Three-Cornered Hat versus Allan Covariance

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***Abstract*—The Three-Cornered Hat is a widely used method to measure the oscillator stability (variance) by comparing three statistically independent units. The Grolambert (two-sample) Covariance (GCOV) is an alternate and equivalent approach, which has the advantage of rejecting the instrument PM noise. This method, left aside in the early time when the time-domain analysis was used only for slow phenomena, is now of renewed interest for recent oscillators (sapphire and photonic oscillators, and femtosecond combs) which exhibit the highest stability at short term.**

We revisit the Groslambert Covariance, we compare it to the Three-Cornered Hat, and we show analytically and experimentally its appealing properties.

Keywords*—*Phase noise, Stability analysis, Time-domain analysis, Covariance.

I. Introduction

The Three-Cornered Hat method addresses the problem of estimating the variance σ_A^2, σ_B^2 and σ_C^2 of three oscillators, based on three measures σ_{ab}^2, σ_{bc}^2 and σ_{ca}^2, each involving two clocks [1]. The solution is

$$\begin{cases} \sigma_A^2 = \frac{1}{2}\left(\sigma_{ca}^2 + \sigma_{ab}^2 - \sigma_{bc}^2\right) \\ \sigma_B^2 = \frac{1}{2}\left(\sigma_{ab}^2 + \sigma_{bc}^2 - \sigma_{ca}^2\right) \\ \sigma_C^2 = \frac{1}{2}\left(\sigma_{bc}^2 + \sigma_{ca}^2 - \sigma_{ab}^2\right) \end{cases}$$

Two obvious hypotheses are required, that the oscillators are statistically independent, and that the three measures overlap in time. This is a textbook case with classical variances, and also applies to Allan variance (AVAR), modified Allan variance (MVAR), parabolic variance (PVAR), and other two-sample variances. For this reason, the Three-Cornered Hat is the standard method for comparing oscillators, when a reference (more stable) oscillator is not available.

The weakness of the Three-Cornered Hat is that the phase noise of the instruments (chiefly white and flicker PM) biases σ_{ab}^2, σ_{bc}^2 and σ_{ca}^2 by a positive amount. Negative values of σ_A^2, σ_B^2 and σ_C^2 are sometimes seen for this reason. Until recently this weakness had small relevance because the AVAR is generally used for timekeeping and for long-term applications ($\tau \gtrsim \ldots 1000$ s), where the instrument phase noise is negligible. In fact, in the presence of white and flicker of phase, AVAR rolls off proportionally to $1/\tau^2$. By contrast, the two-sample variances are progressively being used for the short-term stability of photonic oscillators and femtosecond combs, with τ sometimes even below 1 ms.

A different approach was proposed by Groslambert & al. [2], [3] in 1981, based on the two-sample covariance. In this way, the instrument PM noise is not squared in the calculation of σ_A^2, σ_B^2 and σ_C^2, and therefore the measure is not biased. Unfortunately, this method received little attention in the 1980's, and was soon forgotten. The reason is, again, that the two-sample variances were used for long-τ analysis, while spectral methods were preferred for fast noise phenomena.

We revisit the two-sample Covariance, proving analytically its noise properties and providing experimental evidence of its advantageous features, compared with the Three-Cornered Hat.

Whereas this method could have been called the Allan Covariance and was used by Groslambert as the Cross-Variance by analogy to the Cross-Spectrum, we recommand to denominate it Groslambert Covariance (GCOV) in tribute to Jacques Groslambert (1938–2012).

II. Statement of the problem and notations

A. Measurement principle

We consider the measurement process described in figure 1. We assume that the noises coming from the measuring device (Time Interval Counter, phasemeter, ...) are uncorrelated. As well, we assume that the clocks are uncorrelated.

B. Notations

In the following, we will denote:

- x_{Ik} the time error (phase) of clock I at time t_k ($I \in \{A, B, C\}$)

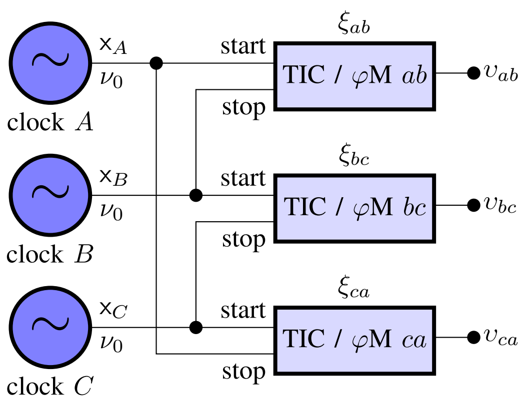

Fig. 1. Measurement principle. TIC stands for Time Interval Counter and φM for phasemeter.

