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On the adequation of dynamic modelling and control of parallel kinematic manipulators

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ABSTRACT

This paper addresses the problem of controlling the dynamics of parallel kinematic manipulators from a global point of view, where modeling, sensing and control are considered simultaneously. The methodology is presented through the examples of the Gough-Stewart manipulator and the Quattro robot.

Keywords: parallel kinematic manipulators, computed torque control, vision-based control, dynamic modeling

1 INTRODUCTION

Robot control has been widely addressed for decades, with a specific focus on serial kinematic manipulators and mobile robots. On the opposite, the control of parallel kinematic manipulators has not received that much interest, despite the kinematic duality they are in with respect to serial kinematic manipulators [12]. Indeed, most of the research concerning parallel kinematic manipulators was devoted to their kinematic analysis and design.

Recently, we have demonstrated [9] that the computed torque control of parallel kinematic manipulators was also stamped by duality: it can be formulated from the computed torque control of serial kinematic manipulators [6] by simply swapping the joint coordinates $\boldsymbol{q}$ and the Cartesian coordinates X, provided that the Cartesian pose and Cartesian velocities of the mobile platform (see Fig. 1). A way to satisfy the latter assumption is to use high-speed vision [3].

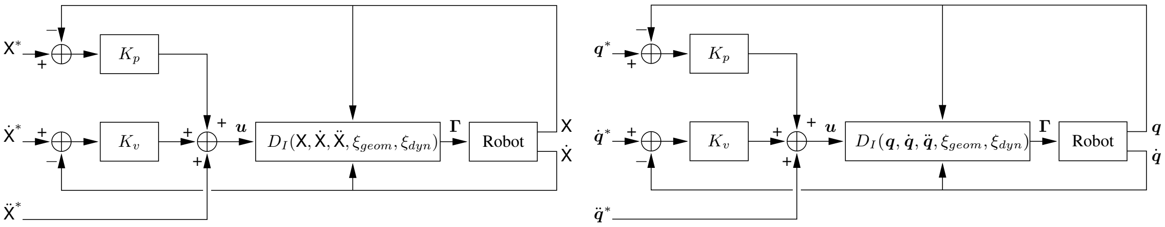

Figure 1. The computed torque control of parallel kinematic manipulators (left) is dual to the one of serial kinematic manipulators (right).

While investigating the use of vision, we exhibited [1] a theoretically elegant kinematic control scheme for parallel kinematic manipulators, based on a formulation of the kinematics in vector form rather than in any coordinate system. In this scheme, the legs of the manipulators are observed by vision, thus providing directly the controller with the inverse kinematic matrix. The latter is the core of a kinematic controller, since it converts desired Cartesian velocities into the joint velocities that are sent to the actuators.

This result asks the question of what is the most adequate sensing for the control of parallel kinematic manipulators. The purpose of the proposed paper is to extend the previously published answer in the case of kinematic control to the case of dynamic control.

To do so, Khalil's [7] and Kane's [5] formulations are merged together, while keeping the underlying expressions in vector form. Khalil's formulation has the advantage of expressing the dynamics of a parallel kinematic manipulator from the sum of all forces applied on the mobile platform. It is thus very intuitive as soon as one has computed the contribution of each leg to this sum. The latter contribution is computed by Khalil *et al.* by considering each leg as an independent serial kinematic chain and thus by applying the famous Luh-Walker-Paul algorithm based on the Newton-Euler formalism. However, this is suboptimal since one first solves the inverse kinematics of each leg then computes the dynamics of each body in the leg from the joint kinematics. Moreover, the computation of the dynamics of each leg looses the intuitiveness of the balance of all the efforts on the mobile platform.

A more straightforward approach to the computation of the dynamics of each body is Kane's method, which calls up to generalized coordinates that must be representative of the state of the system. The dynamics of the system are then computed from the equilibrium between active forces and inertia forces. To do so, one needs to define generalized speeds and to compute the partial derivatives of the translational and rotational velocities of each body with respect to the generalized speeds. This method is rather intuitive for any complex system, but its implementation is very tedious because of the manipulation of scalars.

We propose here to reformulate the dynamics of a leg along this methodology by using generalized vectors in place of the "traditional" generalized coordinates and by using generalized speeds in vector form rather than by using a basis of scalar generalized speeds. This brings a more geometric understanding of the manipulator dynamics and, hopefully, increases the intuitiveness of the latter. It also reduces the computational complexity provided that adequate sensing is used.

On the opposite to the case where joint sensing is considered as the only available perception of the robot state, this paper tries to show that opening to other sensing modalities (such as, but not exclusively to, vision) is fully coherent with this modeling approach and simplifies the control. The approach is derived in the case of the Gough-Stewart platform and browsed through in the case of the Adept Quattro robot and the whole approach, namely modeling, sensing and control, is discussed.

2 METHODOLOGY

This section adresses the question of which methodology should be used for deriving the kinematics and dynamics of a parallel kinematic manipulator for control purposes. There is thus a strong issue, in this question, on efficiency - as defined by Kane *et al.* -, namely "*relative simplicity, ease of manipulation for purposes of designing automatic control systems and minimal consumption of computation time during numerical solution*" [8].

2.1 A comparison of some methodologies

The underlying idea of Kane's method is that a careful choice of the coordinate basis (more precisely, a basis of generalized coordinates and generalized speeds) has a strong effect on the modeling efficiency. Once this choice, which is arbitrary is made, then the methodology he proposes is simply a matter of:

- deriving the kinematics of the mechanism in the coordinate basis ;
- making an inventory of all the forces in the mechanism ;
- projecting these forces on the directions of motion (*i.e.* the directions associated to the generalized speeds) ;
- formal calculus (*i.e.* automatic generation of the model equations).

However, it seems to us that it is still an open question to apply this methodology to parallel kinematic manipulators, namely for the choice of appropriate coordinates. Indeed, expressed in a control manner, this choice is a matter of defining the most appropriate dynamic state variables. It seems to us now certain that taking the actuator positions as the static state variables is not the optimal choice, at least because there is usually not a single solution to the forward kinematic problem. Consequently, taking the actuator positions and velocities as the dynamic state variable is certainly not the appropriate choice either.

