Structure preserving spatial discretization of 2D hyperbolic systems using staggered grids finite difference

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***Abstract*— This paper proposes a finite difference spatial discretization scheme that preserve the port-Hamiltonian structure of 1D and 2D infinite dimensional hyperbolic systems. This scheme is based on the use of staggered grids for the discretization of the state and co state variables of the system. It is shown that, by an appropriate choice of the boundary port variables, the underlying geometric structure of the infinite-dimensional system, *i.e.* its Dirac structure, is preserved during the discretization step. The consistency of the spatial discretization scheme is evaluated and its accuracy is validated with numerical results.**

***Index Terms*— Distributed port-Hamiltonian systems, wave propagation, staggered grids, finite difference method.**

I. Introduction

Port-Hamiltonian systems (PHS) are particularly well suited for the modelling and control of (non-linear) multiphysical systems. They have first been defined in [1], and later used to describe the behavior of complex open physical systems through the study of their internal energy exchanges [2], [3]. The PHS framework has been extended to systems described by boundary controlled partial differential equations (PDEs) in [4], [5] and led to powerful results regarding the analysis [6] and control in infinite dimensions [7], [8], [9].

PHS express the fundamental internal interconnection structure of a system, such as Kirchoffs or Newtons laws, through its geometric structure, defined by a set of structure matrices in the finite dimensional case and differential operators in the infinite dimensional case [3].

In order to perform numerical simulations or implement control schemes for systems governed by PDEs, it is necessary to approximate them by finite-dimensional representations. In this context, preserving the geometric structure of the infinite dimensional system is relevant to preserve the physical properties of the model, such as the conservation of energy, the dissipation profiles and the physical meaning of the inputs and outputs (boundary variables). For PDEs which describe a subsystem of a multiphysical model ([10], [11]) it is even more relevant to preserve the physical properties of the interconnection variables.

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Different structure preserving discretization schemes have been proposed in recent years [12], [13], [14], [15]. In [13], [14] different (mixed) finite elements are used for the approximation of the infinite dimensional state and co-state variables in order to preserve the symplectic structure of the system. This approach has been applied for modeling, reduction and control in [16], [17], [18]. It has also been extended to pseudo spectral approximations by the use of high order polynomial approximations in [15].

A different direction is to incorporate other important numerical methods in this framework, as for instance the finite-difference method which presents (along with the finite volume method) numerous schemes with diverse properties and advantages (see [19] for a recent review). Particularly, schemes presenting staggered grids [20], [21], [22] permit to define different state variables on different grids and thus account for their different geometric nature. A version of finite volume discretization on staggered grids for PHS has recently been proposed in [23] for the 1D case.

This paper shows how the staggered grids finite difference can be used to discretize infinite-dimensional PHS on 1D and 2D spatial domains while preserving its intrinsic PH structure.It is shown that a centered finite difference can be advantageously used to derive a simple and efficient simulator for such system. Moreover, the use of staggered grids permits to directly impose boundary conditions over the effort variables (*e.g* speed and pressure in acoustics) which is not the case with traditional finite difference method (where the boundary conditions would be on pressure and normal acceleration in acoustics).

The paper is organized as follows. Section 2 motivates the use of staggered grids finite difference for the spatial discretization of the wave equation. Section 3 shows that this discretization, with an appropriate choice of discretized port variables, preserves the underlying Dirac structure for the 1D case and Section 4 extends this proof to the 2D case.

II. Staggered grids finite difference

We motivate the use of staggered grids by the 1D wave equation example

$$\frac{\partial^2 x(\xi,t)}{\partial t^2} = \frac{\partial^2 x(\xi,t)}{\partial \xi^2}, \quad \xi \in [0,L] \tag{1}$$

and show how the idea extends naturally to the PH representation. The equation can be (semi-)discretized using the (centered) finite difference approximation at a point ξ_k:

$$\left.\frac{\partial x(\xi,t)}{\partial \xi}\right|_k \approx \frac{x_{k+0.5}(t) - x_{k-0.5}(t)}{h} \tag{2}$$

h is the discretization step and $x_{k\pm 0.5}(t)$ denote the values of x at $\xi_{k\pm 0.5} := \xi_k \pm 0.5h$. To approximate the second order spatial derivative, this approximation is applied again at the half-grid points $\xi_{k\pm 0.5}$ which yields

$$\left.\frac{\partial^2 x(\xi,t)}{\partial t^2}\right|_k \approx \frac{x_{k+1}(t) - 2x_k(t) + x_{k-1}(t)}{h^2} \tag{3}$$

defined at every point ξ_k.

