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Parabolic matching of hyperbolic system using Control by Interconnection

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Abstract: The structural difference between one-dimensional (1D) hyperbolic and parabolic port Hamiltonian system (PHS) is discussed. Then, using a Control by Interconnection (CbI) approach, a distributed state feedback is designed in order to transform an hyperbolic PHS into a parabolic one, the latter being asymptotically stable and even purely dissipative (with no oscillating modes). Distributed wave damping in 1D vibro-acoustic pipes, using piezo actuators, is considered as an illustration example for the proposed control design.
Keywords: *Port Hamiltonian systems (PHS), distributed parameters systems (DPS), Control by Interconnection (CbI), feedback equivalence with distributed control, Interconnection and Damping Assignment Passivity Based Control (IDA-PBC)*

1. INTRODUCTION

Port-Hamiltonian systems (PHS) [van der Schaft and Maschke, 2002] have become quite popular in the system theory community as a class of naturally well-posed (linear) multiphysics systems [Le Gorrec et al., 2005]. Part of this success comes from the possibility to use structural properties of PHS (stability under interconnection, passivity or losslessness, existence of first integrals and structural Casimir's functions) to prove stability results and more generally to design various non linear control such as Interconnection and Damping Assignment Passivity Based Control (IDA-PBC) or energy shaping [Ortega et al., 2002], control based on Casimir Function, Control by Interconnection (CbI) [Macchelli et al., 2004] or CbI together with power shaping [Ortega et al., 2008], ...

Although these control methods have successfully applied to various systems, the controller parameters are only studied case by case. There is still no guideline about how to choose the desired closed-loop interconnection structure, as well as the desired damping or closed-loop energy. The contribution of this paper is to propose a rather simple but natural idea to tune these parameters when dealing with hyperbolic systems with distributed actuators. More precisely, this idea consists in transforming the hyperbolic PHS into a parabolic one, hence ensuring the exponential stability of the closed-loop system which behaves as a purely dissipative system with no oscillation modes.

The considered class of *hyperbolic* PHS (for two conservation laws) is defined using the canonical 1D differential operator and the related Stokes-Dirac interconnection structure [van der Schaft and Maschke, 2002]. We show that a *parabolic* PHS with the same interconnection structure can be obtained by connecting one of two energy accumulators to a resistor, in other words, we aim at canceling "half" of the system's dynamics. This dynamics cancellation appears absurd to the system itself, but it is proved possible to an "augmented plant and controller" system. The controller dynamics will be canceled so-that the closed-loop system is parabolized; which is called "immersion and reduction" technique.

General speaking, the main idea is to eliminate the *hyperbolic* operator and replace it with a *parabolic* one, while keeping the Stokes-Dirac structure embedded in the closed-loop system. Note that this "Stokes-Dirac structure preserving" feedback control allows further possible control design, including boundary control [Macchelli et al., 2004, Vu et al., 2015b]. It also provides some structural robustness properties which are discussed at the end of the paper.

The paper is organized as follows. Section 2 presents the considered classes of controlled hyperbolic PHS and target parabolic PHS. Section 3 briefly recalls the CbI ideas when dealing with PHS. The proposed "parabolizing" controller design is presented in section 4, followed with an analysis of the robustness with respect to some model uncertainty in section 5. The example of the vibro-acoustic system is investigated in section 6.

2. CONTROLLED AND TARGET SYSTEMS CLASSES

This section defines the studied classes of controlled hyperbolic systems and closed-loop parabolic systems in the port Hamiltonian formalism. The Stokes-Dirac intercon-

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nection structure [van der Schaft and Maschke, 2002] is used to underline the structural similarity between these two classes. In turn, this suggests a "parabolizing design" which will be presented in the two following sections.

2.1 Examples of hyperbolic and parabolic systems realizations using the same Stokes-Dirac interconnection structure

A canonical 1D hyperbolic example Consider the canonical Stokes-Dirac structure for the 1D system of two conservation laws in the lossless case [van der Schaft and Maschke, 2002]:

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = -\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_p \end{pmatrix} \tag{2.1}$$

where the flow variables $f_q,\ f_p \in L^2\left([a,b],\mathbb{R}^2\right)$, and the effort variables $e_q,\ e_p \in H^1\left([a,b],\mathbb{R}^2\right)$. One can define a 1D canonical Stokes-Dirac differential operator

$$\mathcal{J} = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} \tag{2.2}$$

with domain $\mathcal{D}(\mathcal{J}) = H^1\left([a,b],\mathbb{R}^2\right)$, which is formally skew-adjoint $\mathcal{J} + \mathcal{J}^* = 0$. A parameterization of the admissible boundary port variables $(e_\partial,\ f_\partial)$ which can be used with (2.1) in order to define a well-posed linear system exists [Le Gorrec et al., 2005]. Including homogeneous boundary conditions $(e_\partial,\ f_\partial) = (0,0)$ in the previous definition of $\mathcal{D}(\mathcal{J})$ it is easily proved using Stokes' theorem that the resulting operator $\mathcal{J}$ becomes (truly) skew-symmetric. For the sake of simplicity, we will consider in the sequel the particular (impedance passive) case:

$$\begin{cases} e_\partial & = e_q|_\partial \\ f_\partial & = e_p|_\partial \end{cases} \tag{2.3}$$