Since the inverse kinematic problem is *usually* well posed, a more efficient choice is to take the end-effector Cartesian pose and velocity as dynamic state variables. This choice is made, for modeling purposes (not control ones!), by several authors [4, 11] which leads to efficient models, *as long as one is able to measure or estimate the pose and velocity of the end-effector in the Cartesian space*. And here is the loss of efficiency: one can not directly measure in the Cartesian space, so one has to estimate such variables, either through a mechanism or optical means (laser, vision). In all cases, except maybe for laser trackers but to which cost?, the estimation is a non linear problem.

Alternately to Kane, Khalil [6] also proposes a methodology for the formal calculus of the model by extending the systematic approach for the modeling of a serial mechanism. This approach has the strong advantage of being intuitive for the handling of the kinematic constraints. Indeed, the latter appears as a simple equilibrium of the forces and torques on the end-effector:

$$\boldsymbol{\Gamma} = \boldsymbol{D}_e^{inv\,(-T)} \left[\mathbb{F}_p + \sum_{i=1}^{m} \left(\boldsymbol{J}_{\mathrm{v}_i}^T \; \boldsymbol{J}_i^{invT} \; \boldsymbol{H}_i(\boldsymbol{\theta}_i, \dot{\boldsymbol{\theta}}_i, \ddot{\boldsymbol{\theta}}_i) \right) \right] \tag{1}$$

where $\boldsymbol{D}_e^{inv}$ is the inverse differential kinematic model, $\boldsymbol{J}_{\mathrm{v}_i}$ is the Jacobian of the terminal point of the ith leg with respect to the end-effector velocity, while $\boldsymbol{J}_i^{inv}$ and $\boldsymbol{H}_i$ are respectively the inverse differential kinematic and inverse dynamic models for the leg i (considered as a fully actuated serial mechanism moving independently from the kinematic constraints) and depend thus on all the (passive and active) joints $\boldsymbol{\theta}_i$ of the leg i and their derivatives. Finally, $\mathbb{F}_p$ contains the platform dynamics which is computed via Newton-Euler formulation.

However, the strong drawback of this method is its loss of efficiency by taking the prerequisite that sensing and actuation are colocated. Yet, as shown in [9], the method becomes extremely efficient when used together with end-effector sensing, namely because it is entirely linear.

2.2 A novel control-oriented methodology

The approach proposed in this paper is to couple Khalil's approach, for its intuitiveness (and efficiency) of equilibrating the forces on the end-effector, and Kane's one, for an efficient derivation of the legs dynamics. Yet, to the difference with Kane's method, where a minimal set of coordinates is chosen, we chose to keep a potentially redundant representation of the kinematics, namely by expressing the kinematics in vector form rather than in scalar form. Doing so, we turn around most of the trouble (and loss of efficiency) associated to the non linearities induced by the choice of a minimal representation and keep the whole problem linear and algebraic.

Moreover, the approach does not make any prior assumption on the kind and location of the sensors. The reasons for that is that such prior assumptions are not necessarily relevant to the global mechatronic design of the parallel kinematic manipulator, which must include control and sensing at early stages.

3 MODELING

The purpose of this section is not to provide the reader with a thorough and detailed model, for the reader is probably more fit to dynamic modeling than the authors. Rather, the purpose of the section is to outline the core equations of the model, in order to prepare for the next section, devoted to control. To do so, two exemplary mechanisms (the well-known Gough-Stewart mechanism and the Quattro robot) are dealt with.

3.1 Gough-Stewart mechanism

3.1.1 Kinematics

Geometry of the mechanism The Gough-Stewart mechanism is composed of a rigid nacelle (end-effector) supported by 6 legs. Each leg is a prismatic joint, connected to the base and the nacelle, by a universal joint

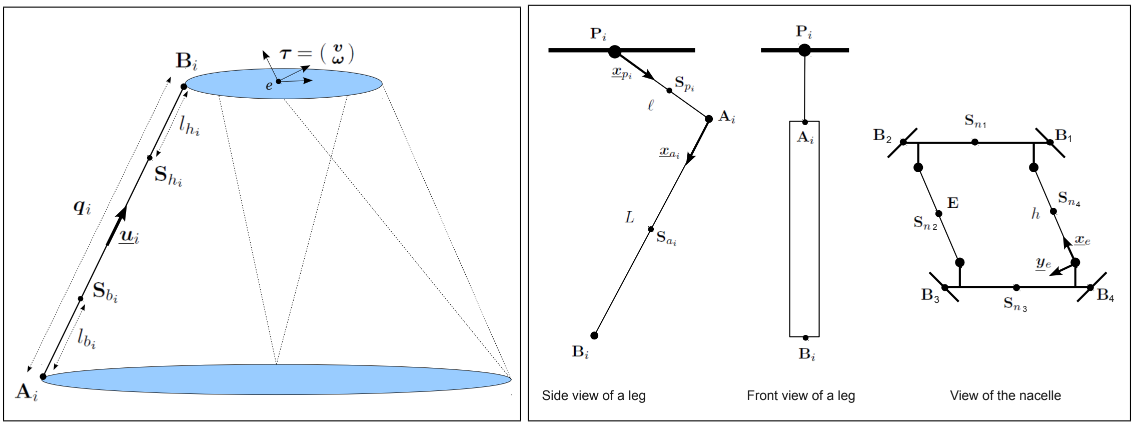

Figure 2. The two manipulators and their notation: Gough-Stewart (left) and Quattro (right)

on the one end and a spherical joint on the other end.

According to the notation in Fig. 2, the center of mass of the lower part of the ith leg i is located in:

$$\mathbf{S}_{b_i} = \mathbf{A}_i + (l_{b_i})\underline{\boldsymbol{u}}_i \tag{2}$$

while the center of mass of the upper part is located in:

$$\mathbf{S}_{h_i} = \mathbf{A}_i + (\boldsymbol{q}_i - l_{h_i})\underline{\boldsymbol{u}}_i \tag{3}$$

The center of mass of the nacelle is assumed to be located at the origin of the reference frame.