To construct finite difference schemes that preserve structural properties (first of all, conservativeness), it is more convenient to consider the first order representation of the wave equation, with the approximations of the first order spatial derivative on the shifted (half) grids as sketched before, see [20], [21]. This point of view fits naturally to the port-Hamiltonian formulation [5] of (1). This formulation is based on the use of the energy variables $x_1(\xi,t) = \frac{\partial x(\xi,t)}{\partial \xi}, x_2(\xi,t) = \frac{\partial x(\xi,t)}{\partial t}$, as state variables in order to rewrite (1) as a first order equation:

$$\underbrace{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}}_{f} = \underbrace{\begin{pmatrix}0 & \frac{\partial}{\partial \xi}\\ \frac{\partial}{\partial \xi} & 0\end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix}x_1\\ x_2\end{pmatrix}}_{e} \tag{4}$$

with $H(x_1,x_2) = \frac{1}{2}\int_0^L \left(x_1^2 + x_2^2\right) d\xi$, and the definition of the boundary port variables f_∂, e_∂ from the evaluation of the effort variables (co-state variables) $e = (e_1, e_2)^T$ at $\xi = 0$ and $\xi = L$:

$$\begin{pmatrix}f_\partial & e_\partial\end{pmatrix}^T = \begin{pmatrix}e_1(0) & -e_2(L) & e_2(0) & e_1(L)\end{pmatrix}^T \tag{5}$$

$$\text{such that : } \frac{dH}{dt} = f_\partial^T e_\partial \tag{6}$$

A geometrical interpretation of (4) and (5) is that the vector of flow variables f and the vector of effort variables e defined in (4) and their extensions to the boundary (5) lie in a Dirac structure $\mathcal{D}$ [18], *i.e.* $(f, f_\partial, e, e_\partial) \in \mathcal{D}$ defined as follows.

Definition 1: The Dirac structure $\mathcal{D}$ is a subspace of $F \times E$ (where $F = L^2([0,L],\mathbb{R}^2) \times \mathbb{R}^2$ is the flow space and $E = H^1([0,L],\mathbb{R}^2) \times \mathbb{R}^2$ the effort space[1]) such as $\mathcal{D} = \mathcal{D}^\perp$ with respect to a canonical product $\langle .|. \rangle$ defined such that :

$$\begin{aligned}\langle (f^1, f^1_\partial, e^1, e^1_\partial) | (f^2, f^2_\partial, e^2, e^2_\partial)\rangle =& \langle f^2, e^1\rangle_{L^2} + \langle f^1, e^2\rangle_{L^2}\\ &- \langle f^2_\partial, e^1_\partial\rangle - \langle f^1_\partial, e^2_\partial\rangle\end{aligned}$$

with $(f^2, f^2_\partial, e^2, e^2_\partial) \in F \times E$, $\langle ., . \rangle_{L^2}$ and $\langle ., . \rangle$ respectively the canonical products on $L^2([0,L],\mathbb{R}^2)$ and $\mathbb{R}^2$, and where:

$$\mathcal{D}^\perp = \{b \in F \times E | \langle b|b'\rangle = 0, \forall b' \in \mathcal{D}\}.$$

All possible parameterizations of the boundary port variables (5) can be found in [5]. Approximating, as in the introductory example,

$$\begin{aligned}\left.\frac{\partial e_2(\xi,t)}{\partial_\xi}\right|_k &\approx \frac{e_2(\xi_{k+0.5},t) - e_2(\xi_{k-0.5},t)}{h}\\ \left.\frac{\partial e_1(\xi,t)}{\partial_\xi}\right|_{k+0.5} &\approx \frac{e_1(\xi_{k+1},t) - e_1(\xi_k,t)}{h}\end{aligned} \tag{7}$$

[1] $H^1([0,L],\mathbb{R}^2)$ denotes the Sobolev space of differentiable functions on the interval $[0,L]$.

corresponds to using different staggered grids for both types of energy and co-energy variables. The aim of the next sections is to make explicit in the 1D and 2D cases the use of such staggered grids to derive a discretized model that keeps a port-Hamiltonian structure *i.e.* such that the discretized version of (6) is satisfied.

III. 1D CASE

Consider the general class of port-Hamiltonian systems defined by (4) where $\begin{pmatrix}e_1 & e_2\end{pmatrix}^T = \mathcal{L}_\xi \begin{pmatrix}x_1 & x_2\end{pmatrix}^T$ with $\mathcal{L}_\xi$ a coercive matrix valued function from $L^2([0,L],\mathbb{R}^2)$ to $L^2([0,L],\mathbb{R}^2)$ (see [6] for more details). The total energy becomes :

$$H = \frac{1}{2}\int_0^L \begin{pmatrix}x_1 & x_2\end{pmatrix} \mathcal{L}_\xi \begin{pmatrix}x_1\\ x_2\end{pmatrix} d\xi \tag{8}$$

A. Discretization scheme

Defining h a spatial step, the state of this system is discretized over the grids described in Fig. 1 and boundary conditions are given by the effort imposed on boundary points numbered 0 and $n+1$.