In this case, the hyperbolic system (2.1) is characterized noticeably by the fact that all eigenvalues of $\mathcal{J}$ lie on the imaginary axis (see [Gorrec et al., 2011]).

A canonical 1D parabolic example We consider, using the same canonical 1D Stokes-Dirac operator, the system defined by:

$$\begin{pmatrix} f_q \\ -f_d \end{pmatrix} = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_d \end{pmatrix} \tag{2.4}$$

where the conjugated dissipative variables $(e_d,\ f_d)$ have replaced the variables (e_p, f_p) in (2.1) associated with the p energy domain which no longer exists.The dissipative variables are related with the dissipation constitutive equations $f_d = {}^1\!/\!_\sigma e_d$. The very same system may also be defined (as in [Gorrec et al., 2011] for the heat conduction problem) as:

$$\begin{pmatrix} f_q \\ e_{int} \end{pmatrix} = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} \begin{pmatrix} e_q \\ f_{int} \end{pmatrix} \tag{2.5}$$

when a gyrator

$$\begin{cases} f_{int} & = e_d \\ e_{int} & = -f_d \end{cases}$$

is applied to the dissipative effort-flow variables $(e_d,\ f_d)$. These systems (2.4 or 2.5) are parabolic since their dynamics reduce to one state variable purely dissipative equation:

$$f_q = -\dot{q} = -\partial_z\left(\sigma \partial_z e_q\right) \tag{2.6}$$

The boundary condition is kept the same as (2.3) with e_d instead of e_p in equation (2.4). With these boundary conditions all the eigenvalues lie on the real negative axis [Gorrec et al., 2011] (no oscillating mode). In the linear case, the equilibrium is exponentially stable, the systems (2.4 or 2.5) being analytic, dissipative and strongly monotone. This property of parabolic system is of course of great interest when dealing with stabilizing control problems.

2.2 The controlled system class

We will investigate hereafter the control problem for a class of distributed parameters PHS defined as:

$$\Sigma : \begin{cases} \dot{x} & = (\mathcal{J} - \mathcal{R})\, \partial_x \mathbb{H} + gu \\ y & = g^T \partial_x \mathbb{H} \end{cases} \tag{2.7}$$

where $x(z,t) \in \mathbb{R}^n$ is depending on space z and time t and $\mathcal{J} = -\mathcal{J}^*$ is a formally skew-symmetric differential operator. For the sake of simplicity we will consider only a restricted class of differential operators of the form $\mathcal{J} = P_1\partial_z + P_0$ where $M_n(\mathbb{R})$ is the class of matrices of dimension n, $P_1 \in M_n(\mathbb{R})$ is a non-singular symmetric matrix and $P_0 = -P_0^T \in M_n(\mathbb{R})$ is a skew-symmetric one, although this class may be generalized to higher order spatial derivatives such as in [Le Gorrec et al., 2005]. We will in particular focus on the canonical Stokes-Dirac (or Hamiltonian) operator (2.2). $\mathcal{R} = \mathcal{R}^T \geqslant 0$ is a symmetric positive semi-definite dissipation matrix, $\mathbb{H}$ is the total energy. $\partial_x \mathbb{H}$ denotes the variational derivative of the energy density with respect to the state x. u is a distributed control, (u, y) is the passive input-output pair of the system. The passivity of this class of systems is proved by computing the power balance equation:

$$\frac{d\mathbb{H}}{dt} = \langle \partial_x \mathbb{H}^T, \dot{x} \rangle \leq \langle y, u \rangle + e_\partial^T f_\partial \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual L^2 inner product on the whole spatial domain, with the appropriate choice of boundary variables $(e_\partial,\ f_\partial)$ [Le Gorrec et al., 2005] such as, for instance $(e_\partial,\ f_\partial)$ defined in (2.3).

