CHEBYSHEV'S BIAS AND GENERALIZED RIEMANN HYPOTHESIS

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ABSTRACT. It is well known that $\mathrm{li}(x) > \pi(x)$ (i) up to the (very large) Skewes' number $x_1 \sim 1.40 \times 10^{316}$ [1]. But, according to a Littlewood's theorem, there exist infinitely many x that violate the inequality, due to the specific distribution of non-trivial zeros γ of the Riemann zeta function $\zeta(s)$, encoded by the equation $\mathrm{li}(x) - \pi(x) \approx \frac{\sqrt{x}}{\log x}[1 + 2\sum_\gamma \frac{\sin(\gamma \log x)}{\gamma}]$ (1). If Riemann hypothesis (RH) holds, (i) may be replaced by the equivalent statement $\mathrm{li}[\psi(x)] > \pi(x)$ (ii) due to Robin [2]. A statement similar to (i) was found by Chebyshev that $\pi(x; 4, 3) - \pi(x; 4, 1) > 0$ (iii) holds for any $x < 26861$ [3] (the notation $\pi(x; k, l)$ means the number of primes up to x and congruent to $l \mod k$). The *Chebyshev's bias* (iii) is related to the generalized Riemann hypothesis (GRH) and occurs with a logarithmic density ≈ 0.9959 [3]. In this paper, we reformulate the Chebyshev's bias for a general modulus q as the inequality $B(x; q, R) - B(x; q, N) > 0$ (iv), where $B(x; k, l) = \mathrm{li}[\phi(k) * \psi(x; k, l)] - \phi(k) * \pi(x; k, l)$ is a counting function introduced in Robin's paper [2] and $R($ resp. $N)$ is a quadratic residue modulo q (resp. a non-quadratic residue). We investigate numerically the case $q = 4$ and a few prime moduli p. Then, we proove that (iv) is equivalent to GRH for the modulus q.

1. Introduction

In the following, we denote $\pi(x)$ the prime counting function and $\pi(x; q, a)$ the number of primes not exceeding x and congruent to $a \mod q$. The asymptotic law for the distribution of primes is the prime number theorem $\pi(x) \sim \frac{x}{\log x}$. Correspondingly, one gets [4, eq. (14), p. 125]

$$\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)} \tag{1.1}$$

that is, one expects the same number of primes in each residue class $a \mod q$, if $(a, q) = 1$. Chebyshev's bias is the observation that, contrarily to expectations, $\pi(x; q, N) > \pi(x; q, R)$ most of the times, when N is a non-square modulo q, but R is.

Let us start with the bias

$$\delta(x, 4) := \pi(x; 4, 3) - \pi(x; 4, 1) \tag{1.2}$$

found between the number of primes in the non-quadratic residue class $N = 3 \mod 4$ and the number of primes in the quadratic one $R = 3 \mod 4$. The values

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$\delta(10^n, 4)$, $n \leq 1$, form the increasing sequence

$$A091295 = \{1,\ 2,\ 7,\ 10,\ 25,\ 147,\ 218,\ 446,\ 551,\ 5960, \ldots\}.$$

The bias is found to be negative in thin zones of size

$$\{2,\ 410,\ 15\ 358,\ 41346,\ 42\ 233\ 786,\ 416\ 889\ 978, \ldots\}$$

spread over the location of primes of maximum negative bias [5]

$$\{26861,\ 623\ 681,\ 12\ 366\ 589,\ 951\ 867\ 937,\ 6\ 345\ 026\ 833,\ 18\ 699\ 356\ 321 \ldots\}.$$

It has been proved that there are are infinitely many sign changes in the Chebyshev's bias (1.2). This follows from the Littlewood's oscillation theorem [6, 7]

$$\delta(x, 4) := \Omega_{\pm}\left(\frac{x^{1/2}}{\log x}\log_3 x\right). \tag{1.3}$$

A useful measure of the Chebyshev's bias is the logarithmic density [3, 6, 8]

$$d(A) = \lim_{x\to\infty}\frac{1}{\log x}\sum_{a\in A, a\leq x}\frac{1}{a} \tag{1.4}$$

for the positive Δ^+ and negative Δ^- regions calculated as $d(\Delta^+) = 0.9959$ and $d(\Delta^-) = 0.0041$.