- y_{Ik} the frequency deviation of clock I averaged over $[t_{k-1}, t_k]$:
$$\mathsf{y}_{Ik} = \frac{\mathsf{x}_{Ik} - \mathsf{x}_{Ik-1}}{\tau} \tag{1}$$
where τ is the sampling period
- z_{Ik} the time differential $\mathsf{z}_{Ik} = \mathsf{y}_{Ik} - \mathsf{y}_{Ik-1}$

Similarly, let's define:

- ξ_{ijk} the phase-time noise induced by the measuring device ij at time t_k $(i, j \in \{a, b, c\})$
- ψ_{ijk} the related fractional frequency noise averaged over τ, i. e. the time derivative of ξ_{ijk}, $\psi_{ijk} = \frac{\xi_{ijk} - \xi_{ijk-1}}{\tau}$ $(i, j \in \{a, b, c\})$
- ζ_{ijk} the time differential $\zeta_{ijk} = \psi_{ijk} - \psi_{ijk-1}$.

At time t_k, the output of the instrument ij connected to the clocks I and J delivers the phase-time measurement υ_{ijk}:

$$\upsilon_{ijk} = \mathsf{x}_{Jk} - \mathsf{x}_{Ik} + \xi_{ijk}. \tag{2}$$

from which we can derive, as we did for (1), the measured frequency deviation of the two clocks η_{ijk}:

$$\eta_{ijk} = \frac{\upsilon_{ijk} - \upsilon_{ijk-1}}{\tau}. \tag{3}$$

From (2) and (3), it comes:

$$\eta_{ijk} = \mathsf{y}_{Jk} - \mathsf{y}_{Ik} + \psi_{ijk}. \tag{4}$$

Finally, let us define γ_{ijk}, the time differential of η_{ijk}:

$$\gamma_{ijk} = \eta_{ijk} - \eta_{ijk-1} = \mathsf{z}_{Jk} - \mathsf{z}_{Ik} + \zeta_{ijk}. \tag{5}$$

C. Neglecting the measurement noises

If we provisionally drop the measurement noises $\{\xi, \psi, \zeta\}$, we may either verify the convergence of the Three-Cornered Hat method or the Groslambert Covariances.

Let us denote $\hat{\sigma}_I^2(\tau)$ the estimate of the Allan variance of clock I obtained by using the Three-Cornered Hat method. For example, for the Three-Cornered Hat method, the Allan variance of clock B may be estimated from:

$$\hat{\sigma}_B^2(\tau) = \frac{1}{2}\left[\sigma_{ab}^2(\tau) + \sigma_{bc}^2(\tau) - \sigma_{ca}^2(\tau)\right] \tag{6}$$

with

$$\sigma_{ab}^2(\tau) = \frac{1}{2}\mathbb{E}[\gamma_{ab}^2] = \frac{1}{2}\left\{\mathbb{E}[\mathsf{z}_B^2] + \mathbb{E}[\mathsf{z}_A^2] - 2\mathbb{E}[\mathsf{z}_B\mathsf{z}_A]\right\}.$$

Since the clocks are uncorrelated, it comes:

$$\sigma_{ab}^2(\tau) = \sigma_B^2(\tau) + \sigma_A^2(\tau).$$

Similarly, $\sigma_{bc}^2(\tau) = \sigma_C^2(\tau) + \sigma_B^2(\tau)$ and $\sigma_{ca}^2(\tau) = \sigma_A^2(\tau) + \sigma_C^2(\tau)$. From these relationships, we can easily verify that (6) is valid.

On the other hand, the Groslambert Covariance of clocks A and B is defined as:

$$\text{GCOV}_{A,B}(\tau) = \frac{1}{2}\mathbb{E}[\mathsf{z}_A\mathsf{z}_B]. \tag{7}$$

Let us denote $\tilde{\sigma}_I^2(\tau)$ the estimate of the Allan variance of clock I obtained by using the Groslambert Covariance method. The use of this covariance relies on:

$$\tilde{\sigma}_B^2(\tau) = -\text{GCOV}_{ab,bc}(\tau) = -\frac{1}{2}\mathbb{E}[\gamma_{ab}\gamma_{bc}] \tag{8}$$

with synchronous measurements of the counters ab and bc. It must be highlighted that this condition is essential for the Groslambert Covariances whereas the Three-Cornered Hat method can skip this condition under the assumption of stationarity.