The PHS system (2.7) may be equivalently represented using Stokes-Dirac interconnection structure associated with a dissipative closure equation of the form:

$$\begin{pmatrix} f \\ e_d \end{pmatrix} = \begin{pmatrix} \mathcal{J} & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} e \\ f_d \end{pmatrix} + \begin{pmatrix} gu \\ 0 \end{pmatrix}, \qquad f_d = \mathcal{R} e_d \tag{2.9}$$

where $f^T = \left(f_q^T, f_p^T\right) = \dot{x}^T$, $e^T = \left(e_q^T, e_p^T\right) = \partial_x^T \mathbb{H}$, $e_d^T = \left(e_{dq}^T, e_{dp}^T\right)$ and $f_d^T = \left(f_{dq}^T, f_{dp}^T\right)$. Homogeneous boundary conditions $(e_\partial,\ f_\partial) = 0$ are considered here.

Proposition 1. In the canonical case of hyperbolic system in the form (2.9) with $\mathcal{R}$ invertible, if half of the system dynamics is canceled (i.e. $f_p = \dot{p} = 0$), the hyperbolic system (2.9) is transformed into a parabolic one.

1. Assume $f_p = 0$, then $f^T = \left(f_q^T, 0\right)$ and $f_d^T = \left(0, f_{dp}^T \neq 0\right)$ and the system *(2.9)* becomes:

$$\begin{pmatrix} f_q \\ f_{dp} \end{pmatrix} = \mathcal{J} \begin{pmatrix} e_q \\ e_{dp} \end{pmatrix} + \begin{pmatrix} gu \\ 0 \end{pmatrix} \tag{2.10}$$

which may be written in explicit form as the parabolic system:

$$f_q = \partial_z\left({}^1\!/\!_\mathcal{R} \partial_z e_q\right) + gu \tag{2.11}$$

Remark 2. In the more general case where both dissipation components may be non null ($f_{dp} \neq 0$ and $f_{dq} \neq 0$), the reduced system becomes:

$$f_q = \partial_z \left({}^1\!/\!_{\mathcal{R}} \partial_z e_q \right) - f_{dq} + gu \tag{2.12}$$

This additional term may arise as a supplementary dissipation term (such as in a reaction-diffusion problem) which can also be considered as a supplementary distributed source term.

The observation in proposition 1 will now be investigated with control purpose, using the transformation of an hyperbolic PHS into a parabolic one to achieve asymptotic stability.

However, it seems to be *unreachable* to design a controller which can annul half of the original system dynamics. To be able to apply the idea in Proposition 1, one of the solutions is to use the "immersion and reduction" technique. First for immersion, we double the dynamics of the closed-loop system using CbI method (Section 3); the IDA-PBC principle, interconnection assignment precisely, is then applied to couple each system dynamics to a controller dynamics, in the manner to reproduce the canonical interconnection structure (2.2) as well as to assure the non null dissipation. Finally, for reduction, the controller dynamics are canceled to *parabolize* the closed-loop system.

3. BACKGROUND OF CBI METHOD

The CbI method [Ortega et al., 2008] uses a controller Σ_C in PHS form coupled to the (PHS) plant Σ via a power preserving interconnection structure Σ_I.

$$\Sigma_C : \begin{cases} \dot{x}_c &= (\mathcal{J}_c - \mathcal{R}_c)\, \partial_{x_c} \mathbb{H}_c + g_c u_c \\ y_c &= g_c^T \partial_{x_c} \mathbb{H}_c \end{cases} \tag{3.1}$$

The feedback is realized as a Dirac interconnection structure of the form:

$$\Sigma_I : \left\{ \begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y_c \end{pmatrix} + \begin{pmatrix} \upsilon \\ 0 \end{pmatrix} \right. \tag{3.2}$$

Then the closed-loop system (or augmented system):

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} \mathcal{J} - \mathcal{R} & -g g_c^T \\ g_c g^T & \mathcal{J}_c - \mathcal{R}_c \end{pmatrix} \begin{pmatrix} \partial_x \mathbb{H} \\ \partial_{x_c} \mathbb{H}_c \end{pmatrix} + \begin{pmatrix} g\upsilon \\ 0 \end{pmatrix} \tag{3.3}$$

is passive since from (2.7,3.1,3.2):

$$\dot{\mathbb{W}} = \dot{\mathbb{H}} + \dot{\mathbb{H}}_c \leq y^T \upsilon \tag{3.4}$$

with the total energy $\mathbb{W}$.

In [Ortega et al., 2008], the Casimir method is used to shape the closed-loop energy function. It states the relation between the controller state and the system state, which allows to reduce the augmented system in 3.3; we call it immersion / reduction method. However, the Casimir method admits a limit for its application due to the dissipation obstacle. Some solutions are also revealed in that work (such as power shaping).

