A port-Hamiltonian formulation of a 2D boundary controlled acoustic system

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Abstract: This paper deals with the port Hamiltonian formulation of a 2D boundary controlled acoustic system. The system under consideration consits of an acoustic wave traveling in a tube equipped with a network of microphones/loudspeakers. The purpose of this *smart skin* is to damp the acoustic wave and reduce its effect at the output of the tube. It is first commented how the original 3D system can be reduced to a 2D system by considering symmetries. Then, the boundary port variables associated to the wave equation are parametrized in order to define a Dirac structure in two dimensions, compatible with the interconnection at the boundaries with the actuation system. The overall system (wave+actuators/sensors) is finally expressed as a port Hamiltonian control system and a first stabilizing distributed control law is proposed.

Keywords: Distributed Port-Hamiltonian systems, passivity based control, wave propagation.

1. INTRODUCTION

Port-Hamiltonian systems (PHS) are derived from the study of energy variations in physical systems. They permit to describe the dynamic behaviour of non-linear and linear systems by the use of skew-symmetric operators which express the energy exchanges between different energy domains of a system. These models were introduced for finite-dimensional systems in Maschke and van der Schaft (1992) and later exploited to develop non-linear passivity based control techniques (van der Schaft, 2000; Ortega et al., 2002). PHS were further extended to infinite-dimensional systems in van der Schaft and Maschke (2002), and control approaches based on differential geometry (Macchelli and Maschke, 2009) and on semi-group theory were later developed for asymptotic and exponential stabilization (Le Gorrec et al., 2005; Villegas, 2007; Ramirez et al., 2014).

Vibration reduction is a technological problem in many engineering appliactions. For instance, in air-planes it is a problematic concerning safety, passenger comfort and noise pollution around airports. A way to reduce the noise is the use of active surfaces (Collet et al., 2011, 2009). A particular concern in this type of application is the control of acoustic waves inside tubular sections (David et al., 2010) using active surfaces to perform boundary control.

This paper addresses the modelling and control of a tubular vibro-acoustic systems actuated with an active surface. It is first shown that a 2D wave propagation process can be modelled as a distributed PHS on a rectangular spatial domain considering the appropriate port variables and by defining a power preserving symmetric pairing. The construction of the model follows the same line of reasoning already presented for 1D distributed PHS (Le Gorrec et al., 2005). It is then shown that the interconnection of the acoustic system (2D wave equation) with an active surface again defines a PHS. The stability of the system is then investigated, and it is shown that under the assumed boundary conditions, a passivity based control (PBC) applied by the active surface reduces the vibrations along the walls and at the output of the acoustic tube.

The paper is organized as follows: Section 2 presents the deduction of the port-Hamiltonian model of wave equation in a tube on a 2D rectangular spatial domain. In Section 3 its is shown that the vibro-acoustic process under consideration fits within the developed framework. In Section 4 the vibro-acoustic process is interconnected with an active surface and it is shown that the interconnected system is a PHS. In Section 5 a control law that eliminates the vibrations along the walls and at the output of the acoustic system is proposed. Finally, some concluding remarks and ideas of future work are given in Section 6.

2. THE 2D WAVE EQUATION AS PHS

The system corresponds to an acoustic wave in a cylindrical tube which evolves without loss of energy. The axisymmetry of the system permits to reduce it from a 3D representation to a 2D representation. Hence, we consider a

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2D linear wave equation over a rectangular spatial domain $(x,y) \in [0,L]\times[0,R]$ equipped with the product $\langle,\rangle_{L_2^2}$ such that $\langle v, w'\rangle_{L_2^2} = \int_0^R \int_0^L v^T w' dxdy$. It is derived from the balance equation on the extensive variables $z = (z_1 \; z_2)^T$ (respectively one and two forms) expressed in term of the intensive variables $\mathcal{L}z = \frac{\delta\mathcal{H}}{\delta z}$ (respectively one and zero forms) where $\mathcal{H} = \int_0^R \int_0^L \left(z^T \mathcal{L} z\right) dxdy$ leading to the infinite dimensional system

$$\dot{z} = \mathcal{J}\mathcal{L}z, \quad \text{with} \quad \mathcal{J} = \begin{pmatrix} 0 & -grad \\ -div & 0 \end{pmatrix}. \tag{1}$$

where 0 represent zero matrices of appropriated dimensions and where $grad$ and div correspond to the gradient and divergence operator respectively. This system can be written in terms of flow and effort variables:

$$f = \mathcal{J}e, \tag{2}$$

Proposition 1. Assume that the effort variables $e = (e_1, e_2, e_3)^T$ of the 2D-wave propagation process have compact domain, then the differential operator $\mathcal{J}$ is skew symmetric.

