ON THE PINNING CONTROLLABILITY OF COMPLEX NETWORKS USING PERTURBATION THEORY OF EXTREME SINGULAR VALUES. APPLICATION TO SYNCHRONISATION IN POWER GRIDS

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ABSTRACT. Pinning control on complex dynamical networks has emerged as a very important topic in recent trends of control theory due to the extensive study of collective coupled behaviors and their role in physics, engineering and biology. In practice, real-world networks consist of a large number of vertices and one may only be able to perform a control on a fraction of them only. Controllability of such systems has been addressed in [16], where it was reformulated as a global asymptotic stability problem. The goal of this short note is to refine the analysis proposed in [16] using recent results in singular value perturbation theory.

1. INTRODUCTION

In recent years, extensive efforts have been devoted to the control of complex dynamical networks [13, 22]. Real networks for which such problems occur are of paramount importance in the natural sciences and engineering [16]. In particular, the examples of contemporaneous beats of the heart cells [25] or the synchronous behaviors of the cells of the suprachiasmatic nucleus in the brain [17] have been considered in the literature. Social networks are also fascinating examples where the formation of mass opinions and the emergence of collective behaviors are frequently observed. One major issue is that real world networks usually consist of a very large number of nodes and links which makes it impossible to apply control actions to all nodes.

Pinning control is a new way to address such control problems by placing local feedback injections on a small fraction of the nodes. Controllability of such systems has been addressed in [16], where it was reformulated as a global asymptotic stability problem. The goal of this short note is to refine the analysis proposed in [16] using recent results in singular value perturbation theory.

1.1. **Notations.** The Kronecker symbol is denoted by $\delta_{i,j}$, *i.e.*, $\delta_{i,j} = 1$ if $i = j$ and is equal to zero otherwise. We denote by $\|x\|_2$ the euclidian norm of a vector x and by $\|A\|$ the associated operator norm (spectral norm) of a matrix A. Its transposition is denoted by A^t.

For any symmetric matrix $B \in \mathbb{R}^{d \times d}$ we will denote its eigenvalues by $\lambda_1(B) \geq \cdots \geq \lambda_d(B)$. The largest eigenvalue will sometimes also be denoted by $\lambda_{\max}(B)$ and the smallest by $\lambda_{\min}(B)$. The smallest nonzero eigenvalue of a positive semi-definite matrix B will be denoted by $\lambda_{\min>0}(B)$. Finally, deg_i is the degree of a vertex i in a given graph, that is, its number of edges.

1.2. **The model.** One considers a set of N n-dimensional oscillators governed by a system of nonlinear differential equations. Moreover, we assume that each oscillator is coupled with a restricted set of other oscillators. This coupling relationship can be efficiently described using a graph where the vertices are indexed by the oscillators and there is an edge between two oscillators if they are coupled. The overall dynamical system is given by the following set of differential equations

$$(1.1) \qquad x_i'(t) \;=\; f(x_i(t)) - \sigma B \sum_{j=1}^{N} l_{ij} x_j(t) + u_i(t), \ t \geq t_0,$$

$i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^n$ is the state of the ith oscillator, $\sigma > 0$, $B \in \mathbb{R}^{n \times n}$, $f : \mathbb{R} \to \mathbb{R}$ describes the dynamics of each oscillator, $L = (l_{ij})_{i,j=1,\ldots,N}$ is the graph Laplacian of the underlying graph, and $u_i(t)$, $i = 1, \ldots, N$ are the controls. For the system to be well defined, we have to specify some initial conditions $x_i(t_0) = x_{i0}$ for $i = 1, \ldots, N$.

1.3. **The control problem.** Assume that we have a reference trajectory $s(t)$, $t \geq t_0$ satisfying the differential equation

$$s'(t) \;=\; f(s(t)).$$

Our goal is to control the system using a limited number of nodes. The selected nodes are called the "pinned nodes". For this purpose, we use a linear feedback law of the form

$$u_i(t) \;=\; p_i K e_i(t),$$

where $e_i(t) = s(t) - x_i(t)$, K is a feedback gain matrix, and where

$$p_i \;=\; \begin{cases} 1 \ \text{ if node } i \text{ is pinned} \\ 0 \ \text{ otherwise.} \end{cases}$$

Let P denote the diagonal matrix with diagonal $(p_1, \ldots, p_N)$.