Otherwise, in this work, we propose to combine the idea of immersion / reduction method with the idea presented in Proposition 1 to transform an hyperbolic system (like system in (3.3)) into a stable parabolic one. The new proposed interconnection will play the role of *Interconnection assignment* (similar to IDA-PBC control while the dissipation injection is accomplished by the CbI damping).

4. CONTROL DESIGN PROPOSITION

The controller structure in (3.1) is employed to form an augmented system:

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \left[\begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}_c \end{pmatrix} - \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R}_c \end{pmatrix} \right] \begin{pmatrix} \partial_x \mathbb{H} \\ \partial_{x_c} \mathbb{H}_c \end{pmatrix} + \begin{pmatrix} gu \\ g_c u_c \end{pmatrix} \tag{4.1}$$

At this stage, we do not use the previous interconnection structure Σ_I defined in (3.2). Instead, using the idea of IDA-PBC [Ortega et al., 2002], we aim to transform the above system (4.1) into a desired system, which is:

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \left[\begin{pmatrix} 0 & \mathcal{J}_1 \\ \mathcal{J}_2 & 0 \end{pmatrix} - \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R}_c \end{pmatrix} \right] \begin{pmatrix} \partial_x \mathbb{H} \\ \partial_{x_c} \mathbb{H}_c \end{pmatrix} \tag{4.2}$$

by using control gu and virtual control $g_c u_c$. The structure of the augmented system (4.1) $\mathcal{J}_{aug} = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}_c \end{pmatrix}$ can be transferred into a new desired one $\mathcal{J}_d = \begin{pmatrix} 0 & \mathcal{J}_1 \\ \mathcal{J}_2 & 0 \end{pmatrix}$ which satisfies $\mathcal{J}_d + \mathcal{J}_d^* = 0$ if (and only if):

$$\begin{cases} gu &= -\mathcal{J} \partial_x \mathbb{H} + \mathcal{J}_1 \partial_{x_c} \mathbb{H}_c \\ g_c u_c &= -\mathcal{J}_c \partial_{x_c} \mathbb{H}_c + \mathcal{J}_2 \partial_x \mathbb{H} \end{cases} \tag{4.3}$$

This condition (matching equation) satisfies the interconnection structure property: $y^T u + y_c^T u_c = 0$.

2. The energy balance in the interconnection element:

$$\begin{aligned} y^T u + y_c^T u_c &= \partial_x \mathbb{H}^T gu + \partial_{x_c} \mathbb{H}_c^T g_c u_c \\ &= \partial_x \mathbb{H}^T \left(-\mathcal{J} \partial_x \mathbb{H} + \mathcal{J}_1 \partial_{x_c} \mathbb{H}_c \right) \\ &\quad + \partial_{x_c} \mathbb{H}_c^T \left(-\mathcal{J}_c \partial_{x_c} \mathbb{H}_c + \mathcal{J}_2 \partial_x \mathbb{H} \right) \\ &= 0 \end{aligned} \tag{4.4}$$

thanks to the skew-symmetric $\mathcal{J}_{aug} = -\mathcal{J}_{aug}^*$ and $\mathcal{J}_d = -\mathcal{J}_d^*$.

In the sequel, in order to apply the idea presented in Section 2.1, the desired interconnection $\mathcal{J}_d$ is chosen as the first order partial derivative operator: $\mathcal{J}_1 = \mathcal{J}_2 = \begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix}$. This interconnection assignment allows to split the desired system (4.2) into two "independent" subsystems:

$$\left\{ \begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_{c2} \end{pmatrix} &= \left[\begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} - \begin{pmatrix} R_1 & 0 \\ 0 & R_{c2} \end{pmatrix} \right] \begin{pmatrix} \partial_{x_1} \mathbb{H} \\ \partial_{x_{c2}} \mathbb{H}_c \end{pmatrix} \\ \begin{pmatrix} \dot{x}_2 \\ \dot{x}_{c1} \end{pmatrix} &= \left[\begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix} - \begin{pmatrix} R_2 & 0 \\ 0 & R_{c1} \end{pmatrix} \right] \begin{pmatrix} \partial_{x_2} \mathbb{H} \\ \partial_{x_{c1}} \mathbb{H}_c \end{pmatrix} \end{aligned} \right. \tag{4.5}$$

Then the closed-loop system (4.5) is parabolized by taking the static control states $\dot{x}_{c1} = \dot{x}_{c2} = 0$. One can notice that without controller damping $\mathcal{R}_c$, the method would not work. In fact, the added damping $\mathcal{R}_c$ takes the role of diffusivity for two new diffusion equations from (4.5).

Remark 3. This distributed control u only impacts the balance equation, hence the boundary condition $(e_\partial,\, f_\partial) = (\partial_{x_1} \mathbb{H},\, \partial_{x_2} \mathbb{H})\,|_\partial$ is still preserved in the closed-loop system.