Proof. One has to show that $\langle e^1, \mathcal{J}e^2\rangle = \langle -\mathcal{J}e^1, e^2\rangle_{L_2^2}$ for any $\{e^1, e^2\} \in L_2^2$. Developing for (1) we have

$$\langle e^1, \mathcal{J}e^2\rangle_{L_2^2} = \int_0^R \int_0^L \left(e_1^1 \; e_2^1 \; e_3^1\right) \begin{pmatrix} 0 & -grad \\ -div & 0 \end{pmatrix} \begin{pmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{pmatrix} dxdy \tag{3a}$$

$$\langle e^1, \mathcal{J}e^2\rangle_{L_2^2} = -\int_0^R \int_0^L \left(e_1^1 \frac{\partial e_3^2}{\partial x} + e_2^1 \frac{\partial e_3^2}{\partial y} + e_3^1 \frac{\partial e_1^2}{\partial x} + e_3^1 \frac{\partial e_2^2}{\partial y}\right) dxdy$$

and integrating by parts

$$\langle e, Je^2\rangle_{L_2^2} = -\int_0^R [e_3^2 e_1^1 + e_1^2 e_3^1]_0^L dy - \int_0^L [e_3^2 e_2^1 + e_2^2 e_3^1]_0^R dx + \int_0^R \int_0^L \left(e_1^2 \frac{\partial e_3^1}{\partial x} + e_2^2 \frac{\partial e_3^1}{\partial y} + e_3^2 \frac{\partial e_1^1}{\partial x} + e_3^2 \frac{\partial e_2^1}{\partial y}\right) dxdy$$

$$\langle -Je^1, e^2\rangle_{L_2^2} = \int_0^R \int_0^L -\left[\begin{pmatrix} 0 & -grad \\ -div & 0 \end{pmatrix} \begin{pmatrix} e_1^1 \\ e_2^1 \\ e_3^1 \end{pmatrix}\right]^T \begin{pmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{pmatrix} dxdy$$

$$\langle -Je^1, e^2\rangle_{L_2^2} = \int_0^R \int_0^L \left(e_1^2 \frac{\partial e_3^1}{\partial x} + e_2^2 \frac{\partial e_3^1}{\partial y} + e_3^2 \frac{\partial e_1^1}{\partial x} + e_3^2 \frac{\partial e_2^1}{\partial y}\right) dxdy$$

which shows that

$$\langle e^1, Je^2\rangle_{L_2^2} = \langle -Je^1, e^2\rangle_{L_2^2} + \int_0^R [e_3^2 e_1^1 + e_1^2 e_3^1]_0^L dy + \int_0^L [e_3^2 e_2^1 + e_2^2 e_3^1]_0^R dx \tag{3b}$$

which becomes in the case of a compact domain $\langle e, \mathcal{J}e^2\rangle_{L_2^2} = \langle -\mathcal{J}e^1, e^2\rangle_{L_2^2}$ which completes the proof. ■

Let us now investigated the existence of a product $\langle,\rangle_+$ on which $\mathcal{J}$ is skew-symmetric for non zero boundary variables.

Proposition 2. Define the symmetric pairing,

$$\langle (f^1, f_\partial^1, e^1, e_\partial^1), (f^2, f_\partial^2, e^2, e_\partial^2)\rangle_+ = \langle e^1, f^2\rangle_{L_2^2} + \langle e^2, f^1\rangle_{L_2^2} - \langle e_\partial^1, f_\partial^2\rangle - \langle e_\partial^2, f_\partial^1\rangle \tag{4}$$

on the bond space $B = L_2^2 \times L_2^2$, where the boundary variables of the system are denoted with a ∂ index and

$$\langle e_\partial^1, f_\partial^2\rangle = \int_0^R e_{\partial 1}^{1T} f_{\partial 1}^2 dx + \int_0^L e_{\partial 2}^{1T} f_{\partial 2}^2 dy.$$

