{array}{rcl} x_0^{n-1}+x_0y_0^2+\lambda_1 x_0^{n-2}+\dots+\lambda_{n-2}x_0+\lambda_{n-1}+\lambda_n y_0 &=& 0,\\ (n-1)x_0^{n-2}+y_0^2+(n-2)\lambda_1 x_0^{n-3}+\dots\lambda_{n-2} &=& 0,\\ 2x_0y_0+\lambda_n &=& 0.\end{array}\right\} \tag{5}$$

Let us assume that $\lambda_n\neq 0$, then if $(\lambda_1,\dots,\lambda_n)\in\Sigma$ there exists t_0 such that the system (4) is satisfied. It is obvious that $\lambda_n\neq 0$ implies $t_0\neq 0$ and thus one can check that the system (5) is also satisfied for $(x_0,y_0)=(t_0,-\frac{\lambda_n}{2t_0})$. This proves that $(\lambda_1,\dots,\lambda_n)\in\Sigma_{D_n}$. On the other hand, if $(\lambda_1,\dots,\lambda_n)\in\Sigma_{D_n}$ and (x_0,y_0) is a solution of (5), then necessarily $y_0=-\frac{\lambda_n}{x_0}$. One can further show that $t_0=x_0$ is a solution of (4) and, therefore, $(\lambda_1,\dots,\lambda_n)\in\Sigma$. Let us now consider the case $\lambda_n=0$. Then $(\lambda_1,\dots,\lambda_n)\in\Sigma$ for a given t_0 implies $(\lambda_1,\dots,\lambda_n)\in\Sigma_{D_n}$ for $(x_0,y_0)=(t_0,0)$. On the other hand, let us assume $(\lambda_1,\dots,\lambda_n)\in\Sigma_{D_n}$ for a given (x_0,y_0). The equation $2x_0y_0+\lambda_n=0$ forces x_0 or y_0 to be zero. But if $x_0=0$ then necessarily also $a_{n-1}=0$ and $t_0=0$ is a solution of (4), proving $(\lambda_1,\dots,\lambda_n)\in\Sigma$. If $x_0\neq 0$, then $y_0=0$ and $t_0=x_0$ is a solution of (4), proving again $(\lambda_1,\dots,\lambda_n)\in\Sigma$. □

B. Hyperdeterminant of format $2\times2\times2\times2$ and D_4-discriminant

The hyperdeterminant of format $2\times2\times2\times2$ is a SLOCC-invariant polynomial generalizing the ideas of Cayley for defining a higher dimensional counterpart of the determinant for multimatrices. From a geometrical perspective, the hyperdeterminant and its generalization have been studied by Gelfand, Kapranov and Zelevinsky[10] in terms of the concept of dual varieties. The geometric definition is the following one: Let $X \subset \mathbb{P}(V)$ be a (smooth) projective variety, we denote by X^* the dual variety of X, defined by

$$X^* = \overline{\{H \in \mathbb{P}(\mathcal{H}^*), \exists x \in X, T_xX \subset H\}}.$$

For the case $X = \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, the dual variety, denoted X^*, is a SLOCC-invariant hypersurface, whose equation is called the hyperdeterminant of format $2\times2\times2\times2$. This invariant polynomial, denoted as Δ_4, is an irreducible polynomial (X^* is irreducible because X is), its degree is 24, and the corresponding hypersurface is singular[21,27] in codimension 1. By definition, X^* parametrizes the singular hyperplane sections of X (alternatively, $H \notin X^*$ is equivalent to saying that $X\cap H$ is a smooth section).

It would be difficult to quote all the papers in QIT (as well as in theoretical physics) referring to the concept of hyperdeterminant[4,5,12,17,18,20,21], but it is clear that this invariant polynomial plays a central role in understanding the symmetries involved in the SLOCC group action.

In the case of four-qubit systems, the ring of polynomials invariant under the group SLOCC was determined by Luque and Thibon[18]. It is a finitely-generated ring with four generators B, L, M and D, of respective degrees $2, 4, 4$ and 6 (explicit expressions, with the same notations, can be found in Ref[13]). In other words, any SLOCC-invariant polynomial P over $\mathcal{H} = \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ belongs to $\mathbb{C}[B, L, M, D]$. In particular, the hyperdeterminant of format $2\times2\times2\times2$ can be expressed as a polynomial in the generators of the ring of invariants and one gets[18]

$$\Delta_4 = \frac{1}{256}(S^3 - 27T^2),$$

with $S = \frac{1}{12}(B^2-4(L+M))^2 - 24(BD+2LM)$ and $T = \frac{1}{216}((B^2-4(L+M))^3 - 3(B^2-48(L+M))(BD+2LM) + 216D^2)$. In his attempt to give a geometric meaning of the invariants of Luque and Thibon, Lévay[16] introduced some alternatives generators which are related to the previous ones as $I_1 = \frac{1}{2}B$, $I_2 = \frac{1}{6}(B^2+2L-4M)$, $I_3 = D + \frac{1}{2}BL$ and $I_4 = L$.