Otherwise, regarding the static controller and the matching equation, a relation between $\mathcal{R}_c$ and $\mathbb{H}_c$ can be revealed from (4.2) and (4.3):

$$\begin{cases} \dot{x}_c = 0 & = \mathcal{J}_2 \partial_x \mathbb{H} - \mathcal{R}_c \partial_{x_c} \mathbb{H}_c \\ gu & = -\mathcal{J} \partial_x \mathbb{H} + \mathcal{J}_1 \partial_{x_c} \mathbb{H}_c \end{cases}$$
$$\Leftrightarrow \begin{cases} \partial_{x_c} \mathbb{H}_c & = \mathcal{R}_c^+ \mathcal{J}_2 \partial_x \mathbb{H} \\ gu & = -\mathcal{J} \partial_x \mathbb{H} + \mathcal{J}_1 \left(\mathcal{R}_c^+ \mathcal{J}_2 \partial_x \mathbb{H} \right) \end{cases} \tag{4.6}$$

where $\mathcal{R}_c^+$ corresponds to the inverse of $\mathcal{R}_c > 0$.

Finally, the proposed method overcomes the difficulty of solving the PDE in CbI method (we don't need to know $\mathbb{H}_c$ and $\mathcal{J}_d$ is already fixed in this method), and restricts the controller parameter choice of CbI control into only one freely designed parameters $\mathcal{R}_c = \mathcal{R}_c^T > 0$.

5. ROBUSTNESS ANALYSIS

The control idea developed in the previous section aims at suppressing all the oscillating mode, and replace with an exponential stable mode. The challenge is not on the design method, but on the robustness properties: in case the oscillating mode is not totally compensated, is the closed-loop system still stable? This section will first deal with the model uncertainties when we try to eliminate the oscillating phenomenon.

Remark 4. The proposed control law in (4.6) requires a full knowledge of the system state. A PHS observer can be considered for this fact. However, the observer itself is not discussed in this paper.

In the sequel we focus only on the error of the observed states and Hamiltonian function, denoted $\hat{x}$ and $\hat{\mathbb{H}}$ respectively. Other disturbances such as $\delta\mathcal{R}$ on dissipation $\mathcal{R}$ or external/random disturbances can also be stabilized with IDA-PBC technique (normally by using enough damping $\mathcal{R}_c$ to dominate these perturbations). Substituting the proposed control law in (4.6) which is now:

$$gu = -\mathcal{J} \partial_{\hat{x}} \hat{\mathbb{H}} + \mathcal{J}_1 \left(\mathcal{R}_c^+ \mathcal{J}_2 \partial_{\hat{x}} \hat{\mathbb{H}} \right) \tag{5.1}$$

the closed-loop system with observation error is:

$$\begin{aligned} \dot{x} &= \quad (\mathcal{J} - \mathcal{R}) \partial_x \mathbb{H} - \mathcal{J} \partial_{\hat{x}} \hat{\mathbb{H}} + \mathcal{J}_1 \left(\mathcal{R}_c^+ \mathcal{J}_2 \partial_{\hat{x}} \hat{\mathbb{H}} \right) \\ &= \underbrace{\mathcal{J}_1 \left(\mathcal{R}_c^+ \mathcal{J}_2 \partial_x \mathbb{H} \right) - \mathcal{R} \partial_x \mathbb{H}}_{\text{desired closed-loop}} + \underbrace{\left(\mathcal{J} - \mathcal{J}_1 \mathcal{R}_c^+ \mathcal{J}_2 \right) \left(\partial_x \mathbb{H} - \partial_{\hat{x}} \hat{\mathbb{H}} \right)}_{\text{error}} \end{aligned} \tag{5.2}$$

The error model is equivalent to a PHS system with $\mathcal{R}_\epsilon \equiv \mathcal{J}_1 \mathcal{R}_c^+ \mathcal{J}_2$ and $\partial_\epsilon \mathbb{H}(\epsilon) \equiv \left(\partial_x \mathbb{H} - \partial_{\hat{x}} \hat{\mathbb{H}} \right)$:

$$\dot{\epsilon} = (\mathcal{J} - \mathcal{R}_\epsilon) \partial_\epsilon \mathbb{H}(\epsilon) \tag{5.3}$$

which is stable in the Lyapunov sense:

$$\frac{d\mathbb{H}(\epsilon)}{dt} = \partial_\epsilon \mathbb{H}(\epsilon) \dot{\epsilon} = \partial_\epsilon \mathbb{H}(\epsilon) (\mathcal{J} - \mathcal{R}_\epsilon) \partial_\epsilon \mathbb{H}(\epsilon) \leq 0 \tag{5.4}$$

thanks to $\mathcal{J} = -\mathcal{J}^*$.