Neglecting the ζ terms in (5), it comes:

$$\begin{aligned}\tilde{\sigma}_B^2(\tau) \;&=\; -\frac{1}{2}\left\{-\mathbb{E}\left[\mathsf{z}_B^2\right] + \mathbb{E}\left[\mathsf{z}_B\mathsf{z}_C\right]\right. \\ &\qquad \left.-\mathbb{E}[\mathsf{z}_A\mathsf{z}_C] + \mathbb{E}[\mathsf{z}_A\mathsf{z}_B]\right\}.\end{aligned} \tag{9}$$

Since the clocks are uncorrelated:

$$\tilde{\sigma}_B^2(\tau) = \frac{1}{2}\mathbb{E}\left[\mathsf{z}_B^2\right] = \sigma_B^2(\tau).$$

III. Influence of the Measurement Noises

A. The Three-Cornered Hat method

Let's resume (6) without neglecting the measurement noise:

$$\sigma_{ab}^2(\tau) = \frac{1}{2}\mathbb{E}[\gamma_{ab}^2] = \frac{1}{2}\mathbb{E}\left\{\left[\mathsf{z}_B - \mathsf{z}_A + \zeta_{ab}\right]^2\right\}$$

Since the clocks and the measurement noise are uncorrelated, only the quadratic terms have a mathematical expectation which is not 0:

$$\sigma_{ab}^2(\tau) = \frac{1}{2}\left\{\mathbb{E}\left[\mathsf{z}_B^2\right] + \mathbb{E}\left[\mathsf{z}_A^2\right] + \mathbb{E}\left[\zeta_{ab}^2\right]\right\}. \tag{10}$$

Similarly,

$$\sigma_{bc}^2(\tau) = \frac{1}{2}\left\{\mathbb{E}\left[\mathsf{z}_C^2\right] + \mathbb{E}\left[\mathsf{z}_B^2\right] + \mathbb{E}\left[\zeta_{bc}^2\right]\right\} \tag{11}$$

and

$$\sigma_{ca}^2(\tau) = \frac{1}{2}\left\{\mathbb{E}\left[\mathsf{z}_A^2\right] + \mathbb{E}\left[\mathsf{z}_C^2\right] + \mathbb{E}\left[\zeta_{ca}^2\right]\right\}. \tag{12}$$

From (6) it comes:

$$\hat{\sigma}_B^2(\tau) = \frac{1}{2}\mathbb{E}\left(\mathsf{z}_B^2\right) + \frac{1}{4}\left[\mathbb{E}\left(\zeta_{ab}^2\right) + \mathbb{E}\left(\zeta_{bc}^2\right) - \mathbb{E}\left(\zeta_{ca}^2\right)\right].$$

If we denote $\sigma_{\psi ij}^2(\tau)$ the Allan variance of the noise of the instrument ij, it comes:

$$\hat{\sigma}_B^2(\tau) = \sigma_B^2(\tau) + \frac{1}{2}\left[\sigma_{\psi ab}^2(\tau) + \sigma_{\psi bc}^2(\tau) - \sigma_{\psi ca}^2(\tau)\right]. \tag{13}$$

Therefore, this estimation of $\sigma_B^2(\tau)$ is affected by the noise of the measuring devices.

B. The Groslambert Covariance

Resuming (8) without neglecting the measurement noise leads to:

$$\tilde{\sigma}_B^2(\tau) = -\frac{1}{2}\mathbb{E}\left[(\mathsf{z}_B - \mathsf{z}_A + \zeta_{ab})(\mathsf{z}_C - \mathsf{z}_B + \zeta_{bc})\right] \quad (14)$$

Since the clocks and the measurement noises are uncorrelated, only the products whose factors are the same have a mathematical expectation which is not 0:

$$\tilde{\sigma}_B^2(\tau) = \frac{1}{2}\mathbb{E}\left[\mathsf{z}_B^2\right] = \sigma_B^2(\tau). \quad (15)$$

Therefore, the estimation of $\sigma_B^2(\tau)$ by using the Groslambert Covariance is not affected by the noise of the measuring devices.