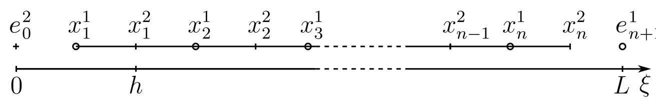

Fig. 1. 1D staggered discretization grids for x_d^1 and x_d^2

In the discretized setting, the continuous (in space) state variables are replaced by the finite-dimensional vector $x_d = \begin{pmatrix}x_d^1 & x_d^2\end{pmatrix}^T \in \mathbb{R}^{2n}$ with $x_d^1 = \begin{pmatrix}x_1^1 & .. & x_n^1\end{pmatrix}^T$, $x_d^2 = \begin{pmatrix}x_1^2 & .. & x_n^2\end{pmatrix}^T$, where the $x_k^{\{1,2\}}$ $(k \in \{1..n\})$ are the approximation of the state $x^{\{1,2\}}$ respectively evaluated at $\xi = \{(k-1)h, (k-0.5)h\}$. e_0^2 and e_{n+1}^1 denote the boundary effort variables. A discrete Hamiltonian which approximates the original energy such that $hH_d \approx H$ can be defined :

$$H_d = \frac{1}{2}x_d^T \mathcal{L}_d x_d := \frac{1}{2}\sum_{i=1}^{n} \begin{pmatrix}x_i^1 & x_i^2\end{pmatrix} \mathcal{L}_{\xi_i} \begin{pmatrix}x_i^1\\ x_i^2\end{pmatrix}, \tag{9}$$

where $\mathcal{L}_d \in \mathbb{R}^{2n \times 2n}$ is a block diagonal matrix, composed of $\mathcal{L}_\xi$, evaluated at the corresponding grid points. Defining the vector of discrete efforts as the gradient of the discrete energy,

$$e_d = \left(\frac{\partial H_d}{\partial x_d}\right)^T = \mathcal{L}_d x_d, \tag{10}$$

one obtains, as the efforts on the k-th grid points,

$$\begin{pmatrix}e_k^1\\ e_k^2\end{pmatrix} = \mathcal{L}_{\xi_k}\begin{pmatrix}x_k^1\\ x_k^2\end{pmatrix}, \tag{11}$$

with the elements of $\mathcal{L}_\xi$ evaluated at the corresponding grid points. Taking into account that

$$\begin{pmatrix}x_k^1\\ x_k^2\end{pmatrix} \approx \begin{pmatrix}x_1(\xi_k)\\ x_2(\xi_{k+0.5})\end{pmatrix} \text{ and } \begin{pmatrix}e_k^1\\ e_k^2\end{pmatrix} \approx \begin{pmatrix}e_1(\xi_k)\\ e_2(\xi_{k+0.5})\end{pmatrix}, \tag{12}$$

we obtain – by central approximation of the spatial derivative – the numerical scheme

$$\begin{pmatrix}f_k^1\\ f_k^2\end{pmatrix} = -\frac{1}{h}\begin{pmatrix}e_k^2 - e_{k-1}^2\\ e_{k+1}^2 - e_k^2\end{pmatrix} \tag{13}$$

where $f_d = \begin{pmatrix} f_d^1 & f_d^2 \end{pmatrix}^T = \begin{pmatrix} f_1^1 & \dots & f_n^1 & f_1^2 & \dots & f_n^2 \end{pmatrix}^T$ is the approximation of $\frac{\partial x_d}{\partial t}$ (evaluated on the same spatial points) *i.e.*

$$\begin{aligned} f_d^1 &= \frac{1}{h} \underbrace{\begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}}_{D} e_d^2 + \frac{1}{h} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e_0^2 \\ f_d^2 &= \frac{1}{h} \underbrace{\begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}}_{-D^T} e_d^1 + \frac{1}{h} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} e_{n+1}^1 \end{aligned} \tag{14}$$

This permits to express the vector of the discretized flow variables f_d :

$$f_d = \underbrace{\begin{pmatrix} 0 & D \\ -(D)^T & 0 \end{pmatrix}}_{J_d} e_d + \underbrace{\frac{1}{h} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & -1 \end{pmatrix}}_{g_{d1}} \begin{pmatrix} e_0^2 \\ e_{n+1}^1 \end{pmatrix} \tag{15}$$

where J_d is skew-symmetric.

Proposition 2: The staggered-grid finite difference spatial discretization of (4) defines a Dirac structure which approximates the original Dirac structure, with explicit representation given by

$$\begin{cases} f_d = J_d e_d + g_d e_d^\partial \\ f_d^\partial = g_d^T e_d \end{cases} \tag{16}$$

with $g_d = g_{d1} U^{-1}$ where U is any invertible transformation and J_d, g_{d1} are defined in (15).