Then, J is a skew-symmetric operator with respect to $\langle,\rangle_+$ and for the set of boundary variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = U \frac{1}{\sqrt{2}} \begin{pmatrix} -e_3(L,y)(L,y) + e_3(0,y) \\ -e_1(L,y) + e_1(0,y) \\ e_1(L,y) + e_1(0,y) \\ e_3(L,y) + e_3(0,y) \\ -e_3(x,R) + e_3(x,0) \\ -e_2(x,R) + e_2(x,0) \\ e_2(x,R) + e_2(x,0) \\ e_3(x,R) + e_3(x,0) \end{pmatrix}. \tag{5}$$

with U a full rank matrix satisfying $U^T \Sigma U = \Sigma$, where $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (I the identity matrix in M_2).

Proof. Denote F_1 and F_2 the terms of the $\langle,\rangle_{L_2^2}$ to compensate, this is

$$F_1 = -\int_0^R [e_3^2 e_1^1 + e_1^2 e_3^1]_0^L dy$$

$$F_2 = -\int_0^L [e_3^2 e_2^1 + e_2^2 e_3^1]_0^R dx$$

which can be written in terms of the $Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ as

$$F_1 = -\int_0^R \left[\left(e_1^1 \; e_3^1\right) Q \begin{pmatrix} e_1^2 \\ e_3^2 \end{pmatrix}\right]_0^L dy$$

$$F_2 = -\int_0^L \left[\left(e_2^1 \; e_3^1\right) Q \begin{pmatrix} e_2^2 \\ e_3^2 \end{pmatrix}\right]_0^R dx$$

Developing the products leads to the definition of a new matrix $Q_{ext} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}$ such that

$$F_1 = \int_0^R \{\left(e_1^1(L,y) \; e_3^1(L,y) \; e_1^1(0,y) \; e_3^1(0,y)\right) Q_{ext} \begin{pmatrix} e_1^2(L,y) \\ e_3^2(L,y) \\ e_1^2(0,y) \\ e_3^2(0,y) \end{pmatrix}\} dy$$

$$F_2 = \int_0^L \{\left(e_2^1(x,R) \; e_3^1(x,R) \; e_2^1(x,0) \; e_3^1(x,0)\right) Q_{ext} \begin{pmatrix} e_2^2(x,R) \\ e_3^2(x,R) \\ e_2^2(x,0) \\ e_3^2(x,0) \end{pmatrix}\} dx$$

Based on Le Gorrec et al. (2005), Definition 3.3 and Lemma 3.4, we define $R_{ext} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}$ (where I is the identity matrix in M_2) which satisfies

$$Q_{ext} = R_{ext}^T \Sigma R_{ext} \tag{6}$$

where $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

$$F_1 = \int_0^R \left(f_{\partial 1}^1 \;\; e_{\partial 1}^1 \right) \Sigma \begin{pmatrix} f_{\partial 1}^2 \\ e_{\partial 1}^2 \end{pmatrix} dy \tag{7a}$$

$$F_2 = \int_0^L \left(f_{\partial 2}^1 \;\; e_{\partial 2}^1 \right) \Sigma \begin{pmatrix} f_{\partial 2}^2 \\ e_{\partial 2}^2 \end{pmatrix} dx \tag{7b}$$

with, for $i \in \{1, 2\}$,

$$\begin{pmatrix} f_{\partial 1}^i \\ e_{\partial 1}^i \end{pmatrix} = UR_{ext} \begin{pmatrix} e_1^i(L, y) \\ e_3^i(L, y) \\ e_1^i(0, y) \\ e_3^i(0, y) \end{pmatrix}, \begin{pmatrix} f_{\partial 2}^i \\ e_{\partial 2}^i \end{pmatrix} = UR_{ext} \begin{pmatrix} e_2^i(x, R) \\ e_3^i(x, R) \\ e_2^i(x, 0) \\ e_3^i(x, 0) \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} f_{\partial 1}^i \\ e_{\partial 1}^i \end{pmatrix} = U \frac{1}{\sqrt{2}} \begin{pmatrix} -e_3^i(L, y) + e_3^i(0, y) \\ -e_1^i(L, y) + e_1^i(0, y) \\ e_1^i(L, y) + e_1^i(0, y) \\ e_3^i(L, y) + e_3^i(0, y) \end{pmatrix} \tag{9a}$$

