Asymptotic stability for a class of boundary control systems with non-linear damping ⋆

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Abstract: The asymptotic stability of boundary controlled port-Hamiltonian systems defined on a 1D spatial domain interconnected to a class of non-linear boundary damping is addressed. It is shown that if the port-Hamiltonian system is approximately observable, then any boundary damping which behaves linear for small velocities asymptotically stabilizes the system.

Keywords: Boundary control systems, infinite-dimensional port Hamiltonian systems, asymptotic stability, non-linear control.

1. INTRODUCTION

Many physical distributed parameter systems can be controlled through their boundaries. This is for instance the case for transmission lines, flexible beams and plates, tubular and nuclear fusion reactors and so on. This class of systems is called Boundary Controlled Systems (BCS). In the linear case the control design for such system can be tackled using the semigroup theory and the associated abstract formulation based on unbounded input/output mappings (Curtain and Zwart, 1995; Staffans, 2005). When asymptotic or exponential stability by non-linear control is concerned, the main difficulty remains in finding the appropriate Lyapunov function candidate to prove the stability. It is usually done on a case by case basis using physical considerations depending on the application field.

In the last decade, an alternative approach has been developed in order to deal with a large class of physical systems. This approach is based on the extension of the Hamiltonian formulation to open distributed parameter systems (van der Schaft and Maschke, 2002). In the 1D linear case it gave rise to the definition of boundary controlled port Hamiltonian systems (Le Gorrec et al., 2004) and allowed to parametrize all the possible boundary conditions that define a boundary control system (Le Gorrec et al., 2005) by using simple matrix conditions. Many variations around these primary works can be found in (Villegas, 2007) and in (Jacob and Zwart, 2012). Well possessedness and stability have been investigated in open-loop and in the case of static boundary feedback control in (Zwart et al., 2010) and (Villegas et al., 2005; Villegas et al., 2009), respectively, and in the case of dynamic linear control in (Ramirez et al., 2014; Augner and Jacob, 2014).

This paper is restricted to the analysis of the asymptotic stability of a port-Hamiltonian system connected to a non-linear damper. It is show that asymptotic stability can be proved whenever the port-Hamiltonian system is approximately observable. In the next section we introduce our class of port-Hamiltonian systems and our class of dampers. In Section 3 we formulate and prove our main theorem.

2. PORT-HAMILTONIAN SYSTEMS

The systems under study are described by the following 1D partial differential equation (PDE):

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right) + P_0\mathcal{H}(\zeta)x(t,\zeta), \qquad (1)$$

$\zeta \in (a,b)$, where $P_1 \in M_n(\mathbb{R})$ [1] is a non-singular symmetric matrix, $P_0 = -P_0^\top \in M_n(\mathbb{R})$, and x takes values in $\mathbb{R}^n$. Furthermore, $\mathcal{H}(\cdot) \in L_2((a,b); M_n(\mathbb{R}))$ is a bounded and measurable, matrix-valued function satisfying for almost all $\zeta \in (a,b)$, $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^\top$ and $\mathcal{H}(\zeta) > mI$, with m independent from ζ.

For simplicity $\mathcal{H}(\zeta)x(t,\zeta)$ will be denoted by $(\mathcal{H}x)(t,\zeta)$. For the above pde we assume that some boundary conditions are homogeneous, whereas others are controlled. Thus there are matrices of appropriate sizes such that

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[1] $M_n(\mathbb{R})$ denote the space of real $n \times n$ matrices

$$u(t) = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix} \tag{2}$$

and

$$0 = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{3}$$

Furthermore, there is a boundary output given by

$$y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{4}$$

To study the existence and uniqueness of solution to the above controlled pde, we follow the semigroup theory, see also Le Gorrec et al. (2005); Jacob and Zwart (2012). Therefor we define the state space X as $X = L_2((a,b);\mathbb{R}^n)$ with inner product $\langle x_1, x_2\rangle_{\mathcal{H}} = \langle x_1, \mathcal{H}x_2\rangle$ and norm $\|x\|^2_{\mathcal{H}} = \langle x, x\rangle_{\mathcal{H}}$. Note that the norm on X and the L_2 norm are equivalent. Hence X is a Hilbert space. The reason for selecting this space is that $\|\cdot\|^2_{\mathcal{H}}$ is related to the energy function of the system, i.e., the total energy of the system equals $E(t) = \frac{1}{2}\|x\|^2_{\mathcal{H}}$. The Sobolev space of order k is denoted by $H^k((a,b),\mathbb{R}^n)$.