Proof: The discretized system defines a Dirac structure if its structure respects an energy balance product :

$$\langle (e_d, e_d^\partial) | (f_d, f_d^\partial) \rangle = \langle e_d, f_d \rangle - \langle e_d^\partial, f_d^\partial \rangle = 0 \tag{17}$$

$$\begin{aligned} \langle (e_d, e_d^\partial) | (f_d, f_d^\partial) \rangle &= e_d^T f_d - (e_d^\partial)^T f_d^\partial \\ &= e_d^T \left[J_d e_d + g_{d1} \begin{pmatrix} e_0^2 \\ e_{n+1}^1 \end{pmatrix} \right] - (e_d^\partial)^T f_d^\partial \\ &= e_d^T g_{d1} \begin{pmatrix} e_0^2 \\ e_{n+1}^1 \end{pmatrix} - (e_d^\partial)^T f_d^\partial = \frac{1}{h} \left(e_1^1 e_0^2 - e_n^2 e_{n+1}^1 \right) - (e_d^\partial)^T f_d^\partial \end{aligned}$$

The Dirac structure is thus defined with respect to the product $\langle | \rangle$ for any $e_d^\partial, f_d^\partial$ that respect $(e_d^\partial)^T f_d^\partial = \frac{1}{h} \left(e_1^1 e_0^2 - e_n^2 e_{n+1}^1 \right)$, which is equivalent to :

$$e_d^\partial = U \begin{pmatrix} e_0^2 \\ e_{n+1}^1 \end{pmatrix} \text{ and } f_d^\partial = U^{-1} \begin{pmatrix} \frac{1}{h} e_1^1 \\ -\frac{1}{h} e_n^2 \end{pmatrix} \tag{18}$$

for any invertible transformation U. The discretized system is then $f_d = J_d e_d + g_d e_d^\partial, \; f_d^\partial = g_d^T e_d$ ■

Remark 3: Choosing the grid such that another effort variable is defined on a boundary permits to define another causality on this boundary and thus different boundary conditions. The proof can easily be extended to cases where any combination of effort variables is defined on the points at the boundaries even if the resulting D matrices may be non-square. As an illustration, the example for the 2D case in the following section has a different causality.

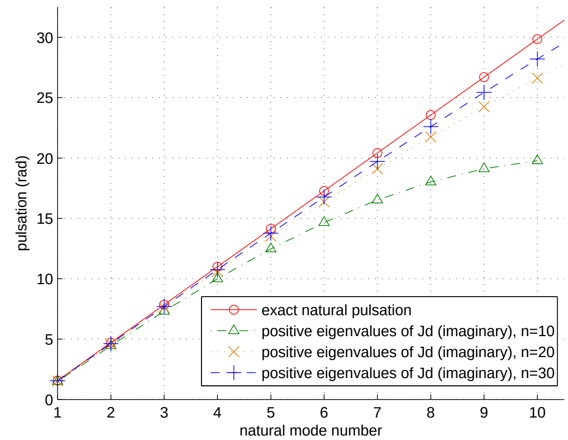

Fig. 2. Eigenvalues of J_d compared to exact values.

B. Numerical results

We consider the discretization of the system (4) along with the boundary conditions $e_2(0) = 0$ and $e_1(L) = 0$ (which corresponds to $e_d^\partial = 0$). In acoustics, the physical meaning of e_1 is the particular speed and the physical meaning of e_3 is the pressure. The theoretical eigenfrequencies of the system are derived from the analytical resolution of the PDE leading for this set of boundary conditions to $f_k = \frac{c_0}{2} \frac{k}{L}$ with $k \in \mathcal{N}$ where c_0 is the velocity of the wave. Fig. 2 shows the imaginary parts of the first eigenvalues of the J_d matrix for $L = 1$, discretized with different spatial steps, compared with the theoretical eigenfrequencies. Since J_d is skew-symmetric, the eigenvalues are exactly on the imaginary axis, which guarantees energy conservation. Fig. 3 shows the Bode gain plot of the discretized system with input e_0^2 and output e_n^2. A cut-off can be observed, for a frequency increasing with the number of discretization points for an increasing roll-off. Such diagram permits to evaluate numerically if the number of points in the discretization is enough to study a the system behavior up to a certain frequency. This is particularly interesting in the case of an acoustic duct where we can take the cutoff frequency of the discrete system higher than the frequency above which the 1D approximation does not hold [24].

IV. 2D CASE

In this section, we consider the 2D-wave equation defined for $\{\xi_1, \xi_2\} \in [0, L_1] \times [0, L_2]$. The aim is to provide a finite dimensional model suitable for simulation and control design purposes using distributed boundary actuation (not developed in this paper). The presentation of such case of study and the associated port Hamiltonian formulation are given in [11]. The model results in

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & \frac{\partial}{\partial \xi_1} \\ 0 & 0 & \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & 0 \end{pmatrix}}_{\mathcal{J}} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{19}$$

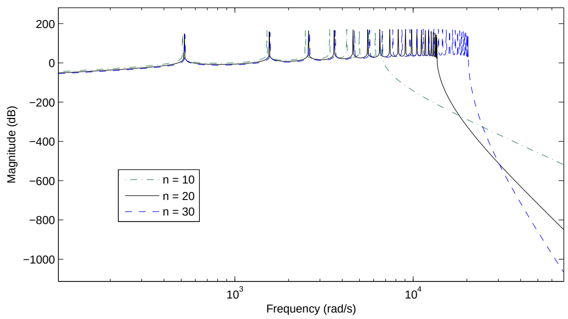

Fig. 3. Magnitude Bode diagram for $TF = \frac{e_n^2}{e_0^2}$.