C. Estimating the noise of each counter

Generalizing (13) and (15) to all clocks, it comes:

$$\begin{cases} 2\left[\hat{\sigma}_A^2(\tau) - \tilde{\sigma}_A^2(\tau)\right] = \sigma_{\psi ab}^2(\tau) - \sigma_{\psi bc}^2(\tau) + \sigma_{\psi ca}^2(\tau) \\ 2\left[\hat{\sigma}_B^2(\tau) - \tilde{\sigma}_B^2(\tau)\right] = \sigma_{\psi ab}^2(\tau) + \sigma_{\psi bc}^2(\tau) - \sigma_{\psi ca}^2(\tau) \\ 2\left[\hat{\sigma}_C^2(\tau) - \tilde{\sigma}_C^2(\tau)\right] = -\sigma_{\psi ab}^2(\tau) + \sigma_{\psi bc}^2(\tau) + \sigma_{\psi ca}^2(\tau). \end{cases}$$

Adding each pair of equations of this system leads to:

$$\begin{cases} \sigma_{\psi ab}^2(\tau) = \hat{\sigma}_A^2(\tau) - \tilde{\sigma}_A^2(\tau) + \hat{\sigma}_B^2(\tau) - \tilde{\sigma}_B^2(\tau) \\ \sigma_{\psi bc}^2(\tau) = \hat{\sigma}_B^2(\tau) - \tilde{\sigma}_B^2(\tau) + \hat{\sigma}_C^2(\tau) - \tilde{\sigma}_C^2(\tau) \\ \sigma_{\psi ca}^2(\tau) = \hat{\sigma}_C^2(\tau) - \tilde{\sigma}_C^2(\tau) + \hat{\sigma}_A^2(\tau) - \tilde{\sigma}_A^2(\tau) \end{cases} \quad (16)$$

Thanks to (16), it is then possible to estimate the Allan variance of the noise of each counter.

D. The closure

Adding the frequency deviations obtained from each counter at a given time t_k leads to a very useful closure relationship:

$$\begin{aligned} \eta_{abk} + \eta_{bck} + \eta_{cak} &= \mathsf{y}_{Bk} - \mathsf{y}_{Ak} + \psi_{abk} \\ &\quad + \mathsf{y}_{Ck} - \mathsf{y}_{Bk} + \psi_{bck} \\ &\quad + \mathsf{y}_{Ak} - \mathsf{y}_{Ck} + \psi_{cak} \\ &= \psi_{abk} + \psi_{bck} + \psi_{cak}. \end{aligned} \quad (17)$$

The Allan variance of the closure, i.e. the Allan variance of the sum of the frequency deviations measured by the counters, is given by:

$$\sigma_{cls}^2(\tau) = \sigma_{\psi ab}^2(\tau) + \sigma_{\psi bc}^2(\tau) + \sigma_{\psi ca}^2(\tau). \quad (18)$$

In addition to (16), the closure relationship (18) gives another way of estimating the sum of the Allan variances of the counter noises.

E. Uncertainty assessment

From these derivation, we can obtain a quick assessment of the variance estimation. A rigorous method providing confidence interval will be developed in a future paper.

1) Three cornered hat method: From (6), we see that the variance estimate $\hat{\sigma}_B^2(\tau)$ is the sum and difference of the three Allan variance estimates of the clock intercomparison. Each intercomparison Allan variance estimate is χ^2-distributed with a number of Equivalent Degrees of Freedom (EDF) which depends on the integration time τ. For the shortest integration times, the EDF number is high enough to approximate the χ^2 distribution to a Laplace-Gauss law. The sum and difference of 3 Gaussian estimate is then also a Gaussian estimate. Let us denote by Δab, Δbc and Δca the uncertainties of, respectively, $\hat{\sigma}_{ab}^2(\tau)$, $\hat{\sigma}_{bc}^2(\tau)$ and $\hat{\sigma}_{ca}^2(\tau)$. The uncertainty over $\hat{\sigma}_B^2(\tau)$ is then:

$$\Delta B = \sqrt{\Delta ab^2 + \Delta bc^2 + \Delta ca^2}. \quad (19)$$

In some cases, ΔB may be greater than $\hat{\sigma}_B^2(\tau)$ causing negative estimates.