$$\begin{pmatrix} f_{\partial 2}^i \\ e_{\partial 2}^i \end{pmatrix} = U \frac{1}{\sqrt{2}} \begin{pmatrix} -e_3^i(x, R) + e_3^i(x, 0) \\ -e_2^i(x, R) + e_2^i(x, 0) \\ e_2^i(x, R) + e_2^i(x, 0) \\ e_3^i(x, R) + e_3^i(x, 0) \end{pmatrix} \tag{9b}$$

Finally, define

$$e_\partial^i = \begin{pmatrix} e_{\partial 1}^i \\ e_{\partial 2}^i \end{pmatrix}, \; f_\partial^i = \begin{pmatrix} f_{\partial 1}^i \\ f_{\partial 2}^i \end{pmatrix}$$

from which $\langle , \rangle_+ = 0$ follows. ■

Remark 1. In practice, it is not usual to use such linear combinations between the boundary variables to develop a control structure. Finding a matrix U permitting to separate them allows to express directly physical variables at the boundary. Such matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Multiplying the previously found boundary variables by U we have

$$\begin{pmatrix} f_{\partial 1}^i \\ e_{\partial 1}^i \end{pmatrix} = \begin{pmatrix} -e_3^i(L, y) \\ e_1^i(0, y) \\ e_1^i(L, y) \\ e_3^i(0, y) \end{pmatrix}, \begin{pmatrix} f_{\partial 2}^i \\ e_{\partial 2}^i \end{pmatrix} = \begin{pmatrix} -e_3^i(x, R) \\ e_2^i(x, 0) \\ e_2^i(x, R) \\ e_3^i(x, 0) \end{pmatrix} \tag{11}$$

or

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} -e_3(L, y) \\ e_1(0, y) \\ -e_3(x, R) \\ e_2(x, 0) \\ e_1(L, y) \\ e_3(0, y) \\ e_2(x, R) \\ e_3(x, 0) \end{pmatrix} \tag{12}$$

from where and using (4) we obtain

$$\langle (f^1, f_\partial^1, e^1, e_\partial^1), (f^2, f_\partial^2, e^2, e_\partial^2) \rangle_+ = 0.$$

The energy balance of the boundary controlled PHS system is simple given by the power exchange through the 2D boundary of the system

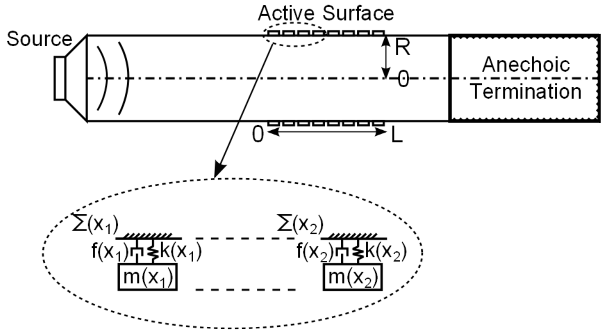

Fig. 1. structure of the setup

$$\dot{\mathcal{H}} = \int_0^L f_{\partial 2}^\top e_{\partial 2} dx + \int_0^R f_{\partial 1}^\top e_{\partial 1} dy. \tag{13}$$

3. APPLICATION TO AN ACOUSTIC PROCESS

The physical application is a cylindrical tube considered in Collet et al. (2011) and Collet et al. (2009) in which an acoustic wave evolves without energy loss. The source of the wave is a non-controlled, but energy bounded loudspeaker. An anechoic chamber avoids any reflection of the wave at the end of the tube (see Fig. 3.1). An active surface covers the tube's walls on part of its length. This active surface permits to damp the wave propagation using boundary control. The symmetry of the problem allows to consider the 3D process using a 2D rectangular spatial domain to describe the behaviour of the fluid considering as a continuity condition the fact that the radial speed of the fluid on horizontal the axis is null.