In essence, Chebyshev's bias $\delta(x, 4)$ is similar to the bias

$$\delta(x) := \mathrm{Li}(x) - \pi(x). \tag{1.5}$$

It is known that $\delta(x) > 0$ up to the (very large) Skewes' number $x_1 \sim 1.40 \times 10^{316}$ but, according to Littlewood's theorem, there also are infinitely many sign changes of $\delta(x)$ [7].

The reason why the asymmetry in (1.5) is so much pronounced is encoded in the following approximation of the bias [3, 9][1]

$$\delta(x) \sim \frac{\sqrt{x}}{\log x}\left(1 + 2\sum_{\gamma}\frac{\sin(\gamma\log x + \alpha_\gamma)}{\sqrt{1/4 + \gamma^2}}\right), \tag{1.6}$$

where $\alpha_\gamma = \cot^{-1}(2\gamma)$ and γ is the imaginary part of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. The smallest value of γ is quite large, $\gamma_1 \sim 14.134$, and leads to a large asymmetry in (1.5).

Under the assumption that the generalized Riemann hypothesis (GRH) holds that is, if the Dirichlet L-function with non trivial real character κ_4

$$L(s, \kappa_4) = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^s}, \tag{1.7}$$

has all its non-trivial zeros located on the vertical axis $\Re(s) = \frac{1}{2}$, then the formula (1.6) also holds for the Chebyshev's bias $\delta(x, 4)$. The smallest non-trivial zero of $L(s, \kappa_4)$ is at $\gamma_1 \sim 6.02$, a much smaller value than than the one corresponding to $\zeta(s)$, so that the bias is also much smaller.

[1]The bias may also be approached in a different way by relating it to the second order Landau-Ramanujan constant [10].

A second factor controls the aforementionned assymmetry of a L-function of real non-trivial character κ, it is the *variance* [11]

$$V(\kappa) = \sum_{\gamma>0} \frac{2}{1/4+\gamma^2}. \tag{1.8}$$

For the function $\zeta(s)$ and $L(s,\kappa_4)$ one gets $V = 0.045$ and V=0.155, respectively.

Our main goal. In a groundbreaking paper, Robin reformulated the unconditional bias (1.5) as a conditional one involving the second Chebyshev function $\psi(x) = \sum_{p^k \leq x} \log p$

(1.9) The equality $\delta'(x) := \mathrm{li}[\psi(x)] - \pi(x) > 0$ is equivalent to RH.

This statement is given as Corollary 1.2 in [12] and led the second and third author of the present work to derive a *good prime counting function*

$$\pi(x) = \sum_{n=1}^{3} \mu(n)\mathrm{li}[\psi(x)^{1/n}]. \tag{1.10}$$

Here, we are interested in a similar method to *regularize* the Chebyshev's bias in a conditional way similar to (1.9). In [2], Robin introduced the function

$$B(x;q,a) = \mathrm{li}[\phi(q)\psi(x;q,a)] - \phi(q)\pi(x;q,a), \tag{1.11}$$

that generalizes (1.9) and applies it to the residue class $a \mod q$, with $\psi(x,q,a)$ the generalized second Chebyshev's function. Under GRH he proved that [2, Lemma 2, p. 265]

$$B(x;q,a) = \Omega_{\pm}\left(\frac{\sqrt{x}}{\log^2 x}\right), \tag{1.12}$$

that is

(1.13) The inequality $B(x;q,a) > 0$ is equivalent to GRH.

For the Chebyshev's bias, we now need a proposition taking into account two residue classes such that $a = N$(a non-quadratic residue) and $a = R$ (a quadratic one).

Proposition 1.1. Let $B(x;q,a)$ be the Robin B-function defined in (1.11), and R (resp. N) be a quadratic residue modulo q (resp. a non-quadratic residue), then the statement $\delta'(x,q) := B(x;q,R) - B(x;q,N) > 0$, $\forall x$ (i), is equivalent to GRH for the modulus q.

The present paper deals about the numerical justification of proposition 1.1 in Sec. 2 and its proof in Sec. 3. The calculations are performed with the software Magma [13] available on a 96 MB segment of the cluster at the University of Franche-Comté.