2) Groslambert Covariance method: Replacing the mathematical expectations of (14) by finite averages yields:

$$\begin{aligned} \tilde{\sigma}_B^2(\tau) &= -\frac{1}{2}\left\langle [\mathsf{z}_B - \mathsf{z}_A + \zeta_{ab}] \times [\mathsf{z}_C - \mathsf{z}_B + \zeta_{bc}] \right\rangle \\ &= \frac{1}{2}\left\langle \mathsf{z}_B^2 \right\rangle - \frac{1}{2}\left\{ \langle \mathsf{z}_A\mathsf{z}_B \rangle + \langle \mathsf{z}_B\mathsf{z}_C \rangle - \langle \mathsf{z}_C\mathsf{z}_A \rangle \right\} \\ &\quad -\frac{1}{2}\left\{ -\langle \mathsf{z}_A\zeta_{bc} \rangle + \langle \mathsf{z}_B\zeta_{bc} \rangle - \langle \mathsf{z}_B\zeta_{ab} \rangle + \langle \mathsf{z}_C\zeta_{ab} \rangle \right\} \\ &\quad -\frac{1}{2}\langle \zeta_{ab}\zeta_{bc} \rangle. \end{aligned} \quad (20)$$

Only the first term is χ^2 distributed, the other ones are centered Bessel probability laws [4].

- Short integration times: In this case, each product is averaged over a large number of terms m and, thanks to the Central Limit Theorem, the average of centered Bessel random variables may be likened to a centered Gaussian with a standard deviation equal to the product of the standard deviation of the factors divided by the square root of m. For instance, $\langle \mathsf{z}_A\mathsf{z}_B \rangle = \frac{1}{2m}\sum_{i=1}^{m} \mathsf{z}_{Ai}\mathsf{z}_{Bi}$ may be assumed as a centered Gaussian random variable of standard deviation $\frac{\sigma_A(\tau)\sigma_B(\tau)}{\sqrt{m}}$.
  On the other hand, $\frac{1}{2}\left\langle \mathsf{z}_B^2 \right\rangle$ is a χ^2 random variable with m degrees of freedom. Since m is large for short integration times, the χ_m^2 may be likened as a Gaussian random variable of mean $\sigma_B^2(\tau)$ and of standard deviation $2\sigma_B^2(\tau)/\sqrt{m}$ [5].
  Thus, the uncertainty over $\tilde{\sigma}_B^2(\tau)$ may be assessed by:

$$\begin{aligned} \Delta B &= \frac{1}{\sqrt{m}}\left[4\sigma_B^4(\tau) + \sigma_A^2(\tau)\sigma_B^2(\tau) \right. \\ &\quad + \sigma_B^2(\tau)\sigma_C^2(\tau) + \sigma_C^2(\tau)\sigma_A^2(\tau) \\ &\quad + \sigma_A^2(\tau)\sigma_{\psi bc}^2(\tau) + \sigma_B^2(\tau)\sigma_{\psi bc}^2(\tau) \\ &\quad + \sigma_B^2(\tau)\sigma_{\psi ab}^2(\tau) + \sigma_C^2(\tau)\sigma_{\psi ab}^2(\tau) \\ &\quad \left. + \sigma_{\psi ab}^2(\tau)\sigma_{\psi bc}^2(\tau)\right]^{1/2}. \end{aligned} \quad (21)$$

- Large integration times: In this case, the counter noise is negligible and the Allan variance estimate obtained by the Groslambert Covariance is very close to the one obtained by the Three-Cornered Hat method. Therefore, the uncertainty assessment described in § III-E1 may be used.

F. *Efficiency of the Groslambert Covariance method*

From (21), we can estimate the efficiency of the Groslambert Covariance method. If we assume that the last term is prevailing and that $\sigma^2_{\psi ab}(\tau) \approx \sigma^2_{\psi bc}(\tau) = \sigma^2_{\psi}(\tau)$, the uncertainty may be simplified as $\Delta B = \sigma^2_{\psi}(\tau)/\sqrt{m}$. The limit of efficiency of the method is then reached if ΔB is of the order of magnitude of $\sigma^2_B(\tau)$. This leads to the following inequality:

$$\sigma^2_{\psi}(\tau) < \sqrt{m}\sigma^2_B(\tau). \qquad (22)$$

For instance, if the number of frequency deviation samples is $N = 10000$, the number of terms in the Allan variance computation for $\tau = \tau_0$ is $m = N - 1$ and it is possible to estimate $\sigma^2_B(\tau)$ even if it is 100 times lower than the counter noise Allan variance.