3.1 Parametrization of the system

Consider a part of the cylindrical tube (length L, radius R) filed with a fluid through which an acoustic wave evolves. The axisymmetry of the problem permits to consider a 2D rectangle (where the x axis represent the length $x \in [0, L]$ and the y axis the width $y \in [0, R]$) to describe the behaviour of the fluid. Consider a small particle of fluid of constant mass M. Define

$$\mu(x, y, t) = \mu_0 + \mu_1(x, y, t)$$
$$v(x, y, t) = v_0 + v_1(x, y, t)$$
$$p(x, y, t) = p_0 + p_1(x, y, t)$$

where $\mu = \mu(x, y, t)$, $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = v(x, y, t)$ and $p = p(x, y, t)$ are, respectively, the volume mass density, the velocity and the pressure of the particle at the point $(x, y) \in [0, L] \times [0, R] \subset \mathbb{R}^2$ at time t, v_0, p_0 and μ_0 are constant values which characterize the fluid without presence of the acoustic wave. v_1, p_1 and μ_1 characterize the acoustic wave. The following assumptions are performed on the properties of the fluid.

Assumption 2.

- There is no internal entropy creation (and thus no internal energy dissipation).
- v_1,p_1,μ_1 are infinitely small and of the same order, as well as their time derivatives and the temporal means of their values are zero (acoustic approximation).
- The input power density of the tube (at $x = 0$) is not controlled but bounded and denoted by $\varsigma_{in}(0, y, t)$.

- The axisymmetry of the horizontal axis is expressed as the boundary condition $v_y(x,0)=0$.
- The anechoic termination imposes the boundary condition (Collet et al., 2009) $\frac{\partial p}{\partial x}(L,y,t) = -\frac{1}{c_0}\frac{dp}{dt}$, where c_0 corresponds to the speed of sound in the air.

3.2 Linear model

The state of the particle is described by Euler's equation and the mass balance (Kinsler et al., 1999), respectively,

$$\mu\left(\frac{\partial v}{\partial t} + (\overrightarrow{v}.\overrightarrow{grad})v\right) = -gradp \tag{15}$$

$$\frac{\partial \mu}{\partial t} + div(\mu v) = 0 \tag{16}$$

Where, with $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, $(\overrightarrow{v}.\overrightarrow{grad})v = \begin{pmatrix} v_x \dfrac{\partial v_x}{\partial x} \\ v_y \dfrac{\partial v_y}{\partial y} \end{pmatrix}$.

From Assumption 2 and (15) (and taking into consideration that if there exist a and b, two infinitely small of the same order, then $ab << a$ and $ab << b$), we have

$$\mu_0 \frac{\partial v_1(x,y,t)}{\partial t} = -gradp_1(x,y,t) \tag{17}$$

and from (16)

$$\frac{\partial \mu_1(x,y,t)}{\partial t} + \mu_0 div(v_1(x,y,t)) = 0. \tag{18}$$

The isentropy assumption allows to define the adiabatic compressibility factor $\chi_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S$ which becomes under the current assumptions :

$$\chi_s = \frac{1}{\mu}\left(\frac{\partial \mu}{\partial p}\right)_S = \frac{1}{\mu_0}\left(\frac{\partial \mu_1}{\partial p_1}\right)_S \tag{19}$$

where S denotes the constant entropy of the process.

3.3 The port-Hamiltonian model

In this subsection it is shown that the linear model of the vibro-acoustic process corresponds to a port-Hamiltonian system. The acoustic energy $E^{particle}$ of one particle is

$$E^{particle} = E_p^{particle} + E_c^{particle} \tag{20}$$

where $E_p^{particle}$ is the potential elastic energy and $E_c^{particle}$ the kinetic energy of the particle. More specifically $E_p^{particle}$ can be deduced by considering the work W of the pressure forces

$$dE_p^{particle} = -dW = -p_1 d(\partial V) = -p_1 V d\alpha$$

where $\alpha = \frac{\partial V}{V}$ is the volume expansion coefficient which from (19) is given by $\alpha = -\chi_s p_1$. Integrating $dE_p^{particle}$ along α we obtain

$$E_p^{particle} = \int_0^\alpha -p_1 V d\alpha = \frac{V}{\chi_s}\int_0^\alpha \alpha d\alpha = V\chi_s \frac{p_1^2}{2} \tag{21}$$