Velocities of the moving bodies The motion of the nacelle is described by its kinematic twist: $\boldsymbol{\tau} = \left(\begin{smallmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{smallmatrix}\right)$ and the velocities of the centers of mass of each leg are given by:

$$\dot{\mathbf{S}}_{b_i} = l_{b_i}\dot{\underline{\boldsymbol{u}}}_i \tag{4}$$

$$\dot{\mathbf{S}}_{h_i} = \dot{\boldsymbol{q}}_i\underline{\boldsymbol{u}}_i + (\boldsymbol{q}_i - l_{h_i})\dot{\underline{\boldsymbol{u}}}_i \tag{5}$$

Since $\underline{\boldsymbol{u}}_i$ is a unit vector, it is orthogonal to its derivative $\dot{\underline{\boldsymbol{u}}}_i$ and so, the above expression is a decomposition of the velocity of the upper center of mass on two orthogonal subspaces. To express it clearly, one rewrites the above equations as:

$$\dot{\mathbf{S}}_{b_i} = l_{b_i}(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i\underline{\boldsymbol{u}}_i^T)\dot{\underline{\boldsymbol{u}}}_i \tag{6}$$

$$\dot{\mathbf{S}}_{h_i} = \dot{\boldsymbol{q}}_i\underline{\boldsymbol{u}}_i + (\boldsymbol{q}_i - l_{h_i})(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i\underline{\boldsymbol{u}}_i^T)\dot{\underline{\boldsymbol{u}}}_i \tag{7}$$

The upper and lower part of the ith leg share the same rotation field, which is the rotation of the direction of the leg.

$$\boldsymbol{\omega}_{h_i} = \boldsymbol{\omega}_{b_i} = \boldsymbol{\omega}_i \tag{8}$$

The latter can easily be expressed as:

$$\boldsymbol{\omega}_i = \underline{\boldsymbol{u}}_i \times \dot{\underline{\boldsymbol{u}}}_i \tag{9}$$

Accelerations of the moving bodies In a straightforward manner, the acceleration of the centers of mass are given by:

- for the nacelle: $\dot{\boldsymbol{\tau}} = \left(\begin{smallmatrix}\dot{\boldsymbol{v}}\\ \dot{\boldsymbol{\omega}}\end{smallmatrix}\right)$
- for the legs:

$$\ddot{\mathbf{S}}_{b_i} = l_{b_i}\ddot{\underline{\boldsymbol{u}}}_i \tag{10}$$

$$\ddot{\mathbf{S}}_{h_i} = \ddot{\boldsymbol{q}}_i\underline{\boldsymbol{u}}_i + 2\dot{\boldsymbol{q}}_i\dot{\underline{\boldsymbol{u}}}_i + (\boldsymbol{q}_i - l_{h_i})\ddot{\underline{\boldsymbol{u}}}_i \tag{11}$$

$$\dot{\boldsymbol{\omega}}_i = \underline{\boldsymbol{u}}_i \times \ddot{\underline{\boldsymbol{u}}}_i \tag{12}$$

As can be seen up to now, there is no difficulty at all for expressing the motion of the mechanism with such a set of variables. Of course, those variables are not independent but are constrained by the closure of the mechanism but, for the moment, we do not eliminate the dependent variables and reduce the kinematics to a set of independent variables. However, even though the kinematic constraints will not be used before the dynamic stage, it is time to explicit them.

Kinematic constraints The implicit kinematic model (or zero-order kinematic constraints) of the Gough-Stewart mechanism writes:

$$\forall i = 1..6, \boldsymbol{q}_i \underline{\boldsymbol{u}}_i = \boldsymbol{R}\mathbf{B}_i + \boldsymbol{t} - \mathbf{A}_i \tag{13}$$

where $\mathbf{B}_i$ is expressed in a reference frame attached to the end-effector and $\mathbf{A}_i$ is expressed in a reference frame attached to the base. From the latter equation, it is easy to derive the implicit differential kinematic model:

$$\boldsymbol{q}_i \dot{\underline{\boldsymbol{u}}}_i + \dot{\boldsymbol{q}}_i \underline{\boldsymbol{u}}_i = \boldsymbol{J}_{\mathbf{B}_i}\boldsymbol{\tau} \quad \text{with} \quad \boldsymbol{J}_{\mathbf{B}_i} = \begin{pmatrix}\boldsymbol{I}_3 & -[\boldsymbol{R}\mathbf{B}_i]_\times\end{pmatrix} \tag{14}$$

which expresses the first-order kinematic constraint, relating the admissible velocities in the mechanism. Matrix $\boldsymbol{J}_{\mathbf{B}_i}$ is actually the same as matrix $\boldsymbol{J}_{\mathbf{v}_i}$ in (1) and the left-hand side of the above equation defines matrix $\boldsymbol{J}_i^{inv}$:

$$\boldsymbol{J}_i^{inv} = \begin{pmatrix} \underline{\boldsymbol{u}}_i^T \\ \frac{1}{\boldsymbol{q}_i}(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i \underline{\boldsymbol{u}}_i^T) \end{pmatrix} \tag{15}$$

because $\underline{\boldsymbol{u}}_i$ and $\dot{\underline{\boldsymbol{u}}}_i$ are orthogonal. Remark again the simple closed form this matrix has once $\underline{\boldsymbol{u}}_i$ is known. The above equation yields the usual inverse differential kinematic model of each leg i:

$$\dot{\boldsymbol{q}}_i = \underline{\boldsymbol{u}}_i^T \boldsymbol{J}_{\mathbf{B}_i}\boldsymbol{\tau} \tag{16}$$

$$\dot{\underline{\boldsymbol{u}}}_i = \frac{1}{q_i}(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i \underline{\boldsymbol{u}}_i^T)\boldsymbol{J}_{\mathbf{B}_i}\boldsymbol{\tau} \tag{17}$$

that will be needed by Khalil's method to convert the inverse dynamic model of a leg into a Cartesian force/torque vector applied on the platform.