1.4. **Controllability.** In [16], the authors propose a definition for (global pinning-) controllability (based on Lyapunov stability criteria):

Definition 1.1. *We say that the system (1.1) is controllable if the error dynamical system $e := (e_i(t))_{1 \leq i \leq N}$ is Lyapunov stable around the origin,*

i.e., there exists a positive definite function V such that $\frac{d}{dt}V(e(t)) < 0$ when $e(0) \neq 0$.

The following result, [16, Corollary 5], provides a sufficient condition for a system to be controllable:

Proposition 1.2 ([16])**.** *Assume that f is such that there exists a bounded matrix $F_{\xi,\widetilde{\xi}}$, whose coefficients depend on ξ and $\tilde{\xi}$, which satisfies*

$$(1.2) \qquad F_{\xi,\widetilde{\xi}}\left(\xi - \widetilde{\xi}\right) \;=\; f(\xi) - f(\widetilde{\xi}), \quad \xi, \widetilde{\xi} \in \mathbb{R}^n.$$

Let $Q \in \mathbb{R}^{n \times n}$ be a positive definite matrix such that

$$
\begin{aligned}
QK + K^t Q^t &= \kappa\left(QB + B^t Q^t\right) \\
\left(QB + B^t Q^t\right) &\succeq 0
\end{aligned}
$$

and

$$(1.3) \qquad \frac{1}{2}\,\lambda_N\left(\sigma L + \kappa P\right)\,\lambda_n\left(QB + B^t Q^t\right) \;>\; \sup_{\xi,\widetilde{\xi}}\|F_{\xi,\widetilde{\xi}}\|\,\|Q\|.$$

Then the system is controllable.

Many systems of interest satisfy the constraint specified in Proposition 1.2, see [10]. This proposition is very useful for node selection via the matrix P. Indeed, assume that Q is selected, then one may try to maximise $\lambda_N\left(\sigma L + \kappa P\right)$ as a function of P, under the constraint that no more than r nodes can be pinned. This is a combinatorial problem that can be relaxed using semi-definite programming or various heuristics [6].

1.5. **Goal of the paper.** Our goal in the present note is to propose an easy controllability condition refining [16, Corollary 7], based on the algebraic connectivity of the graph, the number of pinned nodes, the coupling strength and the feedback gain. Our approach is based on perturbation theory of the extreme singular values of a matrix after appending a column. The basic results of this theory are given in the appendix.

2. Main result

2.1. **A simple criterion for controllability.** Our main result is the following theorem.

Theorem 2.1. *Let $Q \in \mathbb{R}^{n \times n}$ be a positive definite symmetric matrix that satisfies*

$$
\begin{aligned}
QK + K^t Q^t &= \kappa\left(QB + B^t Q^t\right) \\
\left(QB + B^t Q^t\right) &\succeq 0,
\end{aligned}
$$

and assume that

$$(2.4) \qquad \sigma\lambda_{\min>0}(L)\,\lambda_{\min}\left(QB + B^t Q^t\right) \;>\; \sup_{\xi,\widetilde{\xi}}\|F_{\xi,\widetilde{\xi}}\|\,\|Q\|.$$

If κ satisfies

$$(2.5) \qquad \kappa \ \geq \ \frac{\sum_{i=1}^{r} \deg_i}{\sigma\lambda_{\min>0}(L) - \frac{2\,\|F_{\xi,\tilde{\xi}}\|\,\|Q\|}{\lambda_{\min}(QB+B^tQ^t)}} + \sigma\lambda_{\min>0}(L),$$

then the system is controllable.

Proof. We follow the same steps as for the proof of Corollary 7 in [16]. We assume without loss of generality that the first r nodes are the pinned ones. We may write P as

$$P \ = \ \sum_{i=1}^{r} e_i e_i^t,$$

where e_i is the ith member of the canonical basis of $\mathbb{R}^N$, i.e., $e_i(j) = \delta_{i,j}$. We will try to compare $\lambda_N(\sigma L + \kappa P)$ with $\lambda_N(\sigma L)$ and use Proposition 1.2 to obtain a sufficient condition for controllability based on L, i.e. the topology of the network. For this purpose, let us recall that L can be written as

$$L \ = \ \mathcal{I} \cdot \mathcal{I}^t,$$

where $\mathcal{I}$ is the incidence matrix of any directed graph obtained from the system's graph by assigning an arbitrary sign to the edges [2]. Of course L will not depend on the chosen assignment. Using this factorization of L, we obtain that

$$\sigma L + \kappa \sum_{i=1}^{r} e_i e_i^t \ = \ \left[\sqrt{\kappa}\,e_r, \ldots, \sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right] \left[\sqrt{\kappa}\,e_r, \ldots, \sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]^t.$$