Lévay's motivation to define this new set of generators was to obtain a more geometrical and uniform description of those polynomials, as it is shown in his paper[16]. These news invariants I_1, I_2, I_3, I_4 allow one to get a new expression of Δ_4. In particular, Lévay proved (Eq (56)[16]) that

$$\Delta_4 = \frac{1}{256}\Delta(t^4 - (4I_1)t^3 + (6I_2)t^2 - (4I_3)t + I_4^2) \qquad (6)$$

(where Δ is the discriminant of the polynomial in the t variable). This particular finding leads to the following claim:

Proposition III.1. *Let us consider the quotient map $\Phi : \mathcal{H} \to \mathbb{C}^4$ defined by*

$$\Phi(|\Psi\rangle) = (\tilde{I}_1(|\Psi\rangle), \tilde{I}_2(|\Psi\rangle), \tilde{I}_3(|\Psi\rangle), \tilde{I}_4(|\Psi\rangle),$$

where $\tilde{I}_1 = -4I_1$, $\tilde{I}_2 = 6I_2$, $\tilde{I}_3 = -4I_3$ and $\tilde{I}_4 = \frac{i}{2}I_4$. Then, $\Phi(\widehat{X}^*) = \Sigma_{D_4}$.

Proof. According to Lévay's equation for the hyperdeterminant Δ_4, it is clear that our choice of Φ implies that the equation of $\Phi(\widehat{X}^*) \subset \mathbb{C}^4$ is

$$\frac{1}{256}\Delta(t^4 + \lambda_1 t^3 + \lambda_2 t^2 + \lambda_3 t - (\frac{1}{2}\lambda_4)^2) = 0,$$

where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are coordoninates on $\mathbb{C}^4$. But Lemma 1 implies that this zero locus is the discriminant of the D_4 simple singularity, *i.e.* the hypersurface Σ_{D_4}. □

Remark III.2. Propositions II.1 and III.1 prove Theorem 1.

Remark III.3. The quartic $t^4 - (4I_1)t^3 + (6I_2)t^2 - (4I_3)t + I_4^2$ of Eq (6) appears also in the conclusion of a previous paper involving the first two authors[13]. When we evaluate this quartic on the G_{abcd} state, *i.e.* when we consider the quartic $Q(t) = t^4 - (4I_1(G_{abcbd}))t^3 + (6I_2(G_{abcd}))t^2 - (4I_3(G_{abcd}))t + I_4(G_{abcd})^2$, one obtains $Q(t) = (t-a^2)(t-b^2)(t-c^2)(t-d^2)$. The state G_{abcd} will cancel Δ_4 if and only if the quartic Q has (at least) a repeated root, *i.e.* there is (at least) a relation (among the parameters) of type $m = \pm n$ with $m \in \{a, b, c, d\}$ and $n \in \{a, b, c, d\} \setminus m$. Obviously this condition is satisfied by all values of the parameters $\{a, b, c, d\}$ of Tables IV and V because the corresponding states belong to the dual of X (and thus vanish Δ_4). However the relations between the hyperplane sections of Tables IV and V and the number of repeated roots of the quartic Q is probably worth to be further investigate.

Remark III.4. Proposition III.1 establishes a connexion between two types of discriminant. As pointed out earlier, the dual variety of X is a discriminant in the sense that it parametrizes the singular hyperplane sections of X. The D_4-discriminant parametrizes the singular deformation of the germ x^3+xy. The most singular deformation of $x^3+xy^2+\lambda_1x^2+\lambda_2x+\lambda_3+\lambda_4y$ is obtained for $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=(0,0,0,0)$. The preimage via the quotient map of $(0,0,0,0)$ is given by the zero-locus of (all) invariant polynomials

$$\Phi^{-1}(0,0,0,0)=\{|\Psi\rangle,\tilde{I}_1(|\Psi\rangle)=\tilde{I}_2(|\Psi\rangle)=\tilde{I}_3(|\Psi\rangle)=\tilde{I}_4(|\Psi\rangle)=0\}.$$

This set does not depend on our choice of Φ and, after projectivization, it corresponds to a well-known variety $\mathcal{N}\subset\mathbb{P}(\mathcal{H})$, the nullcone, which was already invoked to describe the entanglement classes of a four-qubit system[2,13]. As first pointed out in Ref[2], the nullcone admits a stratification into 9 distinguished classes of orbits which relate to the 9 families of Verstraete *et al.*'s classification. To emphasize the connexion with the D_4 singular type, let us point out that $H=\langle\Psi_2|=\langle 0000|+\langle 1011|+\langle 1101|+\langle 1110|$ (the hyperplane of Example II.2) is a smooth point of $\mathcal{N}$ and this characterizes the hyperplanes of X with a D_4-singular point. This correspondence can diagrammatically be sketched as:

$$\begin{array}{ccc} X\cap H\sim D_4 & \longleftrightarrow & H\in\mathcal{N}_{smooth}\subset X^* \\ & & \downarrow \Phi \\ x^3+xy^2 & \longleftrightarrow & (0,0,0,0)\in\Sigma_{D_4}\subset\mathbb{C}^4. \end{array}$$