The expression of the kinetic energy is simply given by

$$E_c^{particle} = \frac{m}{2} v_1^\top v_1. \tag{22}$$

In order to the consider the energy of the whole system, we proceed to define the energy density $\varepsilon = \frac{E^{particle}}{V}$, which will be integrated over the complete spatial domain to obtain the total energy of the system. Hence we have,

$$\varepsilon = \varepsilon_c + \varepsilon_p = \frac{\chi_s p_1^2}{2} + \frac{\mu_0}{2} v_1^\top v_1 \tag{23}$$

(where we have used $M = \mu_0 V$). Define $\theta = \mu_0 v_1$ (which may be interpreted as density of quantity of movement) and $\Gamma = -\alpha = \chi_s p_1$ as state variables for the port-Hamiltonian system. The energy of the system denoted by $\mathcal{H}$, can then be obtain by integrating ε on the volume of the tube. Considering a 2D-axisymmetric simplification the total energy is given by

$$\mathcal{H} = \int_0^R \int_0^L \left(\frac{\theta^\top \theta}{2\mu_0} + \frac{\Gamma^2}{2\chi_s}\right) dxdy \tag{24}$$

Equations (17),(18) and (19) permit to write the temporal derivatives of the state variables $z = \begin{pmatrix} \theta \\ \Gamma \end{pmatrix}$ in function of $\frac{\partial \varepsilon}{\partial z}$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\Gamma} \end{pmatrix} = -\nabla \begin{pmatrix} \dfrac{1}{\chi_s}\Gamma \\ \dfrac{1}{\mu_0}\theta \end{pmatrix} = \begin{pmatrix} 0 & -grad \\ -div & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \varepsilon}{\partial \theta} \\ \dfrac{\partial \varepsilon}{\partial \Gamma} \end{pmatrix} \tag{25}$$

where 0 represent zero matrices of appropriated dimensions. The previous model corresponds, according to Proposition 2, to a port-Hamiltonian system for the port variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} -e_3(L,y) \\ e_1(0,y) \\ -e_3(x,R) \\ e_2(x,0) \\ e_1(L,y) \\ e_3(0,y) \\ e_2(x,R) \\ e_3(x,0) \end{pmatrix} = \begin{pmatrix} -p(L,y) \\ v_x(0,y) \\ -p(x,R) \\ v_y(x,0) \\ v_x(L,y) \\ p(0,y) \\ v_y(x,R) \\ p(x,0) \end{pmatrix} \tag{26}$$

(25) can be written as (27) and corresponds to the form given in (1) with total energy $\mathcal{H} = \int_0^R \int_0^L \left(z^T \mathcal{L} z\right) dxdy$.

$$\underbrace{\begin{pmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\Gamma} \end{pmatrix}}_{\dot{z}} = \underbrace{\begin{pmatrix} 0 & 0 & -\dfrac{\delta}{\delta x} \\ 0 & 0 & -\dfrac{\delta}{\delta y} \\ -\dfrac{\delta}{\delta x} & -\dfrac{\delta}{\delta y} & 0 \end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix} \dfrac{1}{\mu_0} & 0 & 0 \\ 0 & \dfrac{1}{\mu_0} & 0 \\ 0 & 0 & \dfrac{1}{\chi_s} \end{pmatrix}}_{\mathcal{L}} \underbrace{\begin{pmatrix} \theta_x \\ \theta_y \\ \Gamma \end{pmatrix}}_{z} \tag{27}$$

with energy balance is given by (13)

$$\begin{aligned} \dot{\mathcal{H}} = &\int_0^L (-e_3(x,R)e_2(x,R) + e_2(x,0)e_3(x,0))dx \\ &+ \int_0^R (-e_3(L,y)e_1(L,y) + e_1(0,y)e_3(0,y))dy \end{aligned} \tag{28}$$

Remark 3. The boundary conditions representing the input power density, axisymmetry and the anechoic termination (see Assumption 2) are expressed in terms of the boundary port variables, respectively, as $e_1(0,y)e_3(0,y) = \varsigma_{in}(0,y)$, $v_y(x,0) = e_2(x,0) = 0$ and $v_x(L,y) = c_0\mu_0 p(L,y)$ or equivalently $e_1(L,y) = c_0\mu_0 e_3(L,y)$.