3.1.2 Dynamics

Kane's method abstracted The key formulas in Kane's method are two. The first one states the balance of active forces and torques (generalized active forces, F_{u_r}) with the forces and torques due to the accelerated bodies (generalized inertia forces, $F^*_{u_r}$) along each degree of freedom of the mechanism:

$$F^*_{u_r} + F_{u_r} = 0, \quad r = 1, \ldots, n \tag{18}$$

where n is the number of degrees of freedom. The second one is the most interesting for it is the key to efficiency. It states that the generalized forces are obtained by projecting all the forces and torques acting on each body on the direction of so called partial velocities. These are defined as the partial derivatives of the center of mass velocities $(\mathbf{v}_k, \boldsymbol{\omega}_k)$ of each body k with respect to each generalized speed (u_r):

$$F^*_{u_r} = \sum_{k=1}^{p}\left(\left(\frac{\partial \mathbf{v}_k}{\partial u_r}\right)^T \mathbf{F}^*_k + \left(\frac{\partial \boldsymbol{\omega}_k}{\partial u_r}\right)^T \mathbf{T}^*_k\right), \quad r = 1, \ldots, n \tag{19}$$

$$F_{u_r} = \sum_{k=1}^{p}\left(\left(\frac{\partial \mathbf{v}_k}{\partial u_r}\right)^T \mathbf{F}_k + \left(\frac{\partial \boldsymbol{\omega}_k}{\partial u_r}\right)^T \mathbf{T}_k\right), \quad r = 1, \ldots, n \tag{20}$$

Thus, the key to efficiency lies in finding the generalized speeds that yield the simplest forms for the partial velocities on the one hand and for the forces acting on the bodies.

Dynamics of a leg Looking at the velocities of the bodies (6), (7), (9), a trivial choice for the generalized speeds for each leg is the set $\{\dot{\boldsymbol{q}}_i, \dot{\underline{\boldsymbol{u}}}_i\}$ which yields the following partial velocities:

$$\frac{\partial \dot{\mathbf{S}}_{b_i}}{\partial \dot{\boldsymbol{q}}_i} = \mathbf{0}_{3\times 1} \qquad \frac{\partial \dot{\mathbf{S}}_{h_i}}{\partial \dot{\boldsymbol{q}}_i} = \underline{\boldsymbol{u}}_i \qquad \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\boldsymbol{q}}_i} = \mathbf{0}_{3\times 1} \tag{21}$$

$$\frac{\partial \dot{\mathbf{S}}_{b_i}}{\partial \dot{\underline{\boldsymbol{u}}}_i} = l_{b_i}(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i \underline{\boldsymbol{u}}_i^T) \qquad \frac{\partial \dot{\mathbf{S}}_{h_i}}{\partial \dot{\underline{\boldsymbol{u}}}_i} = (\boldsymbol{q}_i - l_{h_i})(\boldsymbol{I}_3 - \underline{\boldsymbol{u}}_i \underline{\boldsymbol{u}}_i^T) \qquad \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\underline{\boldsymbol{u}}}_i} = [\underline{\boldsymbol{u}}_i]_\times \tag{22}$$

Now, let us have a look at the active and inertia forces and torques acting on each body of leg i:

$$\mathbf{F}_{b_i} = m_{b_i}\boldsymbol{g} \qquad \mathbf{F}^*_{b_i} = -m_{b_i}\ddot{\mathbf{S}}_{b_i} \qquad = -m_{b_i} l_{b_i} \ddot{\underline{\boldsymbol{u}}}_i \tag{23}$$

$$\mathbf{T}_{b_i} = \gamma_{\perp i} \qquad \mathbf{T}^*_{b_i} = -\bar{\boldsymbol{I}}_{b_i}\dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times \bar{\boldsymbol{I}}_{b_i}\boldsymbol{\omega}_i = -\bar{\boldsymbol{I}}_{b_i}\underline{\boldsymbol{u}}_i \times \ddot{\underline{\boldsymbol{u}}}_i - (\underline{\boldsymbol{u}}_i \times \dot{\underline{\boldsymbol{u}}}_i) \times \bar{\boldsymbol{I}}_{b_i}(\underline{\boldsymbol{u}}_i \times \dot{\underline{\boldsymbol{u}}}_i) \tag{24}$$

$$\mathbf{F}_{h_i} = m_{h_i}\boldsymbol{g} + \gamma_i \underline{\boldsymbol{u}}_i \qquad \mathbf{F}^*_{h_i} = -m_{h_i}\ddot{\mathbf{S}}_{h_i} \qquad = -m_{h_i}\left(\ddot{\boldsymbol{q}}_i \underline{\boldsymbol{u}}_i + 2\dot{\boldsymbol{q}}_i \dot{\underline{\boldsymbol{u}}}_i + (\boldsymbol{q}_i - l_{h_i})\ddot{\underline{\boldsymbol{u}}}_i\right) \tag{25}$$

$$\mathbf{T}_{h_i} = \mathbf{0}_{3\times 1} \qquad \mathbf{T}^*_{h_i} = -\bar{\boldsymbol{I}}_{h_i}\dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times \bar{\boldsymbol{I}}_{h_i}\boldsymbol{\omega}_i = -\bar{\boldsymbol{I}}_{h_i}\underline{\boldsymbol{u}}_i \times \ddot{\underline{\boldsymbol{u}}}_i - (\underline{\boldsymbol{u}}_i \times \dot{\underline{\boldsymbol{u}}}_i) \times \bar{\boldsymbol{I}}_{h_i}(\underline{\boldsymbol{u}}_i \times \dot{\underline{\boldsymbol{u}}}_i) \tag{26}$$

where one recognizes, on the right hand side, the forces and torques generated by the accelerated masses and inertias and, on the left hand side, the active forces and torques: the weight of the bodies as well as the force in the actuator $\gamma_i \underline{\boldsymbol{u}}_i$ and the vector of the torques in the passive joint at the base of the leg $\gamma_{\perp_i}$. Note that it is easy to include the friction and compliance effects to those terms.