Remark 5. The ideal control law derived in the second equation of (4.6) seems to be evident when the hyperbolic property is entirely replaced by the parabolic one on preserving the Dirac structure. However, to analytically deduce the control signal u is not trivial, especially in the under-actuated case where g is not full rank. In [Vu et al., 2015b], we propose a simultaneous use of the average distributed control and the boundary one to stabilize the error from this matching equation. First, an average distributed control approach the closed-loop system to a desired one, which admits an error from matching equation. Then the boundary control, using Volterra transformation to describe the propagation of boundary effects, stabilizes the error system.

An example of a vibro-acoustic system is investigated as a demonstration of the proposed control method in the following section.

6. EXAMPLE

6.1 Wave equation for vibro-acoustic system

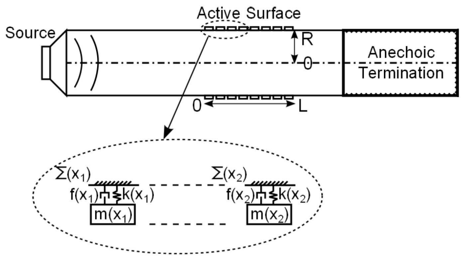

Fig. 1. Vibro-acoustic system

The simplified vibro-acoustic system [Collet et al., 2009] is employed to illustrate the proposed method. This system consists of an acoustic wave traveling in a tube equipped with a network of microphones/ loudspeakers without energy losses . [Collet et al., 2009]. The wave source is the loudspeaker at one side of the tube, and an anechoic chamber preventing any reflection on the other side. The issue is to reduce the acoustic wave inside and at the output of the tube. In [Collet et al., 2009], the active surface partly covered the tube's wall was proposed (see subsection 6.2).

The 2D PHS model is derived in [Trenchant et al., 2015] with the cylindrical symmetry, the 2D operator is then a special "spatial derivative operator" $\mathrm{d} = [\partial_z, \partial_r]^T$:

$$\begin{pmatrix} \partial_t \Phi \\ \partial_t \Gamma \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} v \\ P \end{pmatrix} \tag{6.1}$$

where Φ, v, Γ, P are respectively the kinetic momentum, velocity, volumetric expansion, and the pressure of the particle at the point $(z, r) \in [0, L] \times [0, R] \subset \mathbb{R}^2$.

Nevertheless, to obtain an appropriate control model which will serve as illustration for our proposed control law, we assume that $L \gg R$, meaning the wave is keeping a constant propagation on radius axis r with $\partial_r . = 0$, and all the variables now depend only on z. Hence, we consider till the end the equivalent 1D model:

$$\begin{pmatrix} \partial_t \Phi \\ \partial_t \Gamma \end{pmatrix} = \begin{pmatrix} 0 & -\partial_z \\ -\partial_z & 0 \end{pmatrix} \begin{pmatrix} v \\ P \end{pmatrix} \tag{6.2}$$

with the constitutive relation:

$$\begin{cases} v = & \Phi / \mu_0 \\ P = & \Gamma / \chi_s \end{cases} \tag{6.3}$$

where μ_0 and χ_s are the air mass density and the adiabatic compressibility coefficient respectively. The energy power is thus equivalent to a quadratic form:

$$\mathbb{H} = \frac{1}{2}\int_0^L \left(\frac{\Phi^2}{\mu_0} + \frac{\Gamma^2}{\chi_s}\right) dz \tag{6.4}$$

Finally, the boundary variables are defined (similar in [Trenchant et al., 2015]), with respect to the system derivative operator $\mathcal{J} = \begin{pmatrix} 0 & -\partial_z \\ -\partial_z & 0 \end{pmatrix}$ such that the system passivity is satisfied:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} -P(L) \\ v(0) \\ v(L) \\ P(0) \end{pmatrix} \tag{6.5}$$

The anechoic termination yields the boundary condition [Collet et al., 2009], that is no reflected wave at the termination:

$$\partial_z P|_{z=L} = -1/c_0 \partial_t P|_{z=L} \tag{6.6}$$

where c_0 is the speed of sound in the air.

6.2 Actuator - active surface and control design

The elastic surface is attached to the wall $r = R$ of the cylinder to attenuate the acoustic wave along the wall and at the output of cylinder. The infinite array of parallel mass-spring-damper model used to represent the active wall is also presented in Hamiltonian formulation in [Trenchant et al., 2015].