4. INTERCONNECTION WITH THE ACTIVE SURFACE

To attenuate the acoustic wave along the walls and at the output of the cylinder, an actuated elastic surface is attached to the vibro-acoustic system at the boundary $y = R$ (See Fig. 3.1). The elastic surface is modelled as an infinite array of parallel mass-spring-damper systems, which can be approximated by an infinite dimensional mass-spring-damper system with spatial domain $x \in [0, L]$

$$\dot{w}(x,t) = \begin{pmatrix} \frac{dq}{dt}(x,t) \\ \frac{d\varrho}{dt}(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -f(x) \end{pmatrix} \begin{pmatrix} k(x)q(x,t) \\ \frac{\varrho(x,t)}{m(x)} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(x,t) \tag{29}$$

$$\bar{y}(x,t) = (0 \ 1) \begin{pmatrix} k(x)q(x,t) \\ \frac{\varrho(x,t)}{m(x)} \end{pmatrix}$$

with total energy given by

$$\mathcal{H}_c(x,t) = \frac{1}{2}\int_0^L k(x)q^2(x,t) + \frac{\varrho^2(x,t)}{m(x)} dx, \tag{30}$$

where $w(x,t) \in \mathbb{R}^2 \times [0, L]$ is the state of the system, $q(x,t)$ is the general coordinate (displacement of the mass following the y axis, $\varrho(x,t)$ is the linear momentum, $f(x)$ and $k(x)$ the infinitesimal damping and stiffness constants respectively, and $m(x)$ the linear mass density. The distributed input and output are respectively, $u_1(x,t) = p_R(x,t)$ which corresponds to the pressure density and the distributed velocity density $\bar{y}(x,t)$ at the boundary $y = R$. The energy balance of the boundary layer system is given by

$$\dot{H}_c = \int_0^L u_1(x,t)\bar{y}(x,t) - f(x)\bar{y}^2(x,t)dx. \tag{31}$$

It is observed from the energy balance that (29) defines a dissipative port-Hamiltonian system. The interconnection of the acoustic system and the active surface is performed taking into account the continuity of the velocity of the flux and Newton's third law at the interface, respectively

$$e_2(x,R) = v_y(x,R) = \bar{y}(x,t) \tag{32}$$

$$e_3(x,R) = -u_1 \tag{33}$$

The interconnected system is thus given by

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} kq(x) \\ \frac{\varrho(x)}{m} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (-e_3(x,R))$$

$$e_2(x,R) = (0 \ 1 \ 0 \ 0 \ 0) \begin{pmatrix} kq(x) \\ \frac{\varrho(x)}{m} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} \tag{34}$$

where $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & -f(x) \end{smallmatrix}\right]$ and 0 are zero matrices and vectors of appropriated dimensions. The boundary variables are defined by (26), but now we see that $e_2(x,R)$ and $e_3(x,R)$ relate the dynamic of the boundary layer system and the boundary control system. The energy balance of the interconnected system is simple given by

$$\dot{\mathcal{H}}_T = \dot{\mathcal{H}} + \dot{\mathcal{H}}_c. \tag{35}$$

5. ACTIVE DAMPING CONTROL

The control objective is to to damp the oscillations along the walls and at the output of the tube. This implies to drive the system to a point of bounded energy while minimizing the velocity of the wave at the boundaries $x = L$ and $y = R$. To this end we consider that the active surface is actuated through an external distributed controlled force $u_2(x,t) = f_c(x,t)$. With this control input the interconnected system is given by

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} kq(x) \\ \frac{\varrho(x)}{m} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (-e_3(x,R) + u_2(x,t))$$

$$e_2(x,R) = (0 \ 1 \ 0 \ 0 \ 0) \begin{pmatrix} kq(x) \\ \frac{\varrho(x)}{m} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} \tag{36}$$

with energy balance given by (35) (see (13) and (31))

$$\begin{aligned} \dot{\mathcal{H}}_T &= \dot{\mathcal{H}} + \dot{\mathcal{H}}_c + \int_0^L u_2\bar{y}dx \\ &= \int_0^L f_{\partial 2}^\top e_{\partial 2}dx + \int_0^R f_{\partial 1}^\top e_{\partial 1}dy \\ &+ \int_0^L (u_1\bar{y} - f\bar{y}^2)dx + \int_0^L u_2\bar{y}dx \\ &= \int_0^L (-e_3(x,R)e_2(x,R) + e_2(x,0)e_3(x,0))dx \\ &+ \int_0^R (-e_3(L,y)e_1(L,y) + e_1(0,y)e_3(0,y))dy \\ &+ \int_0^L (u_1\bar{y} - f\bar{y}^2)dx + \int_0^L u_2\bar{y}dx \end{aligned} \tag{37}$$