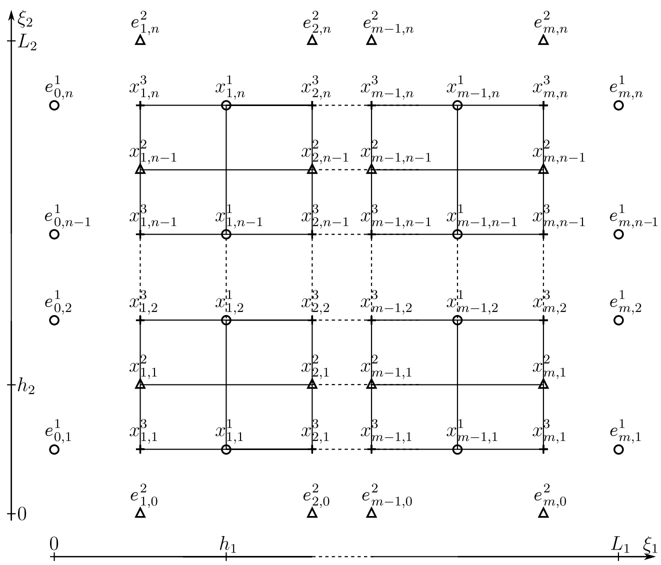

Fig. 4. 2D Discretised grid

with boundary port variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = (-e_3(L_1,\xi_2)\ e_3(0,\xi_2)\ \ -e_3(\xi_1,L_2)\ e_3(\xi_1,0) \\ e_1(L_1,L_2)\ e_1(0,\xi_2)\ e_2(\xi_1,R)\ e_2(\xi_1,0))^T \tag{20}$$

The total energy of this system is :

$$H = \frac{1}{2}\int_0^{L_1}\int_0^{L_2}\left(\begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix}\mathcal{L}_{\xi_1,\xi_2}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}\right) d\xi_2 d\xi_1 \tag{21}$$

where $e = \mathcal{L}_{\xi_1,\xi_2}\begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix}^T$ with $\mathcal{L}_{\xi_1,\xi_2}$ a coercive operator, and its time derivative satisfies $\frac{dH}{dt} = f_\partial^T e_\partial$.

A. Discretization scheme

Defining h_1 (resp. h_2) the spacial step along ξ_1 (resp. ξ_2), the state of the system is discretized over the grids described in Fig.4 and boundary conditions are given by effort imposed on boundary points numbered $(0,k_2)$ and (m,k_2) for conditions on e_1 and $(k_1,0)$ and (k_1,n) for conditions on e_2 (with $k_2 \in \{1..n\}$ and $k_1 \in \{1..m\}$). In the discretesed setting, the continuous state variables are replaced by finite-dimensional vectors :

$$\begin{cases} x_d^1 = \begin{pmatrix} x_{1,1}^1 & x_{1,2}^1 & \dots & x_{m-1,n}^1 \end{pmatrix}^T \in \mathbb{R}^{(m-1)n} \\ x_d^2 = \begin{pmatrix} x_{1,1}^2 & x_{1,2}^2 & \dots & x_{m,n-1}^2 \end{pmatrix}^T \in \mathbb{R}^{m(n-1)} \\ x_d^3 = \begin{pmatrix} x_{1,1}^3 & x_{1,2}^3 & \dots & x_{m,n}^3 \end{pmatrix}^T \in \mathbb{R}^{mn} \\ x_d = \begin{pmatrix} (x_d^1)^T & (x_d^2)^T & (x_d^3)^T \end{pmatrix}^T \in \mathbb{R}^{3mn-(m+n)} \end{cases} \tag{22}$$

The boundary effort variables are denoted :

$$\begin{cases} e_{d\partial}^1 = \begin{pmatrix} e_{0,1}^1 & e_{0,2}^1 & \dots & e_{0,n}^1 & e_{m,1}^1 & e_{m,2}^1 & .. & e_{m,n}^1 \end{pmatrix}^T \\ e_{d\partial}^2 = \begin{pmatrix} e_{1,0}^2 & e_{1,n}^2 & e_{2,0}^2 & e_{2,n}^2 & \dots & e_{m,0}^2 & e_{m,n}^2 \end{pmatrix}^T \\ e_{d\partial}^3 = \{\} \\ e_d^\partial = \begin{pmatrix} (e_{d\partial}^1)^T & (e_{d\partial}^2)^T & (e_{d\partial}^3)^T \end{pmatrix}^T \end{cases} \tag{23}$$