Dynamics of the mechanism From this point, it is only a matter of algebraic manipulation to apply Kane's formulas (18)-(20) to obtain the inverse dynamic model of each leg and to apply Khalil's one (1) to get the inverse dynamic model of the whole mechanism. One interesting fact in the above equations is not only their simplicity but also their clear geometric interpretation, which is lost with other systematic methods. The compactness and algebraicity of the equations are also efficient, *provided that the variables appearing in the equations can efficiently be obtained*. This is the purpose of the discussion in Section 4.

Before starting this discussion, let us consider the Quattro robot and verify the potential versatility of the proposed approach.

3.2 Quattro robot

3.2.1 Kinematics

Geometry of the mechanism The Quattro robot is made of four legs, gathering on an articulated "parallelogrammic" nacelle. Each leg has a R̲Pa kinematic architecture: an actuated revolute joint, followed by a passive parallelogram.

According to the notation in Figure 2, one can write the position of the center of mass of the arm ($\mathbf{S}_{p_i}$) and forearm ($\mathbf{S}_{a_i}$):

$$\mathbf{S}_{p_i} = \mathbf{P}_i + \frac{\ell}{2}\underline{\boldsymbol{x}}_{p_i} \tag{27}$$

$$\mathbf{S}_{a_i} = \mathbf{P}_i + \ell \underline{\boldsymbol{x}}_{p_i} + \frac{L}{2}\underline{\boldsymbol{x}}_{a_i} \tag{28}$$

as well as the position of the center of mass of each body on the nacelle:

$$\mathbf{S}_{n_i} = \mathbf{P}_i + \ell \underline{\boldsymbol{x}}_{p_i} + L\underline{\boldsymbol{x}}_{a_i} + \mathbf{B}_i\mathbf{S}_{n_i} \tag{29}$$

where $\mathbf{B}_1\mathbf{S}_{n_1}$ and $\mathbf{B}_3\mathbf{S}_{n_3}$ are constant (since $\mathbf{S}_{n_1}$ and $\mathbf{S}_{n_3}$ are located on the parts of the nacelle that only translate) and $\mathbf{B}_2\mathbf{S}_{n_2}$ and $\mathbf{B}_4\mathbf{S}_{n_4}$ are the sum of a constant vector and a component along the rotating parts of the nacelle:

$$\mathbf{B}_i\mathbf{S}_{n_i} = const_i + \delta_i \frac{h}{2}\underline{\boldsymbol{x}}_e \tag{30}$$

where $\delta_1 = \delta_3 = 0, \delta_2 = -1, \delta_4 = 1$.

Velocities of the moving bodies Time differentiating the above equations yields the velocities of the mass centers:

$$\dot{\mathbf{S}}_{p_i} = \frac{\ell}{2}\dot{\underline{\boldsymbol{x}}}_{p_i} \qquad \dot{\mathbf{S}}_{a_i} = \ell\dot{\underline{\boldsymbol{x}}}_{p_i} + \frac{L}{2}\dot{\underline{\boldsymbol{x}}}_{a_i} \qquad \dot{\mathbf{S}}_{n_i} = \ell\dot{\underline{\boldsymbol{x}}}_{p_i} + L\dot{\underline{\boldsymbol{x}}}_{a_i} + \delta_i\frac{h}{2}\dot{\underline{\boldsymbol{x}}}_{e} \tag{31}$$

The rotation vectors of each body are also easy to obtain thanks to the use of unit vectors:

$$\boldsymbol{\omega}_{p_i} = \underline{\boldsymbol{x}}_{p_i} \times \dot{\underline{\boldsymbol{x}}}_{p_i} \qquad \boldsymbol{\omega}_{a_i} = \underline{\boldsymbol{x}}_{a_i} \times \dot{\underline{\boldsymbol{x}}}_{a_i} \qquad \boldsymbol{\omega}_{n_i} = \delta_i^2\,\underline{\boldsymbol{x}}_{e} \times \dot{\underline{\boldsymbol{x}}}_{e} \tag{32}$$

Accelerations of the moving bodies The acceleration of each body in the mechanism is thus given by

$$\ddot{\mathbf{S}}_{p_i} = \frac{\ell}{2}\ddot{\underline{\boldsymbol{x}}}_{p_i} \qquad \ddot{\mathbf{S}}_{a_i} = \ell\ddot{\underline{\boldsymbol{x}}}_{p_i} + \frac{L}{2}\ddot{\underline{\boldsymbol{x}}}_{a_i} \qquad \ddot{\mathbf{S}}_{n_i} = \ell\ddot{\underline{\boldsymbol{x}}}_{p_i} + L\ddot{\underline{\boldsymbol{x}}}_{a_i} + \delta_i\frac{h}{2}\ddot{\underline{\boldsymbol{x}}}_{e} \tag{33}$$

$$\dot{\boldsymbol{\omega}}_{p_i} = \underline{\boldsymbol{x}}_{p_i} \times \ddot{\underline{\boldsymbol{x}}}_{p_i} \qquad \dot{\boldsymbol{\omega}}_{a_i} = \underline{\boldsymbol{x}}_{a_i} \times \ddot{\underline{\boldsymbol{x}}}_{a_i} \qquad \dot{\boldsymbol{\omega}}_{n_i} = \delta_i^2\,\underline{\boldsymbol{x}}_{e} \times \ddot{\underline{\boldsymbol{x}}}_{e} \tag{34}$$

Kinematic constraints The closed-loop constraint is given by:

$$\mathbf{P}_i + \ell\underline{\boldsymbol{x}}_{p_i} + L\underline{\boldsymbol{x}}_{a_i} + \mathbf{B}_i\mathbf{E} = \boldsymbol{t}_e \tag{35}$$

where $\mathbf{B}_i\mathbf{E} = const_i + \epsilon_i\frac{h}{2}\underline{\boldsymbol{x}}_e$ and $\epsilon_1 = \epsilon_2 = -1, \epsilon_3 = \epsilon_4 = 1$. It yields the following first-order kinematic constraint:

$$\ell\dot{\underline{\boldsymbol{x}}}_{p_i} + L\dot{\underline{\boldsymbol{x}}}_{a_i} = \boldsymbol{v} - \epsilon_i\frac{h}{2}\dot{\underline{\boldsymbol{x}}}_{e} \tag{36}$$

