Two-letter words and a fundamental homomorphism ruling geometric contextuality

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Avant goût 1: the Bell-Kochen-Specker theorem

- The **Bell-Kochen-Specker proof** demonstrates the impossibility of Einstein's assumption, made in the famous Einstein-Podolsky-Rosen paper [1], that quantum mechanical observables represent 'elements of physical reality'. More specifically, the theorem **excludes hidden variable theories** that require elements of physical reality to be **non-contextual** (i.e. independent of the measurement arrangement) [2]
- **Essence of the BKS proof**: even for compatible observables A and B with values $v(A)$ and $v(B)$, the equations $v(aA+bB)=av(A)+bv(B)$ $(a,b\in\mathbb{R})$ or $v(AB)=v(A)v(B)$ may be violated.

[1] A. Einstein, B. Podolsky and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?" Phys. Rev. 47, 777?80 (1935).

[2] Kochen S. and Specker E. P. 1967 The problem of hidden variables in quantum mechanics *J. Math. Mech.* **17** 59–87.

Avant goût 2: the Peres-Mermin square: an operator parity proof of 2QB BKS theorem [3]

We denote by X, Z and Y the **Pauli spin matrices** in x, y and z directions, and the tensor product is not explicit, i. e. for two qubits one denotes $Z_1 = Z \otimes I$, $Z_2 = I \otimes Z$ and $ZZ = Z \otimes Z$, for three qubits one denotes $Z_1 = Z \otimes I \otimes I$ and so on.

$$
\begin{array}{ccc}
| & | & || \\
-Z_1- & Z_2- & ZZ-_{I} \\
| & | & || \\
-X_2- & X_1- & XX-_{I} \\
| & | & || \\
-ZX- & XZ- & YY-_{I} \\
|_{I} & |_{I} & ||_{-I}
\end{array}
$$

There is **no way** of assigning multiplicative properties to the eigenvalues ± 1 of the nine operators while still keeping the same multiplicative properties for the operators.

[3] Mermin N. D. 1993 Hidden variables and the two theorems of John Bell *Rev. Mod. Phys.* **65** 803-815.

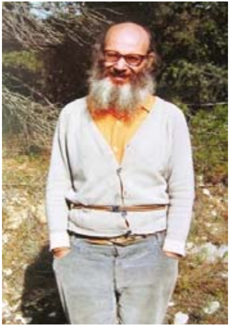
From Einstein-Bohr dialogue to Grothendieck's dessins d'enfants

There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.

Niels Bohr
1885-1962

God does not play dice with the universe.

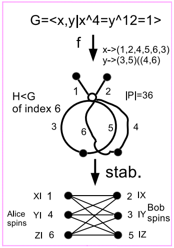
Albert Einstein

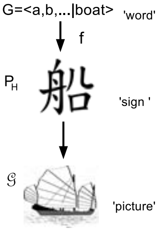


To sum up, I predict that the long-awaited renewal (if it is still coming…) will come from a born mathematician well-informed about the big questions of physics rather than from a physicist. But above all, we will need a man with the kind of 'philosophical openness' necessary to take hold of the heart of the problem. This problem is by no means a technical one, but is rather a fundamental question of 'natural philosophy'.

Harvests and Seeds.
Alexandre GROTHENDIECK 1928-2014

From the quotient group $G = \langle x, y | rels \rangle$ to the Pauli group $\langle X, Z | RELS \rangle$: Todd-Coxeter algorithm, 1936





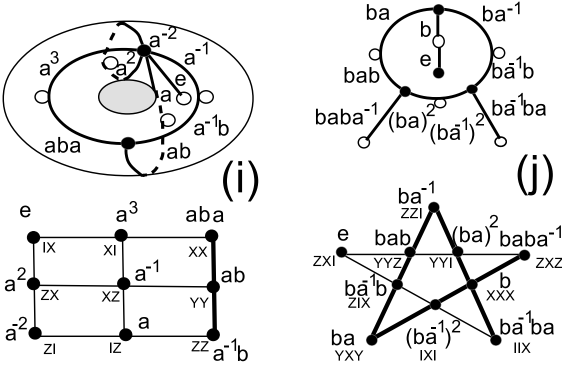
- $G = \langle x, y | rels \rangle$: the **free group on two letters modulo 'rels'**. The structure of cosets gH of a subgroup $H \subset_{index\ n} G$ is equivalent to the **homomorphism** $G \xrightarrow{f} P =< \alpha, \beta >$ induced by $x \to \alpha$ and and $y \to \beta$: permutations on $\{1..n\}$, also called a **dessin d'enfant** (or child's drawing) $\mathcal{D}$.
- Then one defines a map $\mathcal{D}_{edges} \xrightarrow{stab.} \mathcal{G}_{vertices}$ from the **edges of the dessin** $\mathcal{D}$ to the **vertices of a geometry** $\mathcal{G}$.