The discrete Hamiltonian such as $h_1 h_2 H_d \approx H$ can be expressed in terms of the discretized states x_d as (to simplify the notation in the second line of the following equation, consider null the states with index out of their definition set, *e.g.* $x_{m,n}^1 = 0$):

$$\begin{aligned} H_d &= \frac{1}{2} x_d^T \mathcal{L}_d x_d \\ &:= \frac{1}{2}\Sigma_{j=1}^n \Sigma_{i=i}^m \begin{pmatrix} x_{i,j}^1 & x_{i,j}^2 & x_{i,j}^3 \end{pmatrix} \mathcal{L}_{\xi_{1,i},\xi_{2,j}} \begin{pmatrix} x_{i,j}^1 \\ x_{i,j}^2 \\ x_{i,j}^3 \end{pmatrix} \end{aligned} \tag{24}$$

where $\mathcal{L}_d \in \mathbb{R}^{(3mn-(m+n))\times(3mn-(m+n))}$ is a block diagonal matrix, composed of $\mathcal{L}_{\xi_1,\xi_2}$, evaluated at the corresponding grid points. Defining the vector of discrete efforts as the gradient of the discrete energy, $e_d = \left(\frac{\partial H_d}{\partial X_d}\right)^T = \mathcal{L}_d X_d$ one obtains, as the efforts on the (i,j)-indexed grid point, $\begin{pmatrix} e_{i,j}^1 \\ e_{i,j}^2 \\ e_{i,j}^3 \end{pmatrix} = \mathcal{L}_{\xi_{1,i},\xi_{2,j}} \begin{pmatrix} x_{i,j}^1 \\ x_{i,j}^2 \\ x_{i,j}^3 \end{pmatrix}$, with the elements of $\mathcal{L}_{\xi_1,\xi_2}$ evaluated at the corresponding grid points. Taking into account that

$$\begin{pmatrix} x_{i,j}^1 \\ x_{i,j}^2 \\ x_{i,j}^3 \end{pmatrix} \approx \begin{pmatrix} x^1(\xi_{i+0.5}^1, \xi_j^2) \\ x^2(\xi_i^1, \xi_{j+0.5}^2) \\ x^3(\xi_i^1, \xi_j^2) \end{pmatrix}, \quad \begin{pmatrix} e_{i,j}^1 \\ e_{i,j}^2 \\ e_{i,j}^3 \end{pmatrix} \approx \begin{pmatrix} e^1(\xi_{i+0.5}^1, \xi_j^2) \\ e^2(\xi_i^1, \xi_{j+0.5}^2) \\ e^3(\xi_i^1, \xi_j^2) \end{pmatrix} \tag{25}$$

we obtain - by central approximation of the spatial derivative - the numerical scheme

$$\begin{cases} f_{i,j}^1 = -\frac{1}{h_1}(e_{i+1,j}^3 - e_{i,j}^3) \\ f_{i,j}^2 = -\frac{1}{h_2}(e_{i,j+1}^3 - e_{i,j}^3) \\ f_{i,j}^3 = -\frac{1}{h_1}(e_{i,j}^1 - e_{i-1,j}^1) - \frac{1}{h_2}(e_{i,j}^2 - e_{i,j-1}^2) \end{cases} \tag{26}$$

where $f_d = \begin{pmatrix} f_d^1 & f_d^2 & f_d^3 \end{pmatrix}^T =$
$\begin{pmatrix} f_{1,1}^1 & \dots & f_{m-1,n}^1 & f_{1,1}^2 & \dots & f_{m,n-1}^2 & f_{1,1}^3 & \dots & f_{m,n}^3 \end{pmatrix}^T$
is the approximation of $\frac{\partial x_d}{\partial t}$ (evaluated on the same spatial points than x). This leads to :

$$f_d^1 = -\frac{1}{h_1}\underbrace{\begin{pmatrix} -1 & & 1 & \\ & \ddots & & \ddots \\ & & \ddots & & \ddots \\ & & & -1 & & 1 \end{pmatrix}}_{D_1} e_d^3 \tag{27}$$

where $D_1 \in \mathbb{R}^{n(m-1)\times mn}$,

$$f_d^2 = \underbrace{-\frac{1}{h_2}\begin{pmatrix}\alpha & & \\ & \ddots & \\ & & \alpha\end{pmatrix}}_{D_2} e_d^3 \tag{28}$$

where $D_2 \in \mathbb{R}^{m(n-1)\times mn}$ andwith $\alpha \in \mathbb{R}^{(n-1)\times n}$ such that:

$$\alpha = \begin{pmatrix}-1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1\end{pmatrix} \tag{29}$$