Due to the parallelograms, the nacelle stays parallel to its initial plane and thus, the end-effector has a constant axis $\underline{\boldsymbol{z}}_e$ and $\underline{\boldsymbol{x}}_{p_i}$ rotates around the actuator axis $\underline{\boldsymbol{z}}_{p_i}$. Consequently, one has:

$$\dot{\underline{\boldsymbol{x}}}_{e} = \omega_z\underline{\boldsymbol{y}}_e \tag{37}$$

Noting $\boldsymbol{\tau}_r = \left(\begin{smallmatrix}\boldsymbol{v}\\ \omega_z\end{smallmatrix}\right)$, this gives the following matrices needed for Khalil's formula:

$$\boldsymbol{J}_{\mathrm{v}_i} = \begin{pmatrix}\boldsymbol{I}_3 & -\epsilon_i\frac{h}{2}\underline{\boldsymbol{y}}_e\end{pmatrix} \qquad \boldsymbol{J}_i^{inv} = \begin{pmatrix} \frac{1}{\ell}\frac{\underline{\boldsymbol{y}}_{p_i}\underline{\boldsymbol{x}}_{a_i}^T}{\underline{\boldsymbol{y}}_{p_i}^T\underline{\boldsymbol{x}}_{a_i}} \\ \frac{1}{L}\left(\boldsymbol{I}_3 - \frac{\underline{\boldsymbol{y}}_{p_i}\underline{\boldsymbol{x}}_{a_i}^T}{\underline{\boldsymbol{y}}_{p_i}^T\underline{\boldsymbol{x}}_{a_i}}\right)\end{pmatrix} \tag{38}$$

as well as the full inverse differential kinematic model:

$$\dot{\underline{\boldsymbol{x}}}_{p_i} = \frac{1}{\ell}\frac{\underline{\boldsymbol{y}}_{p_i}\underline{\boldsymbol{x}}_{a_i}^T}{\underline{\boldsymbol{y}}_{p_i}^T\underline{\boldsymbol{x}}_{a_i}}\boldsymbol{J}_{\mathrm{v}_i}\boldsymbol{\tau}_r \qquad \dot{\underline{\boldsymbol{x}}}_{a_i} = \frac{1}{L}\left(\boldsymbol{I}_3 - \frac{\underline{\boldsymbol{y}}_{p_i}\underline{\boldsymbol{x}}_{a_i}^T}{\underline{\boldsymbol{y}}_{p_i}^T\underline{\boldsymbol{x}}_{a_i}}\right)\boldsymbol{J}_{\mathrm{v}_i}\boldsymbol{\tau}_r \tag{39}$$

which implies:

$$\boldsymbol{D}_{e_i}^{inv} = \frac{1}{\ell}\frac{\underline{\boldsymbol{x}}_{a_i}^T}{\underline{\boldsymbol{y}}_{p_i}^T\underline{\boldsymbol{x}}_{a_i}}\boldsymbol{J}_{\mathrm{v}_i} \tag{40}$$

3.2.2 Dynamics

The above kinematic analysis suggests that the generalized speeds should be chosen as $\dot{\underline{\boldsymbol{x}}}_{p_i}$, $\dot{\underline{\boldsymbol{x}}}_{a_i}$ and $\dot{\underline{\boldsymbol{x}}}_{e}$. Indeed, this yields the following partial velocities:

$$\frac{\partial\dot{\mathbf{S}}_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = \frac{\ell}{2}\boldsymbol{I}_3 \quad \frac{\partial\dot{\mathbf{S}}_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = \ell\boldsymbol{I}_3 \quad \frac{\partial\dot{\mathbf{S}}_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = \ell\boldsymbol{I}_3 \quad \frac{\partial\omega_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = [\underline{\boldsymbol{x}}_{p_i}]_\times \quad \frac{\partial\omega_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = \mathbf{0}_{3\times3} \quad \frac{\partial\omega_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{p_i}} = \mathbf{0}_{3\times3} \tag{41}$$

$$\frac{\partial\dot{\mathbf{S}}_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = \mathbf{0}_{3\times3} \quad \frac{\partial\dot{\mathbf{S}}_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = \frac{L}{2}\boldsymbol{I}_3 \quad \frac{\partial\dot{\mathbf{S}}_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = L\boldsymbol{I}_3 \quad \frac{\partial\omega_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = \mathbf{0}_{3\times3} \quad \frac{\partial\omega_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = [\underline{\boldsymbol{x}}_{a_i}]_\times \quad \frac{\partial\omega_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{a_i}} = \mathbf{0}_{3\times3} \tag{42}$$

$$\frac{\partial\dot{\mathbf{S}}_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \mathbf{0}_{3\times3} \quad \frac{\partial\dot{\mathbf{S}}_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \mathbf{0}_{3\times3} \quad \frac{\partial\dot{\mathbf{S}}_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \delta_i\frac{h}{2}\boldsymbol{I}_3 \quad \frac{\partial\omega_{p_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \mathbf{0}_{3\times3} \quad \frac{\partial\omega_{a_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \mathbf{0}_{3\times3} \quad \frac{\partial\omega_{n_i}}{\partial\dot{\underline{\boldsymbol{x}}}_{e}} = \delta_i^2[\underline{\boldsymbol{x}}_{e}]_\times \tag{43}$$

and expressions for the forces and torques:

$$\begin{aligned}
\mathbf{G}_{p_i} &= -m_{p_i}\boldsymbol{g} & \mathbf{G}^*_{p_i} &= -m_{pi}\ddot{\mathbf{S}}_{p_i} & &= -\frac{m_{pi}\ell}{2}\dot{\underline{\boldsymbol{x}}}_{p_i} && (44)\\
\mathbf{T}_{p_i} &= \gamma_{p_i}\underline{\mathbf{z}}_{p_i} & \mathbf{T}^*_{p_i} &= -\bar{\boldsymbol{I}}_{p_i}\dot{\boldsymbol{\omega}}_{p_i} - \boldsymbol{\omega}_{p_i}\times\bar{\boldsymbol{I}}_{p_i}\boldsymbol{\omega}_{p_i} & &= -\bar{\boldsymbol{I}}_{p_i}(\underline{\boldsymbol{x}}_{p_i}\times\ddot{\underline{\boldsymbol{x}}}_{p_i}) - (\underline{\boldsymbol{x}}_{p_i}\times\dot{\underline{\boldsymbol{x}}}_{p_i})\times\bar{\boldsymbol{I}}_{p_i}(\underline{\boldsymbol{x}}_{p_i}\times\dot{\underline{\boldsymbol{x}}}_{p_i}) && (45)\\
\mathbf{G}_{a_i} &= -m_{a_i}\boldsymbol{g} & \mathbf{G}^*_{a_i} &= -m_{a_i}\ddot{\mathbf{S}}_{a_i} & &= -m_{a_i}\ell\ddot{\underline{\boldsymbol{x}}}_{p_i} + \frac{m_{a_i}L}{2}\ddot{\underline{\boldsymbol{x}}}_{a_i} && (46)\\
\mathbf{T}_{a_i} &= \boldsymbol{\gamma}_{a_i} & \mathbf{T}^*_{a_i} &= -\bar{\boldsymbol{I}}_{a_i}\dot{\boldsymbol{\omega}}_{a_i} - \boldsymbol{\omega}_{a_i}\times\bar{\boldsymbol{I}}_{a_i}\boldsymbol{\omega}_{a_i} & &= -\bar{\boldsymbol{I}}_{a_i}(\underline{\boldsymbol{x}}_{a_i}\times\ddot{\underline{\boldsymbol{x}}}_{a_i}) - (\underline{\boldsymbol{x}}_{a_i}\times\dot{\underline{\boldsymbol{x}}}_{a_i})\times\bar{\boldsymbol{I}}_{a_i}(\underline{\boldsymbol{x}}_{a_i}\times\dot{\underline{\boldsymbol{x}}}_{a_i}) && (47)\\
\mathbf{G}_{n_i} &= -m_{n_i}\boldsymbol{g} & \mathbf{G}^*_{n_i} &= -m_{n_i}\ddot{\mathbf{S}}_{n_i} & &= -m_{n_i}\ell\ddot{\underline{\boldsymbol{x}}}_{p_i} + m_{n_i}L\ddot{\underline{\boldsymbol{x}}}_{a_i} + \delta_i\frac{m_{n_i}h}{2}\ddot{\underline{\boldsymbol{x}}}_e && (48)\\
\mathbf{T}_{n_i} &= \boldsymbol{\gamma}_{n_i} & \mathbf{T}^*_{n_i} &= -\bar{\boldsymbol{I}}_{n_i}\dot{\boldsymbol{\omega}}_{n_i} - \boldsymbol{\omega}_{n_i}\times\bar{\boldsymbol{I}}_{n_i}\boldsymbol{\omega}_{n_i} & &= -\delta_i^2\bar{\boldsymbol{I}}_{n_i}\underline{\boldsymbol{x}}_e\times\ddot{\underline{\boldsymbol{x}}}_e - \delta_i^4(\underline{\boldsymbol{x}}_e\times\dot{\underline{\boldsymbol{x}}}_e)\times\bar{\boldsymbol{I}}_{n_i}(\underline{\boldsymbol{x}}_e\times\dot{\underline{\boldsymbol{x}}}_e) && (49)
\end{aligned}$$

where γ_{p_i} is the torque in the ith actuator, while $\boldsymbol{\gamma}_{a_i}$ and $\boldsymbol{\gamma}_{n_i}$ are the equivalent torque vectors in the passive joints.

This analysis of the Quattro robot confirms that taking into account the passive joints through a non-minimal vector representation rather than through a minimal coordinate basis allows for efficiency of the model, provided that all the variables appearing in this model can efficiently be obtained.

4 CONTROL

The purpose of this section is to discuss the efficient use of the above methodology at control time. This involves to discuss both the control law and the choice of the sensors.

4.1 Background

The standard method for dynamic control is the so-called computed-torque control [6] which linearizes and decouples the control. Its well-known form, adapted to serial mechanisms, is derived from the Lagrange formulation of the dynamic model (Fig. 1):

$$\boldsymbol{\Gamma} = \boldsymbol{A}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) \triangleq MDI(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) \tag{50}$$

Under the assumption that $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{h}}$ are correct estimates of $\boldsymbol{A}(\boldsymbol{q})$ and $\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}})$, one can build a control torque of the form:

$$\boldsymbol{\Gamma} = \hat{\boldsymbol{A}}\boldsymbol{u} + \hat{\boldsymbol{h}} \tag{51}$$

where $\boldsymbol{u}$ is an auxiliary control vector, equivalent to an acceleration. Indeed, inserting such a control in the direct dynamic model yields a closed-loop equation of the form:

$$\ddot{\boldsymbol{q}} = \underbrace{\boldsymbol{A}(\boldsymbol{q})^{-1}\hat{\boldsymbol{A}}}_{\approx\boldsymbol{I}}\boldsymbol{u} + \boldsymbol{A}(\boldsymbol{q})^{-1}\underbrace{\left(\hat{\boldsymbol{h}} - \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)}_{\approx\boldsymbol{0}_\times} \tag{52}$$

which is a second order system and can therefore be controlled by any linear controller, the latter being often chosen as a PD+feedforward controller in the joint space.

Now, for parallel kinematic manipulators, this control is hardly used because it is computationnally heavy (partly because one has to solve for the forward kinematic problem at each iteration). It is thereby often discarded to the profit of simplified controllers (the worse being a PID controller in the joint space) which can be shown [9] to have poor properties as far as linearization and decoupling are concerned.