Axioms for geometric contextuality[5]

Axiom 1. About the stabilizer subgroups of P [4]: the rank r of P, alias the number of orbits of the one-point stabilizer subgroup of P **and** the number s of two-point stabilizer subgroups of P (up to isomorphism) should be both at least 3 (non-trivial geometries).
Axiom 2. About the normal closure N of the subgroup H in G (that is, the smallest normal subgroup of G containing H): one requires that $N = G$ (critical dessins).
Definition 1. First obey Axioms 1 and 2, then a candidate 'contextual' structure satisfies $f = f_0$ (that is, homomorphisms f and f_0 are the same) while $f \neq f_0$ means non-contextuality. In addition, the elected 'contextual geometry' $\mathcal{G}$ should have the same two-point stabilizer subgroup on a line (representative dessins).

[4]The stabilizer subgroup of G at point x is the set $G_x = \{g \in G | g.x = x\}$.
[5]Planat M. (2016) Two-letter words and a fundamental homomorphism ruling geometric contextuality, to appear in Symmetry, Culture and Science.

Mermin's square and pentagram

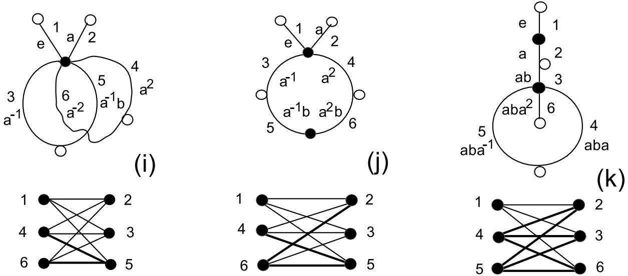


- (i) The contextual map/dessin (top) stabilizing the Mermin square (bottom) with the corresponding coset labelling.(j) The contextual map/dessin (top) stabilizing Mermin pentagram (bottom).

index	all	non-trivial	contextual	types
6	56	7	0	-
8	482	37	0	-
9	1551	53	12 (4)	Mermin sq., $K(3,3)$, Pappus
10	5916	351	2(1)	Mermin penta.
12	90033	3982	many	$K(4,4,4)$, $[12_6, 24_3]_4$

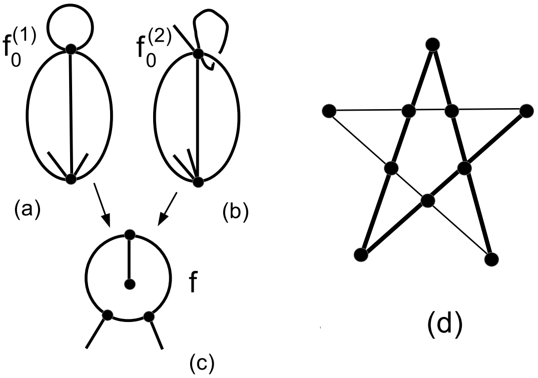
Table: In column 2, all is the **number of conjugacy classes** of subgroups of the corresponding index. In column 3, the **non-trivial** structures with r and $s > 2$ are given, as defined in Axiom 1. In column 3, one finds the number of **candidate** contextual structures (as defined in Definition 1). Inside the parentheses (**one elects**) such geometries $\mathcal{G}$ whose maximum cliques of the collinearity graph have their vertices corresponding to the same stabilizer subgroup of P.

When axiom 2 fails: non-critical dessins



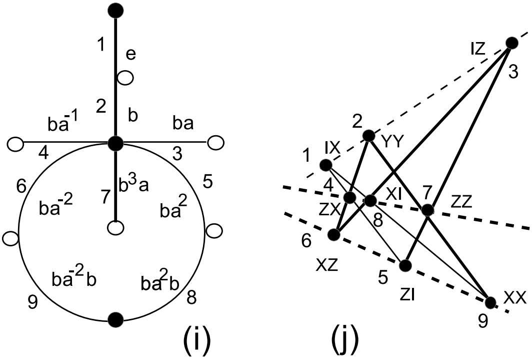
- Three dessins (i), (j) and (k) of index 6 stabilizing the bipartite graph $K(3,3)$. Since Axiom 2 of Sec. 1. is not satisfied, dessins (i)-(k) are considered to be non contextual.

When definition 1 fails: non-representative dessins



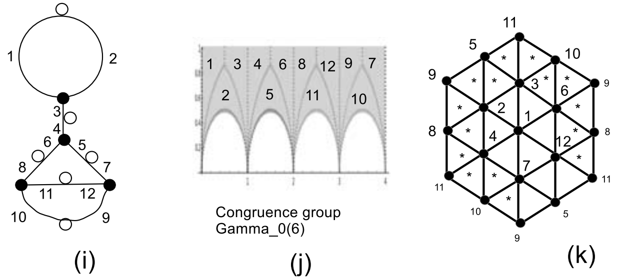
- Both dessins (a) and (b) have the same homomorphism $G \overset{f}{\ } \xrightarrow{mod} \overset{P}{\ } P$ pictured in (c). All stabilize Mermin's pentagram.