2. The regularized Chebyshev's bias

All over this section, we are interested in the prime champions of the Chebyshev's bias $\delta(x,q)$ (as defined in (1.2) or (2.3), depending on the context). We separate the prime champions leading to a positive/negative bias. Thus, the n-th prime champion satisfies

$$\delta^{(\epsilon)}(x_n,q) = \epsilon n,\ \epsilon = \pm 1. \tag{2.1}$$

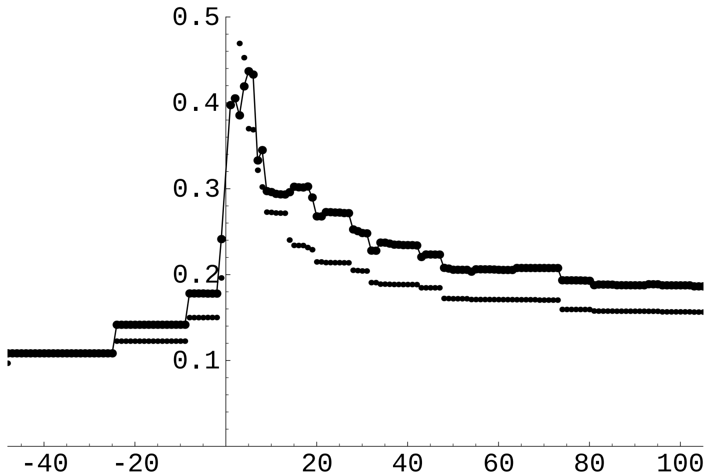

FIGURE 1. The normalized regularized bias $\delta'(x,4)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x,4)$ at the prime champions of $\delta(x,4)$ (when $\delta(x,4) > 0$) and at the prime champions of $-\delta(x,4)$ (when $\delta(x,4) < 0$). The extremal prime champions in the plot are $x = 359327$ (with $\delta = 105$) and $x = 951867937$ (with $\delta = -48$). The curve is asymmetric around the vertical axis, a fact that reflects the asymmetry of the Chebyshev's bias. As explained in the text, a violation of GRH would imply a negative value of the regularized bias $\delta'(x,4)$. The small dot curve corresponds to the fit of $\delta'(x,4)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

We also introduce a new measure of the overall bias $b(q)$, dedicated to our plots, as follows

$$b(q) = \sum_{n,\epsilon} \frac{\delta^{(\epsilon)}(x_n, q)}{x_n}. \tag{2.2}$$

Indeed, smaller is the bias lower is the value of $b(q)$. Anticipating over the results presented below, Table 1 summarize the calculations.

TABLE 1. The new bias (2.2) (column 2) and the standard logarithmic density (1.4) (column 3).

modulus q	bias $b(q)$	log density $d(\Delta^+)$	first zero γ_1
4	0.7926	0.9959 [3]	14.134
11	0.1841	0.9167 [3]	0.2029
13	0.2803	0.9443 [3]	3.119
163	0.0809	0.55 [9]	2.477

Chebyshev's bias for the modulus $q = 4$. As explained in the introduction, our goal in this paper is to reexpress a standard Chebyshev's bias $\delta(x,q)$ into a regularized one $\delta'(x,q)$, that is always positive under the condition that GRH holds. Indeed we do not discover any numerical violation of GRH and we always obtains a positive $\delta'(x,q)$. The asymmetry of Chebyshev's bias arises in the plot δ vs δ', where the fall of the normalized bias $\frac{\delta}{\sqrt{x}}$ is faster for negative values of δ than for positive ones. Fig. 1 clarifies this effect for the historic modulus $q = 4$. We restricted our plot to the champions of the bias δ and separated positive and negative champions.

Chebyshev's bias for a prime modulus p. For a prime modulus p, we define the bias so as to obtain an averaging over all differences $\pi(x;p,N) - \pi(x;p,R)$, where as above N and R denote a non-quadratic and a quadratic residue, respectively

$$\delta(x,p) = -\sum_a \left(\frac{a}{p}\right)\pi(x;p,a), \tag{2.3}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Correspondingly, we define the regularized bias as

$$\delta'(x,p) = \frac{1}{\lfloor p/2 \rfloor}\sum_a \left(\frac{a}{p}\right)B(x;p,a). \tag{2.4}$$

Proposition 2.1. Let p be a selected prime modulus and $\delta'(x,p)$ as in (2.4) then the statement $\delta'(x,p) > 0,\ \forall x$, is equivalent to GRH for the modulus p.

As mentioned in the introduction, the Chebyshev's bias is much influenced by the location of the first non-trivial zero of the function $L(s,\kappa_q)$, κ_q being the real non-principal character modulo q. This is especially true for $L(s,\kappa_{163})$ with its smaller non-trivial zero at $\gamma \sim 0.2029$ [9]. The first negative values occur at $\{15073, 15077, 15083, \ldots\}$.