IV. Application to Real Measurements

We used this method for measuring ultra-stable signal sources, as the ones of three Cryogenic Sapphire Oscillators (CSO) at Femto-ST and the ones generated by an Optical Frequency Comb (OFC) at INRIM.

A. *Application to stability measurement of Cryogenic Sapphire Oscillators*

We used this method for measuring the stability of 3 Cryogenic Sapphire Oscillators (CSO). Fig. 2 shows the Allan Deviation plot of one of the 3 CSO. The advantage of using the Groslambert Covariance is clearly visible on the first measurements (for $\tau < 10$ s). Obviously, this method is very useful when the measurement noise is not negligible regarding the oscillator noise, i.e. at short term and very short term: the more the sampling rate increases, the more the gain of the Groslambert Covariance is conspicuous.

However, for long term, neither method is able to measure the Allan variance of CSO 1 because it is significantly lower than those of the other CSO, causing negative AVAR estimates, i.e. imaginary ADEV estimates (see § III-E).

Using (16) and (18), we notice that the Allan variances of the counter noise of the different channels are oddly scattered:

$$\begin{cases} \sigma^2_{\psi CSO1,CSO2}(\tau) = 0.71\sigma^2_{cls}(\tau) \\ \sigma^2_{\psi CSO2,CSO3}(\tau) = 0.22\sigma^2_{cls}(\tau) \\ \sigma^2_{\psi CSO3,CSO1}(\tau) = 0.07\sigma^2_{cls}(\tau). \end{cases}$$

This discrepancy of the channel noise levels explains that the three cornered hat method failed in computing the Allan variance of CSO3 for short integrations times (see Figure 2 below):

$$\begin{aligned} \hat{\sigma}^2_{CSO3}(\tau) &= \sigma^2_B(\tau) + \frac{1}{2}(-0.71 + 0.22 + 0.07)\sigma^2_{cls}(\tau) \\ &= \sigma^2_B(\tau) - 0.21\sigma^2_{cls}(\tau). \end{aligned}$$

For $\tau = 0.4$ s, $\tilde{\sigma}^2_{CSO3}(\tau) = 1.3 \cdot 10^{-30}$ (i.e. $\tilde{\sigma}_{CSO3}(\tau) = 1.1 \cdot 10^{-15}$), $\sigma^2_{cls}(\tau) = 6.1 \cdot 10^{-29}$ (i. e. $\sigma^2_{cls}(\tau) = 7.8 \cdot 10^{-15}$) and then $\hat{\sigma}^2_{CSO3}(\tau) = -1.2 \cdot 10^{-29}$ (i. e. $\hat{\sigma}_{CSO3}(\tau) = 3.4 \cdot 10^{-15} i$). Thus, $\hat{\sigma}^2_{CSO3}(\tau)$ is negative and $\hat{\sigma}_{CSO3}(\tau)$ is imaginary.

Of course, the estimate $\tilde{\sigma}^2_{CSO3}(\tau)$, obtained by using the Groslambert Covariance method, is not affected by this effect.