Index 10: Pappus configuration



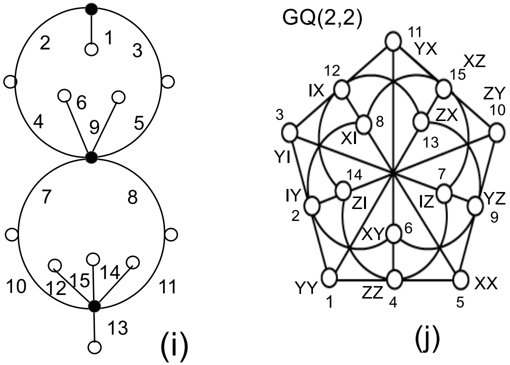
- The contextual map (i) stabilizing Pappus configuration in (j). The configuration embeds the Mermin configuration (plain lines): dotted lines do not belong to the Mermin square.

Modular dessin of index 12: The configuration $\mathcal{G} = [12_6, 24_3]_{(4)}$



- The contextual dessin d'enfant (i) with permutation group of order 72. It corresponds to the congruence subgroup $\Gamma_0(6)$ of modular group Γ as shown in (j). The stabilized configuration is in (k). Triangles with a star inside have non-commuting cosets.

Index 15: The $GQ(2,2)$ generalized quadrangle

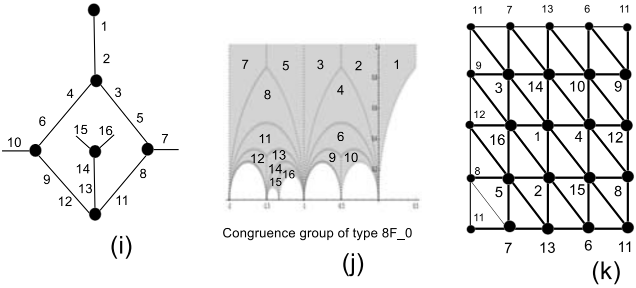


- The contextual dessin d'enfant (i), with $P \cong S_6$, stabilizing $GQ(2,2)$, a model of a two-qubit system. The list of cosets from 1 to 15 is $[e, a, a^{-1}, ab, a^{-1}b, aba, aba^{-1}, b^a, a^{-1}ba^{-1}, aba^{-1}b, a^{-1}bab, aba^{-1}ba, aba^{-1}ba^{-1}, a^{-1}baba^{-1}, aba^{-1}ba^2]$.

Let $\Gamma = PSL(2,\mathbb{Z}) \cong \langle x, y | y^2 = x^3 = e \rangle$. The permutation group $P = \langle \alpha, \beta \rangle$ arises from a subgroup Γ' of the modular group Γ.
How to pass from the topological structure of a **modular dessin** $\mathcal{D}$ to that of a **hyperbolic polygon** $\mathcal{P}$? There are ν_2 elliptic points of order two (resp. ν_3 elliptic points of order three) of $\mathcal{P}$, these points are white points (resp. black points) of valency one of $\mathcal{D}$.
The **genus** of $\mathcal{P}$ equals that of $\mathcal{D}$, a **cusp** of $\mathcal{P}$ follows from a **face** of $\mathcal{D}$, the number B of black (resp. the number W of white) points of $\mathcal{D}$ is given by the relation $B = f + \nu_2 - 1$ (resp. $W = n + 2 - 2g - B - c$), where f is the **number of fractions** and c the number of cusps in $\mathcal{P}$.
The set of cusps for Γ' consists of the Γ'-orbits of $\{\mathbb{Q}\} \cup \{\infty\}$. Γ' is of some type in Cummins-Pauli classification [6]. We use the software Sage to draw the fundamental domain of Γ' thanks to the Farey symbol methodology.

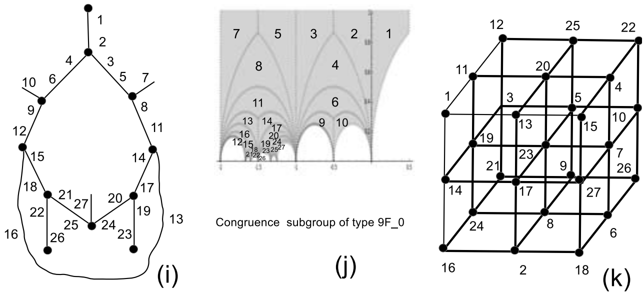
[6] C. Cummins and S. Pauli,
http://www.uncg.edu/mat/faculty/pauli/congruence/congruence.html.