Associated to the (homogeneous) pde we define the operator $Ax = P_1(\partial/\partial\zeta)(\mathcal{H}x) + P_0\mathcal{H}x$ with domain

$$D(A) = \left\{ \mathcal{H}x \in H^1((a,b);\mathbb{R}^n) \;\middle|\; \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \in \ker W_B \right\}$$

where $W_B = \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2} \end{smallmatrix}\right]$. For the rest of the paper we make the following hypothesis

Hypothesis 1. For the operator A and the pde (1)–(4) the following hold:

(1) The matrix W_B is an $n \times 2n$ matrix of full rank;
(2) For $x_0 \in D(A)$ we have $\langle Ax_0, x_0\rangle_{\mathcal{H}} \leq 0$.
(3) The number of inputs and outputs are the same, k, and for classical solutions of (1)–(4) there holds $\dot{E}(t) = u(t)^\top y(t)$.

We remark that from hypothesis (1) and (2) it follows that the system (1)–(4) is a boundary control system (see Le Gorrec et al. (2005); Jacob and Zwart (2012); Jacob et al. (2015)), and so for $u \in C^2([0,\infty);\mathbb{R}^k)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, satisfying (2) and (3) (for $t = 0$), there exists a unique classical solution to (1)–(4). Thus for these dense sets of initial conditions and inputs hypothesis (3) makes sense.

Since, we regard the above port-Hamiltonian system to describe a mechanical system in which u represents (generalized) boundary velocities, and y are the (generalized) boundary forces, the controller is regarded as a generalized mass-damper system. The associated momenta and velocities are denoted by p and v, respectively, and they are related via the mass matrix M, i.e., $p = Mv$. Using Newtons second law, we find

$$\dot{p} = f_{pH} + f_d, \tag{5}$$

where f_{ph} is the force felt from the port-Hamiltonian system, and f_d is the reactive damping force. Based on the interconnecting as discussed above we have the following interconnection relation between the two systems

$$f_{pH} = -y, \qquad v = u. \tag{6}$$

The state space for the closed loop system equals the direct sum of the separate state spaces, i.e. $X_{\text{ext}} = X \oplus \mathbb{R}^k$. The norm is given by

$$\left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|^2 = \|x\|^2_{\mathcal{H}} + v^\top M v. \tag{7}$$

Hence we have that the norm equals twice the total energy.

The closed loop system now becomes

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} P_1 \dfrac{\partial}{\partial\zeta}(\mathcal{H}x) + P_0\mathcal{H}x \\ -M^{-1}W_C \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}f_d \end{bmatrix}. \tag{8}$$

Furthermore, (3) holds together with

$$v(t) = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{bmatrix}. \tag{9}$$

We see that we can write the above as the abstract system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = A_{\text{ext}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f_d(t) \tag{10}$$

with A_{ext} given by the corresponding expression in (8) with domain

$$D(A_{\text{ext}}) = \left\{ \mathcal{H}x \in H^1((a,b);\mathbb{R}^n), v \in \mathbb{R}^k \;\middle|\; v = W_{B,1} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix},\; 0 = W_{B,2} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \right\}.$$

By using similar arguments as in Ramirez et al. (2014) it can be shown that A_{ext} with its domain generates a contraction semigroup on X_{ext}. Moreover, since H^1 is compactly embedded into L_2, we have that A_{ext} has a compact resolvent.

The following energy balance equation will be useful in the next section. Along classical solutions of (8) there holds

$$\begin{aligned} \dot{E}_{\text{tot}}(t) &= \dot{E}(t) + v(t)^\top M\dot{v}(t) \\ &= u(t)^\top y(t) + v(t)^\top M(-M^{-1}y(t) + M^{-1}f_d(t)) \\ &= v(t)^\top f_d(t). \end{aligned} \tag{11}$$

From this equality we see two things. Firstly, when we want to damp the system the damping force needs to be opposite to the velocity. Secondly, when we associate to the system (10) the output operator

$$C_{\text{ext}} = [0 \;\; 1],$$

then $\dot{E}_{\text{tot}}(t)$ is again output times input and $C^*_{\text{ext}} = B_{\text{ext}} := \left[\begin{smallmatrix} 0 \\ M^{-1} \end{smallmatrix}\right]$. Note that the adjoint is calculated with respect to the inner product of X_{ext}.

3. ASYMPTOTIC STABILITY

As we have seen in the previous section, if we want that the energy decays, then we have to inject damping into the system. For the (generalized) damping force we assume the following.

Hypothesis 2. The damping force is a function of the velocity only, i.e., $f_d = -F(v)$. It is opposite the velocity, i.e.,

$$v^\top F(v) \geq 0, \qquad v \in \mathbb{R}^k.$$

Furthermore, the F is a locally Lipschitz continuous function, and there exist positive constants δ, α, γ such that $v^\top F(v) \geq \alpha\|v\|^2$ when $\|v\| < \delta$ and $v^\top F(v) \geq \gamma$ when $\|v\| \geq \delta$.