$f_d^3 = f_{d1}^3 + f_{d2}3$ with :

$$f_{d1}^3 = -(D_1)^T e_d^1 + \underbrace{\frac{1}{h_1}\begin{pmatrix}I_n & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & -I_n\end{pmatrix}}_{g_1} e_{d\partial}^1 \tag{30}$$

where I_m is the identity matrix of size m and the 0 are zero matrices of appropriate size, $g_1 \in \mathbb{R}^{mn\times 2n}$

$$f_{d2}^3 = -(D_2)^T e_d^2 + \underbrace{\frac{1}{h_2}\begin{pmatrix}\beta & & \\ & \ddots & \\ & & \beta\end{pmatrix}}_{g_2} e_{d\partial}^2 \tag{31}$$

where $g_2 \in \mathbb{R}^{mn\times 2m}$ and with $\beta \in \mathbb{R}^{n\times 2}$ such that:

$$\beta = \begin{pmatrix}1 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1\end{pmatrix}^T$$

This permits to express the discrete flow f_d :

$$f_d = \underbrace{\begin{pmatrix}0 & 0 & D_1 \\ 0 & 0 & D_2 \\ -D_1^T & -D_2^T & 0\end{pmatrix}}_{J_d} e_d + \underbrace{\begin{pmatrix}0 & 0 \\ g_1 & g_2\end{pmatrix}}_{g_{d1}} \begin{pmatrix}e_{d\partial}^1 \\ e_{d\partial}^2 \\ e_{d\partial}^3\end{pmatrix} \tag{32}$$

Where J_d is skew-symmetric.

Proposition 4: A staggered-grid finite difference spatial discretization of the 2D system (19)-(21) defines a Dirac structure which approximates the original Dirac structure, with explicit representation given by

$$f_d = J_d e_d + g_d e_d^\partial, \; f_d^\partial = g_d^T e_d \tag{33}$$

with $g_d = g_{d1} U^{-1}$ where U is any invertible transformation.

Proof: To define a Dirac structure, the system has to respect the energy balance product (17).

$$\begin{aligned}\langle (e_d, e_d^\partial) | (f_d, f_d^\partial)\rangle &= e_d^T f_d - (e_d^\partial)^T f_d^\partial \\ &= e_d^T \left[J_d e_d + g_{d1}\begin{pmatrix}e_{d\partial}^1 \\ e_{d\partial}^2 \\ e_{d\partial}^3\end{pmatrix}\right] - (e_d^\partial)^T f_d^\partial \\ &= e_d^T g_{d1} \begin{pmatrix}e_{d\partial}^1 \\ e_{d\partial}^2 \\ e_{d\partial}^3\end{pmatrix} - (e_d^\partial)^T f_d^\partial \\ &= \Sigma_{i=1}^m \frac{1}{h_1}\left(e_{i,0}^2 e_{i,1}^3 - e_{i,n}^2 e_{i,n}^3\right) + \\ &\quad \Sigma_{j=1}^n \frac{1}{h_2}\left(e_{0,j}^1 e_{1,j}^3 - e_{m,j}^1 e_{m,j}^3\right) - (e_d^\partial)^T f_d^\partial\end{aligned} \tag{34}$$

The Dirac structure is defined with respect to $\langle | \rangle$ for any e_d^∂,f_d^∂ that respect :

$$(e_d^\partial)^T f_d^\partial = \Sigma_{i=1}^m \frac{e_{i,0}^2 e_{i,1}^3 - e_{i,n}^2 e_{i,n}^3}{h_1} + \Sigma_{j=1}^n \frac{e_{0,j}^1 e_{1,j}^3 - e_{m,j}^1 e_{m,j}^3}{h_2} \tag{35}$$

Let consider: $f_{d1}^\partial =$

$$\big(\tfrac{1}{h_1}e_{1,1}^3 \; \tfrac{1}{h_1}e_{1,2}^3 \cdots \tfrac{1}{h_1}e_{1,n}^3 \; -\tfrac{1}{h_1}e_{m,1}^3 \; -\tfrac{1}{h_1}e_{m,2}^3 \cdots -\tfrac{1}{h_1}e_{m,n}^3 \\ \tfrac{1}{h_2}e_{1,1}^3 \; -\tfrac{1}{h_2}e_{1,n}^3 \; \tfrac{1}{h_2}e_{2,1}^3 \; -\tfrac{1}{h_2}e_{2,n}^3 \cdots \tfrac{1}{h_2}e_{m,1}^3 \; -\tfrac{1}{h_2}e_{m,n}^3\big)^T \tag{36}$$

(35) is equivalent to $e_d^\partial = U e_{d1}^\partial$ and $f_d^\partial = U^{-1} f_{d1}^\partial$ for any invertible transformation U. Define $g_d = g_{d1}U^{-1}$, the discretized system is then derived as:

$$f_d = J_d e_d + g_d e_d^\partial, \; f_d^\partial = g_d^T e_d \tag{37}$$

■

Remark 5: Choosing the grid such that another effort variable is defined on a boundary permits to define another causality on this boundary and thus other boundary conditions. The proof is extended straightforwardly to cases where any combination of effort variables can be defined on the points at the boundaries. It is the case in the 1D example in the previous section where the causality is not the same.