Moreover, $\lambda_{\min>0}(\sigma L + \kappa P)$ can be expressed easily as the smallest nonzero eigenvalue of the rth term of a sequence of matrices with shape (A.16) for which we can use Theorem A.2 iteratively. Indeed, we have

$$\lambda_{\min>0}(\sigma L + \kappa e_1) \ = \ \lambda_{\min>0}\left(\left[\sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]^t \left[\sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]\right).$$

Let us denote by x the vector $\sqrt{\kappa}\,e_1$ and by X the matrix $[\sqrt{\sigma}\mathcal{I}]$. Then, we have that

$$\left[\sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]^t \left[\sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right] \ = \ \left[\begin{array}{cc} x^t x & x^t X \\ X^t x & X^t X \end{array}\right].$$

Therefore, Theorem A.2 gives

$$\lambda_{\min>0}\left(\sigma L + \kappa e_1 e_1^t\right) \ \geq \ \sigma\lambda_{\min>0}(L) - \frac{\deg_1}{(\kappa - \sigma\lambda_{\min>0}(L))},$$

where $\deg_1$ is the degree of node number 1.

Let us now consider $\lambda_{\min>0}(\sigma L + \kappa\,e_1 + \delta_2 e_2)$. We have that

$$\lambda_{\min>0}(\sigma L + \kappa\,e_1 + \delta_2 e_2) \ = \ \lambda_{\min>0}\left(\left[\sqrt{\kappa}\,e_2, \sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]^t \left[\sqrt{\kappa}\,e_2, \sqrt{\kappa}\,e_1, \sqrt{\sigma}\mathcal{I}\right]\right).$$

Let us denote by x the vector $\sqrt{\kappa}\ e_2$ and by X the matrix $[\sqrt{\kappa}\ e_1, \sqrt{\sigma}\mathcal{I}]$. Then, we have that

$$\left[\sqrt{\kappa}\ e_2, \sqrt{\kappa}\ e_1, \sqrt{\sigma}\mathcal{I}\right]^t \left[\sqrt{\kappa}\ e_2, \sqrt{\kappa}\ e_1, \sqrt{\sigma}\mathcal{I}\right] \quad = \quad \left[\begin{array}{cc} x^t x & x^t X \\ X^t x & X^t X \end{array} \right]$$

and using Theorem A.2 again, we obtain

$$\lambda_{\min>0}\left(\sigma L + \kappa e_1 e_1^t + \kappa e_2 e_2^t\right) \quad \geq \quad \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) - \frac{\deg_2}{(\kappa - \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t))}.$$

Since $\lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) \leq \lambda_{\min>0}(\sigma L)$, we thus obtain

$$\lambda_{\min>0}\left(\sigma L + \kappa e_1 e_1^t + \kappa e_2 e_2^t\right) \quad \geq \quad \lambda_{\min>0}(\sigma L + \kappa e_1 e_1^t) - \frac{\deg_2}{(\kappa - \sigma\lambda_{\min>0}(L))}.$$

We can repeat the same argument r times and obtain

$$(2.6) \qquad \lambda_{\min>0}\left(\sigma L + \kappa P\right) \quad \geq \quad \sigma\lambda_{\min>0}(L) - \frac{\sum_{i=1}^{r} \deg_i}{\kappa - \sigma\lambda_{\min>0}(L)}.$$

Finally, by Proposition 1.2, we know that the following constraint is sufficient for preserving controllability

$$(2.7) \qquad \lambda_{\min>0}\left(\sigma L + \kappa \sum_{i=1}^{r} e_i e_i^t\right) \quad \geq \quad \frac{2\,\|F_{\xi,\tilde{\xi}}\|\,\|Q\|}{\lambda_{\min}\left(QB + B^t Q^t\right)}.$$