Fig. 2 represents the Chebyshev's bias δ' for the modulus $q = 163$ versus the standard one δ (thick dots). Tha asymmetry of the Chebyshev's bias is revealed at small values of $|\delta|$ where the the fit of the regularized bias by the curve $2/\log x$ is not good (thin dots).

For the modulus $q = 13$, the imaginary part of the first zero is not especially small, $\gamma_1 \sim 3.119$, but the variance (1.8) is quite high, $V(\kappa_{-13}) \sim 0.396$. The first negative values of $\delta(x,13)$ at primes occur when $\{2083, 2089, 10531, \ldots\}$. Fig. 3 represents the Chebyshev's bias δ' for the modulus $q = 13$ versus the standard one δ (thick dots) as compared to the fit by $2/\log x$ (thin dots).

Finally, for the modulus $q = 11$, the imaginary part of the first zero is quite small, $\gamma_1 \sim 0.209$, and the variance is high, $V(\kappa_{-11}) \sim 0.507$. In such a case, as shown in Fig. 4, the approximation of the regularized bias by $2/\log x$ is good in the whole range of values of x.

3. Proof of proposition 1.1

For approaching the proposition 1.1 we reformulate it in a simpler way as

Proposition 3.1. One introduces the regularized couting function $\pi'(x;q,l) := \pi(x;q,l) - \psi(x;q,l)/\log x$. The statement $\pi'(x;q,N) > \pi'(x;q,R),\ \forall x$ (ii), is equivalent to GRH for the modulus q.

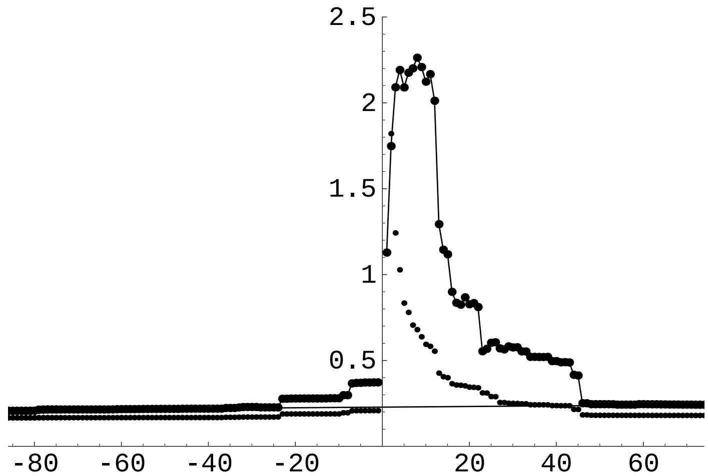

FIGURE 2. The normalized regularized bias $\delta'(x,163)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x,163)$ at all the prime champions of $|\delta(x,163)|$ [from $|\delta(x,163)| > 74$ the bias is $\delta(x,163) < 0$ negative], superimposed to the curve at the prime champions of $-\delta(x,163)$ (when $\delta(x,163) < 0$). The extremal prime champions in the plot are $x = 68491$ (with $\delta = 74$) and $x = 174637$ (with $\delta = -86$). The asymmetry is still clearly visible in the range of small values of $|\delta|$ but tends to disappear in the range of high values of $|\delta|$. The small dot curve corresponds to the fit of $\delta'(x,163)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

Proof. First observe that proposition 1.1 follows from proposition 3.1. This is straightforward because according to [2, p. 260], the prime number theorem for arithmetic progressions leads to the approximation

$$\mathrm{li}[\phi(q)\psi(x;q,l)] \sim \mathrm{li}(x) + \frac{\phi(q)\psi(x;q,l) - x}{\log x}. \tag{3.1}$$

As a result

$$\begin{aligned} \delta'(x,q) &= B(x;q,R) - B(x;q,N) \\ &= \mathrm{li}[\phi(q)\psi(x;q,R)] - \mathrm{li}[\phi(q)\psi(x;q,N)] + \phi(q)\delta(x,q) \\ &\sim \phi(q)[\pi'(x;q,N) - \pi'(x;q,R)]. \end{aligned}$$