B. Consistency

Each spatial derivative operator is approximated by a centred scheme such that $\frac{\partial e_i(\xi_j)}{\partial \xi_j} = \frac{e_i(\xi_{j+1}) - e_i(\xi_{j-1})}{2h_j} + \epsilon$ with $i \in \{1,2,3\}$ and $j \in \{1,2\}$ and where ϵ is the local consistency error in space. A Taylor series expansion of $e_i(\xi_{j+1})$ and $e_i(\xi_{j-1})$ shows that $\epsilon = \mathcal{O}(h_j^2)$. Furthermore, the definition of the effort e_d^∂ variables outside of the grid permits to define the differential operators on the boundaries without loss of consistency order. The local consistency error in space is thus of order 2.

C. Numerical results

We consider the discretization of the PH system (19) along with the boundary conditions $e_1(0,\xi_2) = 0$, $e_1(L_1,\xi_2) = 0$, $e_2(\xi_1, 0) = 0$ and $e_2(\xi_1, L_2) = 0$ (which correspond to $e_d^\partial = 0$). In acoustics, e_1 and e_2 are the components of the particular speed and e_3 the pressure. The theoretical eigenfrequencies of the system are derived from the analytical resolution of the PDE leading for this set of boundary conditions to $f_{k_1 k_2} = \frac{c_0}{2}\sqrt{\left(\frac{k_1}{L_1}\right)^2 + \left(\frac{k_2}{L_2}\right)^2}$ with $\{k_1, k_2\} \in \mathcal{N}^2$ Fig. 5 shows the imaginary parts of the first eigenvalues of the J_d matrix for $L_1 = 1$ and $L_2 = 0.8$, discretized with different spatial steps on each axis, compared with the theoretical eigenfrequencies. Since J_d is skew-symmetric, the eigenvalues are exactly on the imaginary axis, which guarantees energy conservation. Figure 6 shows the Bode gain plot of the discretized system with input $e_{0,int(n/2)}^1$ and output $e_{m,int(n/2)}^3$ (with $int(n/2)$ the integer part of $n/2$). A cut-off can be observed, for a frequency increasing with the number of discretization points for an increasing roll-off. Such diagram permits to evaluate numerically if the number of points in the discretization is enough to study a the system behavior up to a certain frequency.

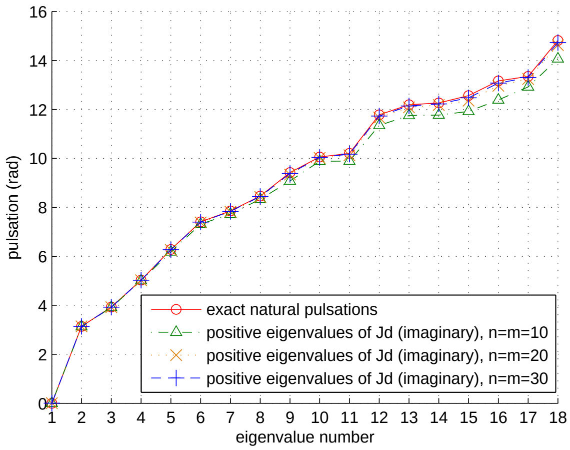

Fig. 5. Eigenvalues of J_d compared to exact values.

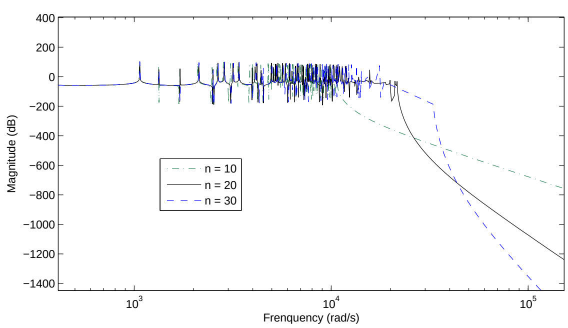

Fig. 6. Magnitude Bode diagram for $TF = \frac{e^3_{m,int(n/2)}}{e^1_{0,int(n/2)}}$.

V. Conclusion

In this paper a structure preserving spatial discretization scheme based on staggered grids and finite difference has been proposed for port-Hamiltonian systems (PHS) defined on 1D and 2D spatial domains. It has been shown how to define the discretized state and co-state variables and their extension to the boundary such that the PHS structure is preserved. This leads to a consistent balance equation on the discretized energy. A strong advantage of this approach is the preservation of the physical interpretation of the boundary port variables, which can be used for the interconnection of the hyperbolic system with its environment through its boundaries. The same is true when considering boundary control. Another important feature of the proposed method is its simplicity, inherited from the underlying finite-difference scheme. Future work will deal with the full discretization, i.e., in space and time, and its extension for control design.

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