By (2.6), it is sufficient to guarantee the controllability of our system to impose

$$\sigma\lambda_{\min>0}(L) - \frac{\sum_{i=1}^{r} \deg_i}{\kappa - \sigma\lambda_{\min>0}(L)} \quad \geq \quad \frac{2\,\|F_{\xi,\tilde{\xi}}\|\,\|Q\|}{\lambda_{\min}\left(QB + B^t Q^t\right)}.$$

Then, combining (2.7) with (2.4) implies that

$$\kappa \quad \geq \quad \frac{\sum_{i=1}^{r} \deg_i}{\sigma\lambda_{\min>0}(L) - \frac{2\,\|F_{\xi,\tilde{\xi}}\|\,\|Q\|}{\lambda_{\min}(QB + B^t Q^t)}} + \sigma\lambda_{\min>0}(L)$$

is a sufficient condition for controllability. $\square$

2.2. **Consequence of the formula.** What the formula (2.5) tells us is that the nodes of smallest degree should be selected before the others. The importance of considering the smallest degrees first comes as a natural consequence of our formula in a very explicit fashion, and the proof of the formula is slightly more transparent than the argument provided, *e.g.*, in [16].

2.3. **Comparison with previous works.** Closely related statements in the context of numerical and theoretical optimization of pinning strategies have already been established in the literature (cf. [19] and [23]), while in [20] a more recent work on edge snapping provides interesting insight into optimal evolution of pinning gains. For instance, the idea of using the node degrees for choosing the pinned nodes of the network already appeared in [16] but not as explicitly as in our work, as it is a consequence of a subsequent qualitative discussion. Let us also mention that the importance of considering the degree sequence of the graph also appears in the more recent works [18] but only

via the maximum and minimum degree and [23], where the analysis is done via computational experiments.

Additionally, we used a perturbation theorem that is more precise than Weyl's perturbation bound, and therefore, we end up with tighter inequalities. Finally, κ can be explicitly found with our approach, while it is implicit in [16].

3. Application to synchronisation in power grids

Our methods can be applied to power network control and synchronization as described in [5, 15, 14]. We consider an AC power network modelled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $n = |\mathcal{V}|$ where the vertices represent the nodes and the edges represent the transmission lines. The network is also defined by a symmetric admittance matrix Y.

The circuit equations have a solution V_i at each node $i = 1, \ldots, N$. The set $\mathcal{V}$ of nodes is divided into three different subsets. The set $\mathcal{V}_1$ contains the index set of the load buses, $\mathcal{V}_2$ is the index set of the synchronous generators, and $\mathcal{V}_3$ is the index set of the grid connected direct current (DC) power sources. For lossless networks, the active power flow from node j to node j' is given by $a_{jj'} \sin(\theta_j - \theta_{j'})$ where θ_j is the argument of V_j and

$$(3.8) \qquad a_{jj'} = |V_j|\,|V_{j'}|\,\mathfrak{I}(Y_{jj'}).$$

The system is governed by the following equations. For load buses, we have

$$(3.9) \qquad D_j \frac{d\theta_j}{dt} + P_j^{(1)} = -\sum_{j'=1}^{N} a_{jj'} \sin(\theta_j - \theta_{j'}), \quad j \in \mathcal{V}_1,$$

where $P_j^{(1)} > 0$ is the active power drawn at node j (1 in the exponent is a notation to recall that we consider the set $\mathcal{V}_1$) and $D_j > 0$ is the damping factor. In the case of synchronous generators, we have

$$(3.10) \qquad M_j \frac{d^2\theta_j}{dt^2} + D_j \frac{d\theta_j}{dt} = P_j^{(m)} - \sum_{j'=1}^{N} a_{jj'} \sin(\theta_j - \theta_{j'}), \quad j \in \mathcal{V}_2,$$

where $P_j^{(m)} > 0$ is the mechanical power output (m in exponent is for *mechanical*) and $M_j > 0$ is the inertia factor. Finally, for DC power sources $\mathcal{V}_3$, we have:

$$(3.11) \qquad D_j \frac{d\theta_j}{dt} = P_j^{(d)} - \sum_{j'=1}^{N} a_{jj'} \sin(\theta_j - \theta_{j'}), \quad j \in \mathcal{V}_3,$$

where we assumed that each DC source is connected to the AC grid via and DC/AC inverter and that each inverter $j \in \mathcal{V}_3$ has a nominal value $P_j^{(d)} > 0$ and is equipped with a droop controller with droop slope $1/D_j$, following the model of [21].