The asymtotic equivalence in (3.1) holds up to the error term [2, p. 260] $O(\frac{R(x)}{x\log x})$, with

$$R(x) = \min\left(x^{\theta_q}\log^2 x, xe^{-a\sqrt{\log x}}\right),\ a > 0,$$

$$\theta_q = \max_{\kappa \mod q}(\sup \Re(\rho),\ \rho \text{ a zero of } L(s,\kappa)).$$

Let us now look at the statement GRH $\Rightarrow (i)$. Following [3, p 178-179], one has

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\kappa \mod q} \bar{\kappa}(a)\psi(x,\kappa)$$

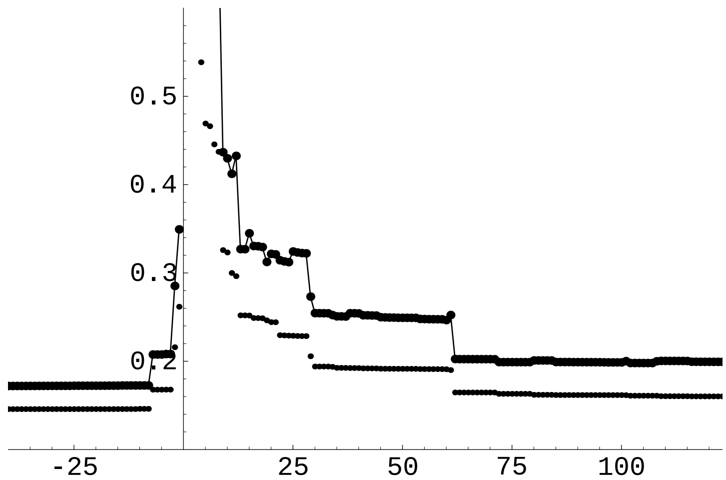

FIGURE 3. The normalized regularized bias $\delta'(x, 13)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 13)$ at the prime champions of $\delta(x, 13)$ (when $\delta(x, 13) > 0$), and the curve at the prime champions of $-\delta(x, 13)$ (when $\delta(x, 13) < 0$). The extremal prime champions in the plot are $x = 263881$ (with $\delta = 123$) and $x = 905761$ (with $\delta = -40$). The small dot curve corresponds to the fit of $\delta'(x, 13)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

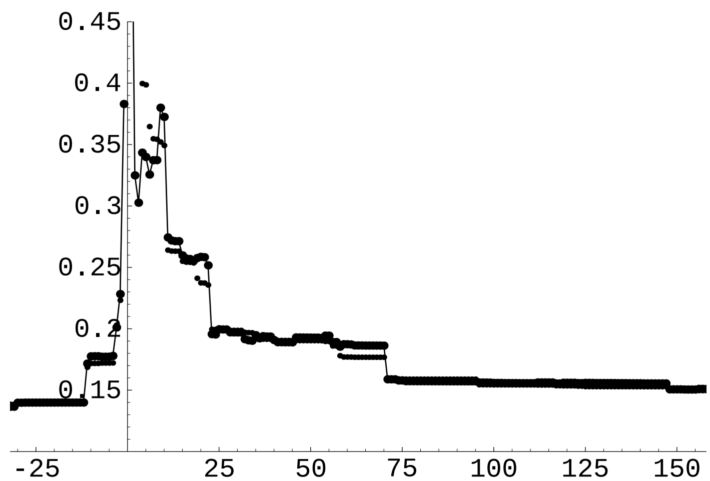

FIGURE 4. The normalized regularized bias $\delta'(x, 11)/\sqrt{x}$ versus the Chebyshev's bias $\delta(x, 11)$ at the prime champions of $\delta(x, 11)$ (when $\delta(x, 11) > 0$), and the curve at the prime champions of $-\delta(x, 11)$ (when $\delta(x, 11) < 0$). The extremal prime champions in the plot are $x = 638567$ (with $\delta = 158$) and $x = 1867321$ (with $\delta = -32$).The small dot curve corresponds to the (very good) fit of $\delta'(x, 11)/\sqrt{x}$ by $2/\log x$ calculated in Sec. 3.

and under GRH

$$\pi(x;q,a)=\frac{\pi(x)}{\phi(q)}-\frac{c(q,a)}{\phi(q)}\frac{\sqrt{x}}{\log x}+\frac{1}{\phi(q)\log x}\sum_{\kappa\neq\kappa_0}\bar{\kappa}(a)\psi(x,\kappa)+O(\frac{\sqrt{x}}{\log^2 x}),$$

where κ_0 is the principal character modulo q and

$$c(q,a)=-1+\#\{1\leq b\leq q:b^2=a\mod q\}$$

for coprimes integers a and q. Note that for an odd prime $q=p$, one has $c(p,a)=\left(\frac{a}{p}\right)$.