Alternately to such a joint-space computed torque control, whose essential drawback is to rely on a dynamic model expressed in the joint space, one can set up a Cartesian-space computed torque control according to many recommendations for expressing the dynamic model in the Cartesian space [4, 2]:

$$\boldsymbol{\Gamma} = \widehat{\boldsymbol{A}(\mathsf{X})}\boldsymbol{u} + \widehat{\boldsymbol{h}(\mathsf{X},\dot{\mathsf{X}})} \tag{53}$$

where $\boldsymbol{u}$ is now equivalent to a Cartesian acceleration and (X,$\dot{\mathsf{X}}$) are the Cartesian pose and velocity of the end-effector to be estimated. This control is a state feedback in the case where the inverse kinematic problem has a single solution.

However, if the dynamic model is lighter, the estimation of X and $\dot{\mathsf{X}}$ is not easy and, consequently, one can set up a sensor-based computed torque control:

$$\mathbf{\Gamma} = \widehat{\boldsymbol{A}(\boldsymbol{s})}\boldsymbol{u} + \widehat{\boldsymbol{h}(\boldsymbol{s}, \dot{s})} \tag{54}$$

where $\boldsymbol{u}$ is now equivalent to the second-order time-derivative of the sensor signal $\boldsymbol{s}$.

This control encompasses the former two since a joint-space computed torque control can be seen as a sensor-based control where the sensor signal is given by the joint encoders and a Cartesian-space control as a sensor-based control where the sensor signal is the end-effector pose. In fact, the theoretical condition for the validity of a sensor-based control is that there exists a diffeomorphim (*i.e.* a differentiable bijective mapping) between the sensor space and the state space of the system [10]. And that is where it hurts in parallel kinematics, especially when one only considers the actuator positions for sensing: the mapping is neither bijective (several solutions to the forward kinematic problem) nor differentiable (singularities of any type). Then, which sensor signal shall be used ?

4.2 Choosing a sensor signal

The choice of the sensor is led by the state of the technology, but more relevantly, since the latter is ever improving, by the control algorithm and, in turn, the control algorithm depends on the model it is built upon. In correlation with the proposed methodology, this comes down to examining the variables in the model and to determining how to get them efficiently.

There are three kinds of variables involved in the above models:

- variables related to the actuators: $\{\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i\}$ for the Gough-Stewart mechanism and $\{\underline{\boldsymbol{x}}_{p_i}, \dot{\underline{\boldsymbol{x}}}_{p_i}\}$ for the Quattro
- variables related to the end-effector: respectively $\{\boldsymbol{R}, \boldsymbol{\tau}\}$ and $\{\underline{\boldsymbol{x}}_e, \dot{\underline{\boldsymbol{x}}}_e\}$
- redundant variables: respectively $\{\underline{\boldsymbol{u}}_i, \dot{\underline{\boldsymbol{u}}}_i\}$ and $\{\underline{\boldsymbol{x}}_{a_i}, \dot{\underline{\boldsymbol{x}}}_{a_i}\}$

Which kind, if any, is the best suited for efficient control ? Of course, if one can sense all of these variables, the problem is completely solved, up to itching calibration and data coherence issues. For the same reason, a combination of two of the three kinds is left out of the discussion here (although there might lie the practical optimum).

The variables related to the actuators are to be definitely discarded, since they face the forward kinematic problem, which is not only a non-linear but also a square problem. The variables related to the end-effector are not necessarily the answer due to either their technological cost (laser) or algorithmic cost (vision). The latter case is nevertheless better than the forward kinematic problem since the non-linear problem of pose estimation is not square but over-constrained, which make it numerically more robust, and since it relies on optics rather than mechanical parts.

Focusing on the use of redundant variables only, we have found out that they provide us with a linear formulation of the whole problem. Indeed, the proposed models only make use of linear algebra once all the variables are known *and* the non-redundant variables can be expressed linearly from the redundant ones. To do so, one has to exploit a little bit further the kinematic constraints (13) and (35). Indeed, considering pairs of legs (no bad joke intended), we can write:

$$\forall i, j = 1..6, i \neq j, \quad \boldsymbol{q}_i \underline{\boldsymbol{u}}_i - \boldsymbol{q}_j \underline{\boldsymbol{u}}_j = \boldsymbol{R}(\mathbf{B}_i - \mathbf{B}_j) - (\mathbf{A}_i - \mathbf{A}_j) \tag{55}$$

$$\forall i, j = 1..4, i \neq j, \quad (\mathbf{P}_i - \mathbf{P}_j) + \ell\underline{\boldsymbol{x}}_{p_i} - \ell\underline{\boldsymbol{x}}_{p_j} + L\underline{\boldsymbol{x}}_{a_i} - L\underline{\boldsymbol{x}}_{a_j} + (\mathbf{B}_i\mathbf{B}_j) = \mathbf{0}_{3\times 1} \tag{56}$$

These two sets of equations are linear in their unknowns, namely $\{\boldsymbol{q}_i, \boldsymbol{q}_j, \boldsymbol{R}\}$ for the Gough-Stewart mechanism and $\{\underline{\boldsymbol{x}}_{p_i}, \underline{\boldsymbol{x}}_{p_j}\}$ for the Quattro, because the other terms are either constant values or the redundant variables. Time differentiating the above two equations yields relationships between the variable derivatives that are naturally also linear.

Consequently, as soon as one can sense the redundant variables, one can derive a control using only linear algebra. And so, the only remaining question is how to measure those redundant variables. Our answer is, unsurprisingly, to observe by vision the associated mechanical elements in the legs, preferably as revolute cylinders, as we did it in kinematics.

5 CONCLUSION

The proposed methodology appears to be coherent from many viewpoints. It is rather intuitive to the understanding of the dynamic modeling. It seems rather efficient, in Kane's meaning. It merges dynamic modeling and dynamic control in a single framework.

Yet, there are a couple of issues to be dealt within the next future for a formal validation. First of all, we need to extend our work on high-speed vision to the observation of the manipulator legs (both by increasing the observation frequency to cope with the real-time constraints and by including an estimation of the legs velocity from the image). Then it will be the time for validating experimentally the approach. Finally, the proposed choice of variables is made upon kinematic considerations whereas Kane suggests to choose the generalized speeds upon dynamic ones. Is there an even better framework ? This paper is here to open the discussion.

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