Since $\mathcal{H}_T$ is the total mechanical energy of the interconnected system, it is strictly positive definite with minimum at the mechanical equilibrium $(z, w) = (0, 0)$. Hence $\mathcal{H}_T$ qualifies as a Lyapunov function candidate for the interconnected system. However, we don't wish to drive the system to its mechanical equilibrium, but to a position of partial equilibrium (bounded energy) that tends to minimize $v_y(x,R)$ and $v_x(L,y)$. To this end u_2 should be chosen such that $\dot{\mathcal{H}}_T < 0$ for large $v_y(x,R)$ and $v_x(L,y)$ and such that $\mathcal{H}_T(z,w)$ remains bounded for small $v_y(x,R)$ and $v_x(L,y)$. Taking into account the boundary conditions due to the unknown but bounded input density power of the wave $e_1(0,y)e_3(0,y) = \varsigma_{in}(0,y)$, the symmetry axis $e_2(x,0) = 0$, anechoic termination $e_1(L,y) = \mu_0 c_0 e_3(L,y)$ and the interconnections $e_3(x,R) = -u_1$ and $e_2(x,R) = \bar{y}$, we obtain from (37) the following energy balance

$$\dot{\mathcal{H}}_T = \int_0^L (e_3(x,R)e_2(x,R) - f(x)e_2^2(x,R) + u_2 e_2(x,R))dx + \int_0^R (\varsigma_{in}(0,y) - \tfrac{1}{\mu_0 c_0}e_1^2(L,y))dy. \tag{38}$$

Hence a possible choice for the control is

$$u_2 = -e_3(x,R) - k(x,y)e_2(x,R) \tag{39}$$

with $k(x, y, t) \geq 0$ a bounded scalar function which may be function of for instance $\varsigma_{in}(0, y)$. Notice that the control law expresses a passivity based control (PBC) law that shifts the equilibrium (first term) and injects damping (second term). The control yields the following closed-loop energy balance

$$\dot{\mathcal{H}}_T = -\int_0^L (f(x) + k(x, y))e_2^2(x, R)dx + \int_0^R (\varsigma_{in}(0, y) - \tfrac{1}{\mu_0 c_0} e_1^2(L, y))dy. \quad (40)$$

It is observed from the closed-loop energy balance that the system will be driven to a point of bounded energy. Indeed, (40) contains two dissipative terms and one bounded non signed-defined term that is function of the input power density $\int_0^R (\varrho_{in}(0, y)dy$. The dissipative terms are functions of the wave's velocity densities along the walls and at the end of the acoustic tube, $v_y(L, y)$ and $v_x(L, y)$ respectively. Since the velocities express the oscillations of the wave, which in turn depend on the energy of the wave at the input of the tube, the energy balance (40) will become negative if these are big, and will tend to zero when they are small enough to equal the non-dissipative term which depends on the input power. The dissipation rate can be tuned by increasing or decreasing the contribution of the control function $k(x, y)$ in (39), making the PB controller more or less sensible to large oscillations.

6. CONCLUSION

A 2D boundary controlled port-Hamiltonian model of a an acoustic wave process in an actuated tube has been proposed. The boundary controlled model permits to express the interconnection between the acoustic wave and the actuator, defined by an active surface situated on the walls of the tube and modelled as a distributed mass-spring-damper system. The complete control system, given by the interconnection of the wave and the active surface, has been shown to be in port-Hamiltonian format as well. This has allowed to study the attenuation of wave oscillations along the walls and output of the tube using passivity based control techniques. As first result a control strategy which actively damps the oscillations has been proposed. Future work will deal with the discretization and simulation of the control system and the implementation of the proposed control strategy.

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