Thus, under GRH

$$\begin{aligned}\pi(x;q,N)-\pi(x;q,R)&=\tfrac{1}{\phi(q)\log x}[\sqrt{x}(c(q,R)-c(q,N))\\ &+\textstyle\sum_{\kappa\mod q}(\bar{\kappa}(N)-\bar{\kappa}(R))\psi(x,\kappa)+O(\left(\tfrac{\sqrt{x}}{\log^2 x}\right)].\end{aligned}\tag{3.2}$$

The sum could be taken over all characters because $\bar{\kappa}_0(N)=\bar{\kappa}_0(R)$. In addition, we have

$$\psi(x;q,N)-\psi(x;q,R)=\frac{1}{\phi(q)}\sum_{\kappa\mod q}[\bar{\kappa}(N)-\bar{\kappa}(R)]\psi(x,\kappa).\tag{3.3}$$

Using (3.2) and (3.3) the regularized bias reads

$$\begin{aligned}\delta'(x,q)&\sim\pi'(x;q,N)-\pi'(x;q,R)\\ &=\tfrac{\sqrt{x}}{\log x}[c(q,R)-c(q,N)]+O\left(\tfrac{\sqrt{x}}{\log^2 x}\right).\end{aligned}\tag{3.4}$$

For the modulus $q=4$, we have $c(q,1)=-1+2=1$ and $c(q,3)=-1$ so that $\delta'(x,4)=\frac{2\sqrt{x}}{\log x}$ The same result is obtained for a prime modulus $q=p$ since $c(p,N)=-1$ and $c(p,R)=c(p,1)=\left(\frac{1}{p}\right)=1$.

This finalizes the proof that under GRH, one has the inequality $\pi'(x;q,N)>\pi'(x;q,R)$.

If GRH does not hold, then using [2, lemma 2], one has

$$B(x;q,a)=\Omega_{\pm}(x^{\xi})\text{ for any }\xi<\theta_q.$$

Applying this assymptotic result to the residue classes $a=R$ and $a=N$, there exist infinitely many values $x=x_1$ and $x=x_2$ satisfying

$$B(x_1;q,R)<-x_1^{\xi}\text{ and }B(x_2;q,N)>x_2^{\xi}\text{ for any }\xi<\theta_q,$$

so that one obtains

$$B(x1;q,R)-B(x_2;q,N)<-x_1^{\xi}-x_2^{\xi}<0.\tag{3.5}$$

Selecting a pair (x_1,x_2) either

$$B(x_1;q,R)>B(x_2;q,R)$$

so that $B(x_2;q,R)-B(x_2;q,N)<0$ and (i) is violated at x_2, or

$$B(x_1;q,R)<B(x_2;q,R).\tag{3.6}$$

In the last case, either $B(x_1;q,N)>B(x_2;q,N)$, so that $B(x_1;q,R)-B(x_1;q,N)<0$ and the inequality (i) is violated at x_1, or simultaneously

$$B(x_1;q,N)<B(x_2;q,N)\text{ and }B(x_1;q,R)<B(x_2;q,R),$$

which implies (3.5) and the violation of (i) at $x=x_1=x_2$.

To finalize the proof of 3.1, and simultaneously that of 1.1, one makes use of the asymptotic equivalence of (i) and (ii), that is if GRH is true $\Rightarrow$ (ii) $\Rightarrow$ (i), and if GRH is wrong, (i) may be violated and (ii) as well.

Then, proposition 2.1 also follows as a straigthforward consequence of proposition 1.1.

□

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4. Summary

We have found that the asymmetry in the prime counting function $\pi(x;q,a)$ between the quadratic residues $a = R$ and the non-quadradic residues $a = N$ for the modulus q can be encoded in the function $B(x;q,a)$ [defined in (1.11)] introduced by Robin the context of GRH [2], or into the regularized prime counting function $\pi'(x;q,a)$, as in Proposition 3.1. The bias in π' reflects the bias in π conditionaly under GRH for the modulus q. Our conjecture has been initiated by detailed computer calculations presented in Sec. 2 and proved in Sec. 3. Further work could follow the work about the connection of π, and thus of π', to the sum of squares function $r_2(n)$ [10].

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