We shall show that when this damping force is applied the closed loop system is asymptotically stable, provided the system (1)–(4) is approximately observable. For the proof of this result, the following theorem from Oostveen (Oostveen, 2000, Chapter 2) is extremely useful.

Theorem 3. Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable on infinite time. Then

(a) for all $\kappa > 0$, the operator $A - \kappa BB^*$ generates a strongly stable semigroup, $T_{-\kappa BB^*}(t)$;

(b) the closed-loop system $\Sigma(A-\kappa BB^*, B, B^*, 0)$ is input stable, i.e., for $u \in L_2((0,\infty); U)$

$$\| \int_0^\infty T_{-\kappa BB^*}(s)Bu(s)ds\|^2 \leq \tfrac{1}{2}\|u\|^2_{L_2((0,\infty);U)}.$$

(c) for all $u \in L_2((0,\infty); U)$ we have

$$\int_0^t T_{-\kappa BB^*}(t-s)Bu(s)ds \to 0 \text{ as } t \to \infty.$$

In the following corollary we show that the results remain valid when $\Sigma(A, B, B^*, 0)$ is approximately observable on infinite time.

Corollary 4. Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and the state linear system $\Sigma(A, B, B^*, 0)$ is approximately observable on infinite time, then the three items as formulated in Theorem 3 hold.

Proof. If $\Sigma(A, B, B^*, 0)$ is approximately observable on infinite time, then $\Sigma(A^*, B, B^*, 0)$ is approximately controllable on infinite time. Since A^* has also a compact resolvent and is the infinitesimal generator of a contraction semigroup, we have by the above theorem that the operator $A^* - \kappa BB^*$ generates a strongly stable semigroup. This implies that its dual generates a weakly stable semigroup. However, since the resolvent of $A - \kappa BB^*$ is compact, this semigroup is strongly stable as well. Now the other two assertions follow as in (Oostveen, 2000, Chapter 2).

Our main result is presented next.

Theorem 5. Consider the system (9) satisfying Hypothesis 1, with the non-linear feedback $f_d = -F(v)$ with F satisfying Hypothesis 2. This closed-loop system is globally asymptotically stable if and only if the system (1)–(4) is approximately observable.

For the proof of this result we need a couple of lemmas. The first lemma gives that the closed loop system possesses a unique global solution for all initial conditions.

Lemma 6. The system (9) satisfying Hypothesis 1 with the non-linear feedback $f_d = -F(v)$ with F satisfying Hypothesis 2 possesses for every initial condition a unique mild solution. Furthermore,

$$E_{\text{tot}}(t) = E_{\text{tot}}(0) - \int_0^t v(\tau)^\top F(v(\tau))d\tau. \qquad (12)$$

Proof. Since F is is a Lipschitz continuous function on $\mathbb{R}^k$, and since B_{ext} and C_{ext} are bounded linear mappings, it follows from e.g. (Pazy, 1983, Theorem 6.1.5)) that for every initial condition, the closed loop equation possesses a unique mild solution on some time interval $[0, t_{\max})$. By (11), we have that for classical solutions

$$\dot{E}_{\text{tot}}(t) = v(t)f_d(t) = -v(t)^\top F(v(t)).$$

Thus

$$E_{\text{tot}}(t) - E_{\text{tot}}(0) = -\int_0^t v(\tau)^\top F(v(\tau))d\tau. \qquad (13)$$

Since classical solutions form a dense set, we see that the above equality holds for all initial conditions. So (12) is shown. Since $2E_{\text{tot}}(t)$ equals the norm, we conclude from (13) that the norm of the state is uniformly bounded by the norm of the initial state. Now (Pazy, 1983, Theorem 6.1.4) gives that $t_{\max} = \infty$, and so we have global existence.

The second lemma concerns observability. Recall that a system is approximately observable on infinite time, when for the system with zero input the following holds; if the output is identically zero on $[0, \infty)$, then so is the initial state.

Lemma 7. The system (8) with output (9) is approximately observable on infinite time if and only if the system (1)–(3) with output (4) is approximately observable on infinite time.

Proof. *if*: Assume that the output (9) is identically zero. By definition this gives that $v \equiv 0$, and thus by (8) we find that $W_C \left[\begin{smallmatrix} (\mathcal{H}x)(t,b) \\ (\mathcal{H}x)(t,a) \end{smallmatrix}\right] \equiv 0$ ($f_d \equiv 0$ by assumption). So we have that

$$0 = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix} \text{ and } 0 = W_C\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix}.$$

By the approximate observability on infinite time of the system (1)–(3) with output (4), this implies that $x(0) = 0$. We already had that $v(0) = 0$, and thus the system (8) with output (9) is approximately observable on infinite time.

only if: Assume that the system (1)–(3) has its output (4) identically equal to zero. Choosing now as initial condition for (8) the same x and $v(0) = 0$, it is not hard to see that

$$x_{\text{ext}}(t) = \begin{bmatrix} x(t) \\ 0 \end{bmatrix}$$

is a solution of (8). Furthermore, the corresponding output is identically zero. By the approximate observability of (8), (9) we see that $x(0) = 0$, and thus the system (1)–(3) with output (4) is approximately observable on infinite time.