In this work, we only consider the wall effect as a pressure force along the wall, which is a distributed control $u(z)$. The system with active surface can be considered as:

$$\begin{pmatrix} \partial_t\Phi \\ \partial_t\Gamma \end{pmatrix} = \begin{pmatrix} 0 & -\partial_z \\ -\partial_z & 0 \end{pmatrix} \begin{pmatrix} v \\ P \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{6.7}$$

The main objective of the control is to minimize the acoustic wave at the anechoic terminator $z = L$, as well as along the wall. In this purpose, the proposed controller which makes the closed-loop system parabolic is a promising solution.

The CbI-IDA control law (4.6) is applied to this system *i.e.*:

$$gu = -\mathcal{J}\partial_x\mathbb{H} + \mathcal{J}_1\left(\mathcal{R}_c^+\right)\mathcal{J}_2\partial_x\mathbb{H} \tag{6.8}$$

with $\partial_x\mathbb{H} = \begin{pmatrix} v \\ P \end{pmatrix}$ and $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The control parameters $\mathcal{J}_1 = \mathcal{J}_2$ are fixed to $\begin{pmatrix} 0 & \partial_z \\ \partial_z & 0 \end{pmatrix}$, the only tunable parameter is $\mathcal{R}_c^+$ which is chosen in a simple form $\mathcal{R}_c^+ = \begin{pmatrix} R_{c1} & 0 \\ 0 & R_{c2} \end{pmatrix} > 0$, that implies $R_{c1}, R_{c2} > 0$. We then have:

$$\begin{cases} u &= \partial_z P + \partial_z\left(R_{c2}\partial_z v\right) \\ 0 &= \partial_z v + \partial_z\left(R_{c1}\partial_z P\right) \end{cases} \tag{6.9}$$

Note that we are dealing with an under-actuated system, where the distributed control only acts on a half of system states . In (6.9), when the control law u satisfies the first equation with a free choice of R_{c2}, Φ is stabilized:

$$\partial_t\Phi = \partial_z\left(R_{c2}\partial_z v\right) = \partial_z\left(R_{c2}\partial_z\frac{\Phi}{\mu_0}\right) \tag{6.10}$$

The second one is not fulfilled. However, it is not really necessary in this particular case since Γ is indeed stabilized because the relation:

$$\partial_t\Gamma = -\partial_z v = -\partial_z\frac{\Phi}{\mu_0} \tag{6.11}$$

still stands. The only action u cannot fulfill the control objective: transform a wave equation into a pure diffusion equation. However, in this particular example, we can easily prove that Γ is diffusive according to Φ by taking the spatial derivative of equation (6.10) and then using the relation (6.11):

$$\begin{aligned} \partial_z\left(\partial_t\Phi\right) &= \partial_z\partial_z\left(R_{c2}\partial_z\frac{\Phi}{\mu_0}\right) \\ \Leftrightarrow\ \partial_t\left(-\mu_0\partial_t\Gamma\right) &= -\partial_z\left(\partial_z\left(R_{c2}\partial_t\Gamma\right)\right) \\ \Leftrightarrow\ \partial_t\Gamma - \partial_t\Gamma_0 &= \frac{1}{\mu_0}\partial_z\left(\partial_z\left(R_{c2}\Gamma\right)\right) \end{aligned} \tag{6.12}$$

where the diffusivity coefficient is the same as in (6.10).

Remark 6. The result in equation (6.12) yields the diffusion of the time variation $\partial_t\Gamma$. Γ itself is also governed by a diffusion equation but its initial condition is not affected by the feedback control. This point will be shown in the following simulation part.

Remark 7. This example is a good illustration of the under-actuated control case. Yet another actuator limit can be considered. The ideal case assumes that u is infinite distributed control, while in practice, there is only a limit number of actuators along the wall. We call it a finite rank distributed control. How to choose a reasonable approximation of the feedback control u in equation (6.9) is an open issue for some future works. However, one can consider the solution using piecewise constant approximations method [Macchelli et al., 2015] to be able to locally linearize the infinite system at the corresponding actuator position.

6.3 Simulation

The vibro-acoustic model is simulated using Matlab. The system parameters used in this example can be found in table 1.

L	$1.84\,[m]$	tube length
μ_0	$1.204\,\left[kg/m^3\right]$	air mass density
χ_S	$7.0432 \times 10^{-6}\,\left[Pa^{-1}\right]$	adiabatic compressibility
c_0	$343.4\,[m/s]$	speed of sound

Table 1. System parameters.

Open-loop and closed-loop (with the aforementioned control strategy) simulations are shown in Figure 2 where a sound source is imposed at $z = 0$ and initial conditions are chosen equal to zero.

The condition at the anechoic termination (no reflected wave) is correctly simulated.