A modular dessin of index 16: the Schrikhande graph $[16_6, 32_3]_{(5)}$



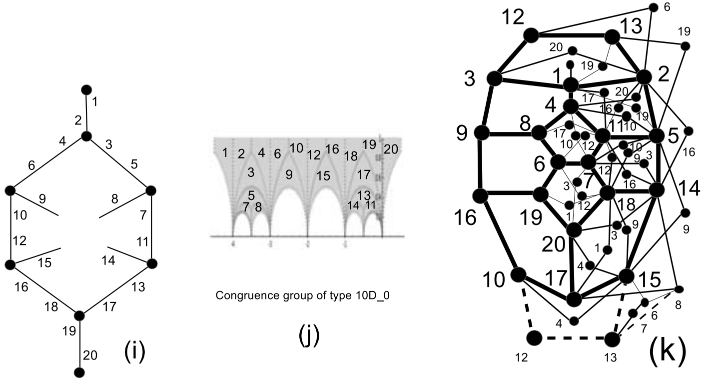
- The dessin (i) for the Shrikhande graph (k) and the corresponding modular polygon (j).

A modular dessin of index 27: the $3 \times 3 \times 3$-grid



- The dessin (i) and the corresponding modular polygon (j), of type $9F_0$, stabilizing the smallest slim dense near hexagon (alias the $3 \times 3 \times 3$ grid) $[27_3]_{(5)}$ shown in (k).

A modular dessin of index 20: a 6-regular triangle free graph with $P = \mathbb{Z}_2 \times A_5$



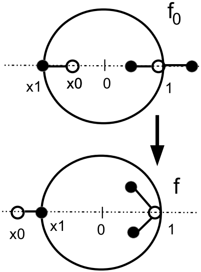
- The dessin (i) and the corresponding modular polygon (j) stabilizing a 6-regular graph on 20 vertices and 60 edges shown in (k).

Contextuality on finite simple groups [7]

group	index	configuration	names	κ
$S_4'(2)$	15	$[15_3]_{(3)}$	$GQ(2,2)$: 2QB	0.800
	30	$[30_{16}, 160_3]_{(7)}$		0.900
$S_4(3)$	27	$[27_5, 45_3]_{(3)}$	$GQ(2,4)$	0.867
	40	$[40_4]_{(3)}$	$GQ(3,3)$: 2QT	0.900
$S_6(2)$	63	$[63_{15}, 135_7]_{(3)}$	$W_5(2)$: 3 QB	*
	120	$[120_{28}, 1120_3]_{(3)}$		0.975
	126	$[126_{64}, 2688_3]_{(5)}$		0.972
	135	$[135_7, 315_3]_{(4)}$	DQ(6,2)	0.978
	315	$[315_3, 135_7]_{(5)}$		*
	336	$[336_{10}, 1120_3]_{(5)}$		0.991
$U_3(3)$	63	$[63_3]_{(5)}$	$GH(2,2)$; 3QB	0.952
	63	$[63_3]_{(4)}$	$GH(2,2)$ dual	0.936
$U_3(4)$	208	$[208_6, 416_3]_{(5)}$	config. over $\mathbb{F}_{16}$	0.971
$0_8^+(2):2$	120	$[120_{28}, 1120_3]_{(3)}$	$N0^+(8,2)$	0.952

[7] Planat, M. (2016) Zoology of Atlas-groups: dessins d'enfants, finite geometries and quantum commutation, Preprint 1601.04865 [quant-ph].

Two dessins in $\mathbb{Q}(\sqrt{6})$ [8]



- Both dessins have $P = S_5$, $\mathcal{G} = K_5$, $s = (B, W, F, g) = (3, 2, 2, 0)$. Belyi function: $f(x) = \frac{k(x-1)^4(x-x_0)}{x^3}$ with $x_0 = -10 \pm 4\sqrt{6}$ and $x_1 = -3 \pm \sqrt{6}$.
- Both curves are over $\bar{\mathbb{Q}} = \mathbb{Q}\sqrt{6}$ in the same class.
- $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ and contextuality?

[8] Galois actions on regular dessins of small genera, M. D.E. Conder, Rev. Mat. Iberoam. 29, 163-181 (2013).

- A striking comparison between the commutativity of quantum observables and that of cosets of subgroups of the two-generator free group. This parallel allowed us to propose **a new definition of contextuality** based on the coset structure of Grothendieck's **dessins d'enfants**, in good correspondance with the standard quantum one.
- Since a complex algebraic curve defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers [that is a Belyi function $f(x)$] is attached to any dessin d'enfant $\mathcal{D}$, it is expected that **the contextuality criterion features specific curves** through the action of the group of automorphisms Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ of the field $\bar{\mathbb{Q}}$ (the absolute Galois group) on dessins that would be helpful to recognize.