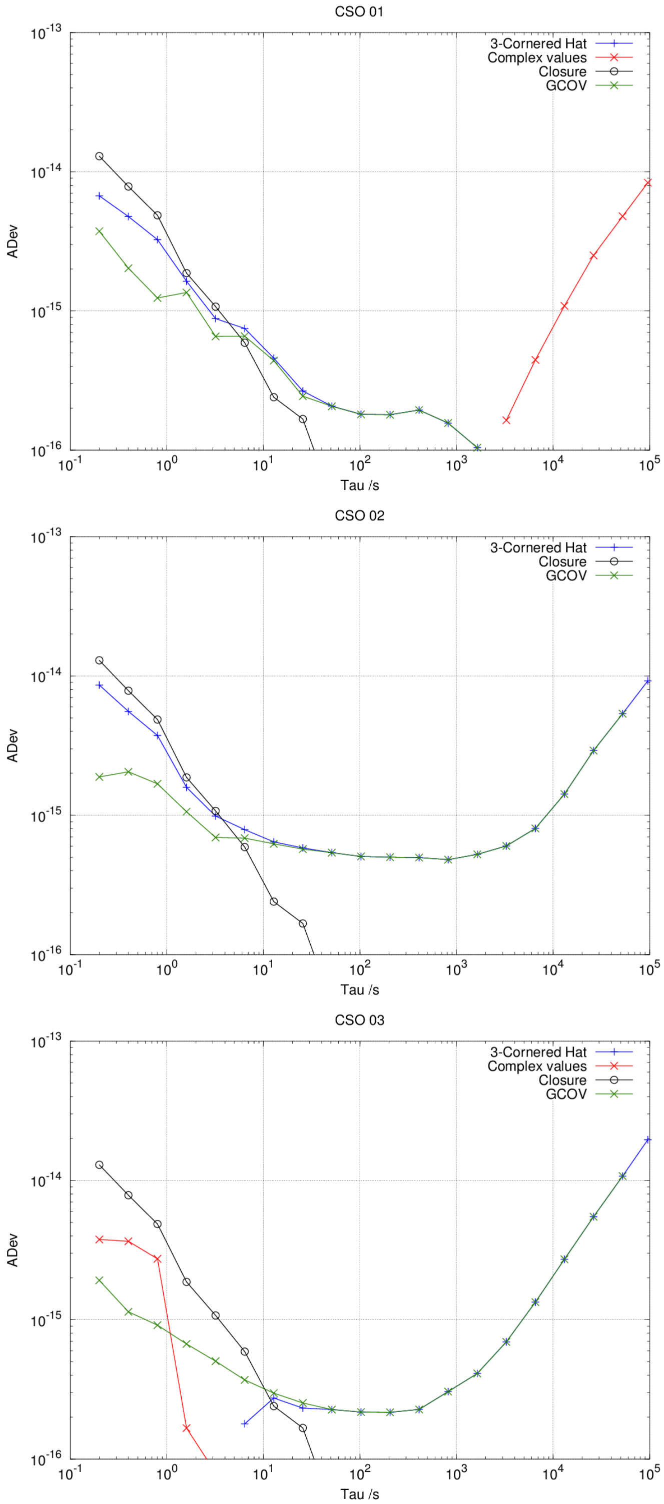

Fig. 2. Allan Deviation plot of the 3 CSO obtained with the classical Three-Cornered Hat (blue) and with the Groslambert Covariances (green). The curves captioned as "Complex values" shows negative Allan variance estimates and then note measurable variances. For CSO3, the short term stability is only measurable with the Groslambert Covariance because the noise levels of the 3 counters are completely different. For CSO1, its long term instabilities are significantly lower than those of the other CSO and are not measurable by any of the 2 methods.

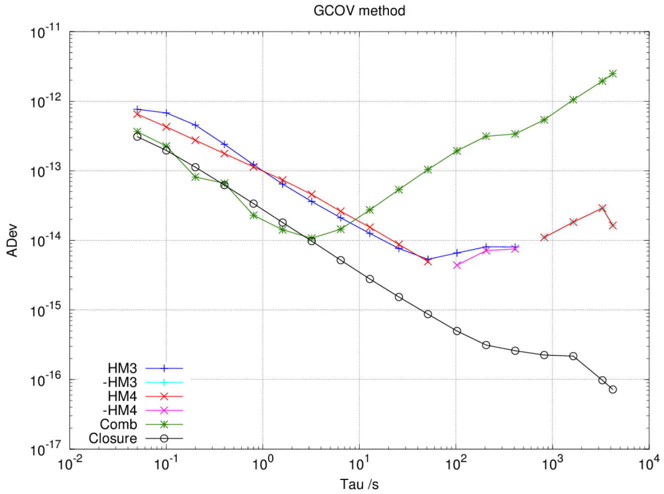

Fig. 3. Allan deviation plot of the comb, the reference Hydrogen Masers and the closure obtained with the Groslambert Covariance method. The curves denoted "-HM3" (cyan) and "-HM4" (magenta) show negative variance values (i.e. imaginary deviation values) resulting from the inability of the method to detect the H-Maser instabilities, completely negligible compared to those of the OFC.