Figure 3 shows the feedback control result with the initial condition. The result illustrates the remark 6, in which it is stated that Γ is diffusive with the feedback control, but the initial condition stands still. Otherwise, the variation $\partial_t\Gamma$ is diffusive despite of the initial conditions.

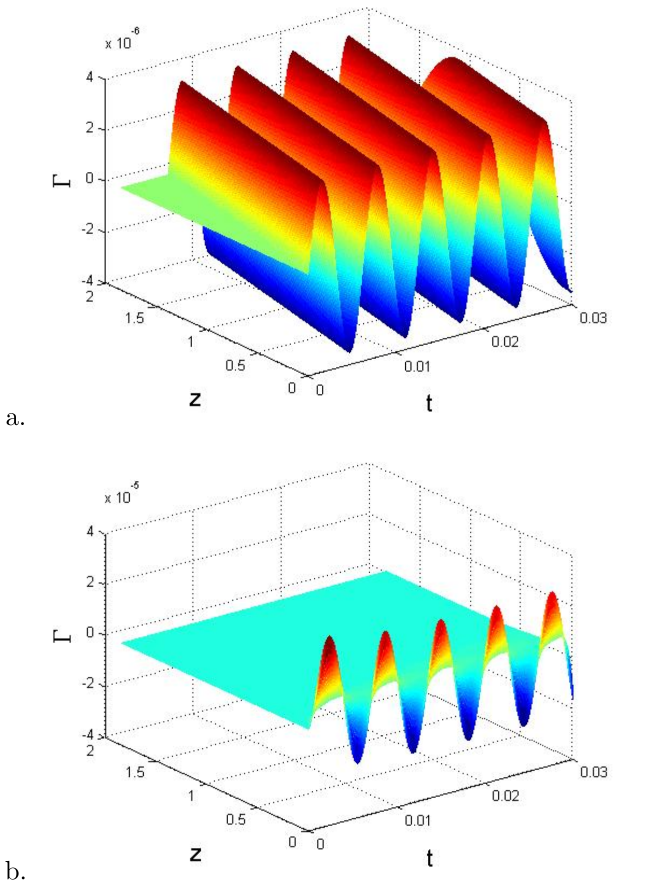

Fig. 2. Γ in open and closed-loop with sound source at $z = 0$ and without initial condition

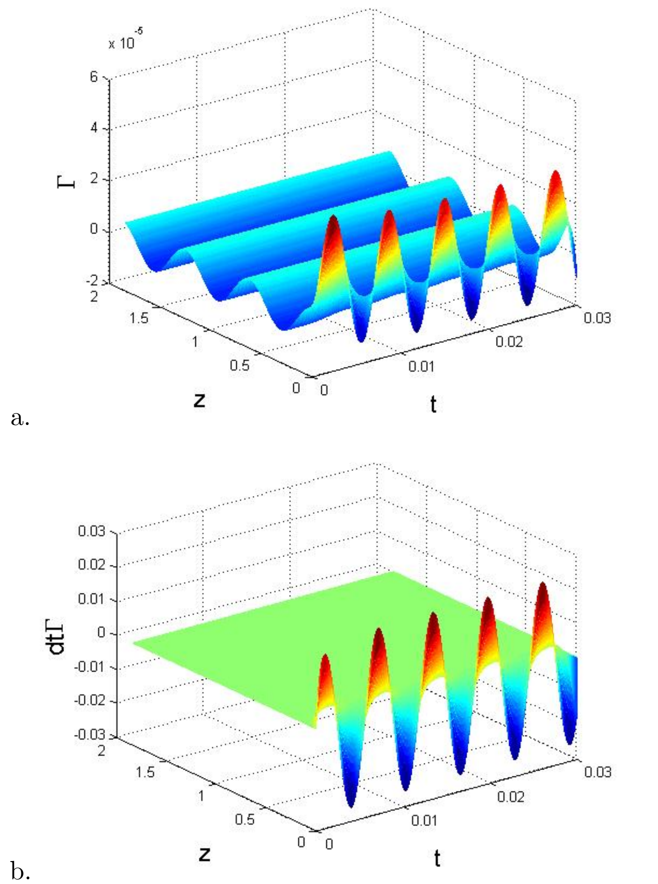

Fig. 3. Γ and $\partial_t\Gamma$ with sound source at $z = 0$ and initial condition

7. CONCLUSION

A distributed controller which transforms a hyperbolic PHS system into a parabolic one has been proposed. The control design has been proved to be robust with uncertainties as well as external disturbances. The proposed method is illustrated on a vibro-acoustic example, where an exponential convergence has been obtained as expected. Among the immediate prospects of this work are the design of finite rank distributed control approximations and two-dimensional extensions.

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