After linearisation, we obtain the system

(3.12)
$$\frac{dx_j}{dt} = -\left[\begin{array}{c} 0 \\ \frac{P_j^{(1)}}{D_j} \end{array}\right] + \sum_{j'=1,\ j'\neq j}^{n} \frac{1}{D_j} \left[\begin{array}{cc} 0 & 0 \\ 0 & a_{jj'} \end{array}\right] x_{j'} - \left(\sum_{j'=1,\ j'\neq j}^{n} \frac{1}{D_j}\left[\begin{array}{cc} 0 & 0 \\ 0 & a_{jj'} \end{array}\right]\right) x_j, \quad j \in \mathcal{V}_1,$$

$$\frac{dx_j}{dt} = \left[\begin{array}{c} 0 \\ \frac{P_2^{(m)}}{M_2} \end{array}\right] \sum_{j'=1,\ j'\neq j}^{N} \left[\begin{array}{cc} 0 & 0 \\ \frac{a_{2,j}}{M_2} & 0 \end{array}\right] x_{j'}$$

(3.13)
$$- \left(\sum_{j'=1}^{N} \left[\begin{array}{cc} 0 & 0 \\ \frac{a_{j,j'}}{M_2} & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & \frac{D_2}{M_2} \\ 0 & 0 \end{array}\right] +\right) x_j, \quad j \in \mathcal{V}_2,$$

and

(3.14)
$$\frac{dx_j}{dt} = -\left[\begin{array}{c} 0 \\ \frac{P_j^{(1)}}{D_j} \end{array}\right] + \sum_{j'=1,\ j'\neq j}^{n} \frac{1}{D_j} \left[\begin{array}{cc} 0 & 0 \\ 0 & a_{jj'} \end{array}\right] x_{j'} - \left(\sum_{j'=1,\ j'\neq j}^{n} \frac{1}{D_j}\left[\begin{array}{cc} 0 & 0 \\ 0 & a_{jj'} \end{array}\right]\right) x_j, \quad j \in \mathcal{V}_3.$$

Denote by P the vector whose j^{th} component is $P_j^{(1)}$ if $j \in \mathcal{V}_1$, $P_j^{(m)}$ if $j \in \mathcal{V}_2$, and $P_j^{(d)}$ otherwise. The results of the previous section readily apply to the system described by (3.12), (3.13) and (3.14) if we duplicate the components of P in order to take into account that the state at each bus is two-dimensional.

Figures 1, 2, and 3 show the evolution of the phase at each node for the system governed by (3.12), (3.13), and (3.14). The way we obtained them is described in details hereafter. As stated previously, we implemented our method in Python. In particular, we used the NetworkX [7] library for graph modelling, we used a SciPy [11] library module for simulating the power grid's dynamics, and Numpy [24] to compute with matrices and generate random objects. We generated the network based on 3 main parameters:

- the total number n of nodes (or buses, $n = 50$ in the simulations shown in the figures),
- a threshold T under which the coupling strength is ignored (set to 0 in the figures), and
- the number of nodes (ordered by increasing degree) that are pinned (10 here).

The vectors P, D, M and K which appear in formulas (3.9), (3.10), and (3.11) are chosen at random in [0,1]. The only components of K considered in the simulations are the ones associated by the pinned nodes. The coupling strength matrix a is random with i.i.d. components in [0,1]. This latter is used as connectivity matrix, except that edges with weight lower than T are removed. The n nodes are then sorted in increasing degree order, and divided into three sets, namely the $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$. The ODE's are then defined and solved numerically using the "integrate" SciPy module. The
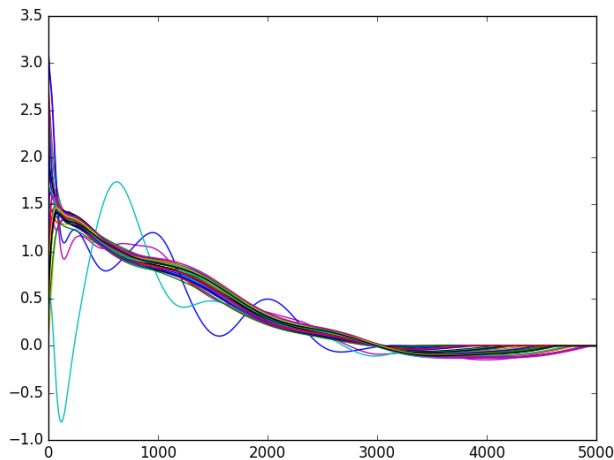
FIGURE 1. Evolution of the phase at each bus as a function of time for a random network