Proof of Theorem 5

Let us first assume that the system (1)–(3) with output (4) is approximately observable on infinite time. Then by Lemma 7 the same holds for the system (8) with output (9).

Since $E_{\text{tot}}(t)$ is always positive, we conclude from (12) and Hypothesis 2 that

$$\int_0^\infty v(t)^\top F(v(t))dt < \infty. \qquad (14)$$

Let $\Omega_1 := \{t \in [0,\infty) : \|v(t)\| > \delta\}$ and $\Omega_2 := \{t \in [0,\infty) \mid \|v(t)\| \leq \delta\}$. So by the assumptions of F, see Hypothesis 2, we obtain

$$\int_{\Omega_1} v(t)^\top F(v(t))dt \geq \gamma\mu(\Omega_1),$$

and (14) implies that Ω_1 has finite measure. Moreover,

$$\infty > \int_{\Omega_2} v(t)^\top F(v(t))dt \geq \alpha \int_{\Omega_2} \|v(t)\|^2 dt.$$

Thus

$$\int_0^\infty \|v(t)\|^2 dt = \left(\int_{\Omega_1} + \int_{\Omega_2}\right) \|v(t)\|^2 dt < \infty.$$

Since $C_{\text{ext}} = B_{\text{ext}}^*$, and since $v = B_{\text{ext}}^* \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = B_{\text{ext}}^* x_{\text{ext}}$, we can reformulate the closed-loop system as

$$\begin{aligned} \dot{x}_{\text{ext}}(t) &= (A_{\text{ext}} - B_{\text{ext}} B_{\text{ext}}^*) x_{\text{ext}}(t) + \\ &\qquad [B_{\text{ext}} B_{\text{ext}}^* x_{\text{ext}}(t) - B_{\text{ext}} F(B_{\text{ext}}^* x_{\text{ext}}(t))], \\ x_{\text{ext}}(0) &= \left[\begin{smallmatrix} x_0 \\ v_0 \end{smallmatrix}\right]. \end{aligned}$$

So the closed-loop solution is also given by

$$\begin{aligned} x_{\text{ext}}(t) &= T_{-BB^*}(t) x_{\text{ext}}(0) + \\ &\qquad \int_0^t T_{-BB^*}(t-s) B_{\text{ext}} [B_{\text{ext}}^* x_{\text{ext}}(s) \\ &\qquad\qquad - F(B_{\text{ext}}^* x_{\text{ext}}(s)]\, ds \\ &= T_{-BB^*}(t) x_{\text{ext}}(0) + \\ &\qquad \int_0^t T_{-BB^*}(t-s) B_{\text{ext}} [v(s) - F(v(s))]\, ds, \quad (15) \end{aligned}$$

where $T_{-BB^*}(t)$ is the semigroup generated by $A_{\text{ext}} - B_{\text{ext}} B_{\text{ext}}^*$. By Corollary 4 the semigroup $T_{-BB^*}(t)$ is strongly stable.

Since $v(t)$ is bounded (see (12)) and F is (locally) Lipschitz, we find that $F(v(t))$ is bounded. Combining this with the fact that the measure of Ω_1 is finite, we have

$$\int_{\Omega_1} \|F(v(s))\|^2 ds < \infty.$$

For $s \in \Omega_2$ we have $\|v(s)\| \leq \delta$ and so

$$\int_{\Omega_2} \|F(v(s))\|^2 ds \leq L(\delta)^2 \int_{\Omega_2} \|v(s)\|^2 ds < \infty,$$

where $L(\delta)$ is the Lipschitz contant for elements in the ball with radius δ. Using the expression (15) and Corollary 4 completes the proof.

Hence it remains to show that if the system (1)–(3) with output (4) is not approximately observable on infinite time, then the closed loop system is not asymptotically stable. If the system (1)–(4) is not approximately observable in infinite time, then there exists an initial condition $x(0)$ such that the solution, $x(t)$ of (1)–(3) with this initial conditions has output identically zero. Now it is not hard to see that $\left[\begin{smallmatrix} x(t) \\ 0 \end{smallmatrix}\right], t \geq 0$ is a solution of the closed loop system. It remains only to show that this solution does not converge to zero. By (12), we have that the energy stays constant, and thus the solution cannot converge to zero.

4. CONCLUSIONS

For a (generalized) mechanical, undamped, distributed parameter system we show that any damper will asymptotically stabilize it, provided the damper acts linearly for small velocities, and the distributed parameter system is approximately observable. Furthermore, we showed that asymptotic stability is impossible when this observability condition does not hold.

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