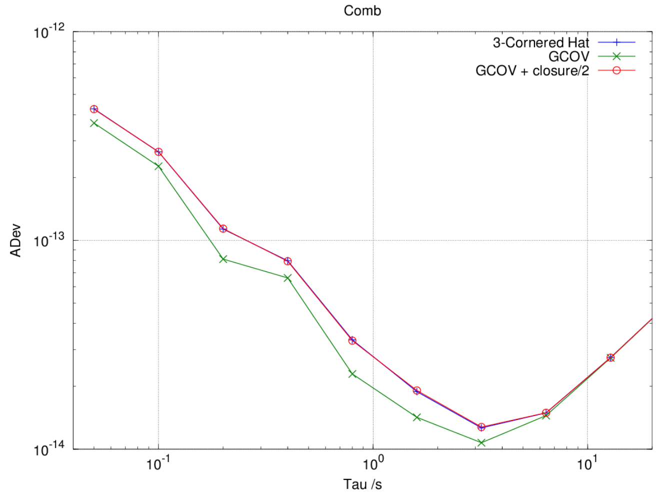

Fig. 4. Enlargement of the Allan deviation plot of the comb for the shortest time integrations obtained with the classical Three-Cornered Hat (blue) and with the Groslambert Covariances (green). The difference between the estimates of the Allan variance of the comb obtained by these two methods is the Allan variance of the closure divided by 2.

B. Application to stability measurement of an Optical Frequency Comb

We characterized the 100 MHz signal generated by a MENLO OFC referred to a 1542 nm ultra-stable laser with respect to four active hydrogen masers [6]. The instrument we used is a 6-channel synchronous phasemeter that implements the tracking DDS technique. The single-channel residual phase noise with respect to the internal reference is $2.1 \cdot 10^{-14}$@1 s ($f_h = 10$ Hz, flicker phase noise).

The signal at the output of the comb was split to feed two channels, because the noise of the comb was expected to be of the same order of the single channel one and we intended to take full advantage of the 2-sample covariance. The masers, instead, were connected without splitting, being their noise higher. Here, for brevity, we consider only the two best masers we have: CH1-75A from Kvarz [7] here referred as HM3 and iMaser 3000 from T4Science [8], indicated as HM4.

The results of the Groslambert Covariance method are presented on Figure 3. For large integration times, the Allan variance estimates, whatever the method (i.e. 3-cornered hat as well as GCOV) of the H-Masers becomes negative because their instabilities are far below the ones of the comb. Therefore, the large uncertainties over the AVAR estimates can lead to negative values when (6) is applied (see § III-E).

Moreover, Figure 3 shows that, for short integration times, the closure Allan variance is of the order of magnitude of the OFC AVAR and significantly lower than the ones of the Hydrogen Masers. Here also, applying (16) shows that the measurement noises are quite different from one channel to the other:

$$\begin{cases} \sigma^2_{\psi HM4,OFC}(\tau) = 0.485\sigma^2_{cls}(\tau) \\ \sigma^2_{\psi OFC,HM3}(\tau) = 0.515\sigma^2_{cls}(\tau) \\ \sigma^2_{\psi HM3,HM4} \approx 0. \end{cases}$$

This is due to the measurement scheme which is different from the one described in this paper: with a 5-channel scheme, it is not possible to discriminate the instrument noise from the oscillator one (if only one channel per oscillator is used) and so HM4-HM3 appears to be measured with a noiseless instrument.

As a consequence, the Allan variance of the OCF obtained by using the Three-Cornered Hat method is equal to the Allan variance of the OCF obtained by using the GCOV method plus half of the Allan variance of the closure (see Figure 4):

$$\begin{aligned} \hat{\sigma}^2_{OCF}(\tau) &= \tilde{\sigma}^2_{OCF}(\tau) + \frac{1}{2}\left[\sigma^2_{\psi HM4,OFC}(\tau)\right. \\ &\quad \left. + \sigma^2_{\psi OFC,HM3}(\tau) - \sigma^2_{\psi HM3,HM4}\right] \\ &= \tilde{\sigma}^2_{OCF}(\tau) + \frac{1}{2}\left[0.485 + 0.515 - 0\right]\sigma^2_{cls}. \end{aligned}$$

Thus, the estimation using GCOV is better because the contribution of the instrument noise averages to zero.

V. Conclusion

The Groslambert Covariance presents the advantage of rejecting the measurement noise and then optimizes short term stability measurements. This method is not more difficult to implement than the Three-Cornered Hat but must respect the simultaneous measurements of all channels of the measuring instrument. Whenever this is possible, it is then highly recommended to use the Groslambert Covariances.

Acknowledgement

This work is partially funded by the ANR "Programme d'Investissement d'Avenir" (PIA) under the Oscillator-IMP project and First-TF network, and by grants from the Région Franche Comté intended to support the PIA.

We are grateful to V. Giordano (PI), B. Dubois and C. Fluhr, FEMTO-ST Institute, for providing the CSO data; and to the Optics Division of INRIM for providing the comb measurements.

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