integrator chosen for these experiments was vode [3], which is a real-valued variable-coefficient ODE solver with fixed-leading-coefficient implementation, with a backward differentiation formulas method. The iteration method of the ODE solver's correction step is "chord iteration" with an internally generated full Jacobian. The maximum order used by the integrator was set to 5, while maximum number of (internally defined) steps allowed during one call to the solver was equal to 500,000. Finally, evolution of the phase at each bus has been plotted using matplotlib [9] library on data produced by the integrated ode object. As depicted in the 3 figures provided below, the method successfully controls the random power grid [1] in this randomly drawn example. We observed that the procedure successfully solved the control problem in most random instances.

4. CONCLUSION

In this article, an easy controllability condition refining [16, Corollary 7], based on the algebraic connectivity of the graph, the number of pinned nodes, the coupling strength, and the feedback gain, has been proposed. The result is based on perturbation theory of the extreme singular values of a matrix after appending a column. The method has finally been applied to power network control and synchronization, showing that the proposed results can readily be applied to such systems.

The application considered in the paper, which was an important motivation for our results, still raises important remaining questions. The most difficult is the one of how to handle nonlinear settings of the type arising in power grids. Future work will be devoted to this problem with many expected potential application to other types of networks as well.
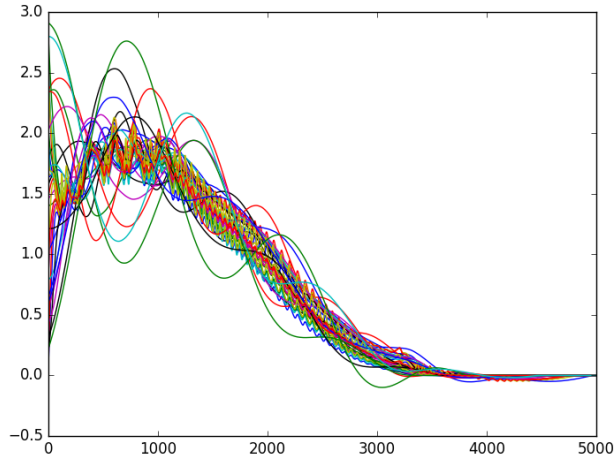
FIGURE 2. A second example of evolution of the phase at each bus as a function of time for a random network, with same setup
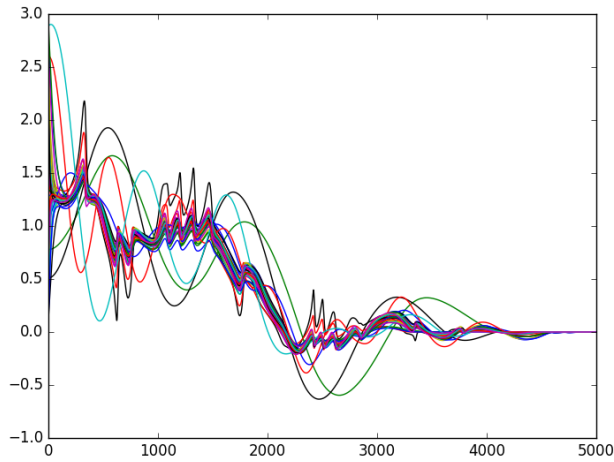

FIGURE 3. A last evolution of the phase at each bus, same configuration

APPENDIX A. PERTURBATION THEORY OF EXTREME SINGULAR VALUES AFTER APPENDING A COLUMN

A.1. **Framework.** Let d be an positive integer. Let $X \in \mathbb{R}^{d \times n}$ be a $d \times n$-matrix and let $x \in \mathbb{R}^d$ be a column vector. We denote by a subscript ^{t} the transpose of vectors and matrices. There exist at least two ways to study

the singular values of the matrix (x, X) obtained by appending the column vector x to the matrix X:

(A1) Consider the matrix

(A.15)
$$A = \begin{bmatrix} x^t \\ X^t \end{bmatrix} \begin{bmatrix} x & X \end{bmatrix} = \begin{bmatrix} x^t x & x^t X \\ X^t x & X^t X \end{bmatrix};$$

(A2) Consider the matrix

$$\widetilde{A} = \begin{bmatrix} x & X \end{bmatrix} \begin{bmatrix} x^t \\ X^t \end{bmatrix} = XX^t + xx^t.$$

On one hand, one may study in **(A1)** the eigenvalues of the $(n+1) \times (n+1)$ hermitian matrix A, i.e., the matrix $X^t X$ augmented with an arrow matrix.

On the other hand, one will deal in **(A2)** with the eigenvalues of the $d \times d$ hermitian matrix $\widetilde{A}$, which may be seen as a rank-one perturbation of XX^t. The matrices A and $\widetilde{A}$ have the same non-zeros eigenvalues, and in particular $\lambda_{\max}(A) = \lambda_{\max}(\widetilde{A})$. Moreover, the singular values of the matrix (x, X) are the square-root of the eigenvalues of the matrix A.

Equivalently, the problem of a rank-one perturbation can be rephrased as the one of controlling the perturbation of the singular values of a matrix after appending a column.

A.2. **A theorem of Li and Li.** In this paper, we use a slightly more general framework than **(A1)**, that is the case of a matrix

(A.16)
$$A = \begin{bmatrix} c & a^t \\ a & M \end{bmatrix},$$

where $a \in \mathbb{R}^d$, $c \in \mathbb{R}$ and $M \in \mathbb{R}^{d \times d}$ is a symmetric matrix.

The following theorem provides sharp upper bounds for $\lambda_{\max}(A)$, and lower bounds on $\lambda_{\min}(A)$, depending on various information on the submatrix M of A. As discussed above, this problem has close relationships with our problem of appending a column to a given rectangular matrix, because $\lambda_1(\widetilde{A}) = \lambda_1(A)$.

Theorem A.1 (Li-Li's inequality and a lower bound). *Let d be a positive integer and let $M \in \mathbb{C}^{d \times d}$ be an Hermitian matrix, whose eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_d$ with corresponding eigenvectors $(V_1, \cdots, V_d)$. Set $c \in \mathbb{R}$, $a \in \mathbb{C}^d$. Let A be given by (A.16). Therefore:*

(A.17)
$$\frac{2\langle a, V_1 \rangle^2}{\eta_1 + \sqrt{\eta_1^2 + 4\langle a, V_1 \rangle^2}} \leq \lambda_1(A) - \max(c, \lambda_1) \leq \frac{2\|a\|^2}{\eta_1 + \sqrt{\eta_1^2 + 4\|a\|^2}},$$

with

$$\eta_1 = |c - \lambda_1|.$$

A.3. **Perturbation of the smallest nonzero eigenvalue.** The same technics used to prove Theorem A.1 also give lower bounds for the smallest nonzero eigenvalue, which are also direct consequences of Li-Li's inequality. For more details, we refer the reader to [4].

Theorem A.2. *Let d be a positive integer and let $M \in \mathbb{C}^{d \times d}$ be a positive semi-definite Hermitian matrix, whose eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_d$ with corresponding eigenvectors $(V_1, \cdots, V_d)$. Set $c \in \mathbb{R}$, $a \in \mathbb{C}^d$. Let A be given by (A.16). Assume that M has rank $r \leq d$. Therefore:*

$$(A.18) \qquad \lambda_{r+1}(A) \geq \min(c, \lambda_r) - \frac{2\|a\|^2}{\eta_r + \sqrt{\eta_r^2 + 4\|a\|^2}},$$

with

$$\eta_r \;=\; |c - \lambda_r|.$$

In particular, the following perturbation bounds of Weyl and Mathias hold:

Corollary A.3.

$$(A.19) \qquad \lambda_{r+1}(A) \;\geq\; \min(c, \lambda_r) - \|a\|_2$$

$$(A.20) \qquad \lambda_{r+1}(A) \;\geq\; \min(c, \lambda_r) - \frac{\|a\|_2^2}{|c - \lambda_r|}.$$

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