IV. CONCLUSION

We have introduced a new construction that assigns to any quantum state $|\Psi\rangle$ a complex hypersurface defined by the hyperplane section $X\cap\langle\Psi|$ of the set X of all separable states. This hypersurface may have singular points, which can be studied using the theory of singularity. Because the variety of separable states is G-homogeneous, this construction is G-invariant and two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ which do not define equivalent (singular) hyperplane sections will not be SLOCC equivalent. For four qubits, this construction allowed us to realize the singularity of type D_4 as a specific hyperplane section and we also proved that no "higher" isolated singularities can be obtained by this construction.

The D_4 singularity is obtained only when we consider the section $X\cap\langle\Psi|$, where $|\Psi\rangle$ is a point of an orbit of maximal dimension of the nullcone[13] (*i.e.* a smooth point of the nullcone).

This is emphasized when we rephrase the notion of Cayley $2 \times 2 \times 2 \times 2$ hyperdeterminant, *i.e.* the dual equation of the set of separable states, in terms of the discriminant of a D_4-singularity. The stratification of the discriminant Σ_{D_4} in terms of mutiplicities induces a stratification of the dual variety X^* — a variety that is of great relevance in the study of entanglement of four qubits, as pointed out by Miyake[20,21].

Although the correspondence between four qubits and simple Lie algebra of type D_4 is now clear from the action of the SLOCC group, the correspondence established in this paper between four qubits and a simple singularity of type D_4 is rather surprising and points out to a novel relationship between simple Lie algebra and simple singularity of type D_4.

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Appendix A: Hyperplane sections of type G_{abcd}

In this appendix, we will give the different values of the parameters a, b, c, d of the hyperplanes of type G_{abcd} which lead either to hyperplane sections with only A_1 singular points (Table IV) or hyperplane sections with non-isolated singularities (Table V).

$\{a=a, b=b, c=a, d=d\}, \{a=a, b=b, c=c, d=b\},$
$\{a=a, b=b, c=c, d=c\}, \{a=a, b=b, c=c, d=-b\},$
$\{a=a, b=b, c=c, d=-c\}, \{a=a, b=b, c=-a, d=d\},$
$\{a=a, b=c, c=c, d=d\}, \{a=a, b=-a, c=c, d=d\},$
$\{a=a, b=-c, c=c, d=d\}, \{a=b, b=b, c=c, d=d\},$
$\{a=c, b=0, c=c, d=d\}, \{a=d, b=b, c=c, d=d\},$
$\{a=-c, b=0, c=c, d=d\}, \{a=-d, b=b, c=c, d=d\}$

Table IV. Hyperplane sections of type G_{abcd} with only A_1 singularities.

$\{a=0,b=0,c=0,d=d\},\{a=0,b=0,c=c,d=0\},$
$\{a=0,b=b,c=0,d=0\},\{a=a,b=0,c=0,d=0\},$
$\{a=a,b=d,c=d,d=d\},\{a=a,b=-c,c=c,d=-c\},$
$\{a=a,b=-d,c=d,d=d\},\{a=a,b=-d,c=-d,d=d\},$
$\{a=b,b=b,c=0,d=b\},\{a=b,b=b,c=0,d=-b\},$
$\{a=c,b=0,c=c,d=c\},\{a=c,b=0,c=c,d=-c\},$
$\{a=c,b=c,c=c,d=d\},\{a=c,b=-c,c=c,d=d\},$
$\{a=d,b=b,c=d,d=d\},\{a=d,b=d,c=c,d=d\},$
$\{a=d,b=d,c=d,d=d\},\{a=d,b=-d,c=d,d=d\},$
$\{a=-b,b=b,c=0,d=b\},\{a=-b,b=b,c=0,d=-b\},$
$\{a=-b,b=b,c=c,d=-b\},\{a=-c,b=0,c=c,d=c\},$
$\{a=-c,b=0,c=c,d=-c\},\{a=-c,b=b,c=c,d=-c\},$
$\{a=-c,b=c,c=c,d=d\},\{a=-c,b=c,c=c,d=-c\},$
$\{a=-c,b=-c,c=c,d=d\},\{a=-c,b=-c,c=c,d=-c\},$
$\{a=-d,b=b,c=d,d=d\},\{a=-d,b=b,c=-d,d=d\},$
$\{a=-d,b=d,c=c,d=d\},\{a=-d,b=d,c=d,d=d\},$
$\{a=-d,b=d,c=-d,d=d\},\{a=-d,b=-d,c=c,d=d\},$
$\{a=-d,b=-d,c=d,d=d\},\{a=-d,b=-d,c=-d,d=d\}$

Table V. Hyperplane sections of type G_{abcd} with non-isolated singularities.

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