Stabilization of infinite dimensional port-Hamiltonian systems by nonlinear dynamic boundary control

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Abstract

The conditions for existence of solutions and stability, asymptotic and exponential, of a large class of boundary controlled systems on a 1D spatial domain subject to nonlinear dynamic boundary actuation are given. The consideration of such class of control systems is motivated by the use of actuators and sensors with nonlinear behavior in many engineering applications. These nonlinearities are usually associated to large deformations or the use of smart materials such as piezo actuators and memory shape alloys. Including them in the controller model results in passive dynamic controllers with nonlinear potential energy function and/or nonlinear damping forces. First it is shown that under very natural assumptions the solutions of the partial differential equation with the nonlinear dynamic boundary conditions exist globally. Secondly, when energy dissipation is present in the controller, then it globally asymptotically stabilizes the partial differential equation. Finally, it is shown that assuming some additional conditions on the interconnection and on the passivity properties of the controller (consistent with physical applications) global exponential stability of the closed-loop system is achieved.

Keywords: Boundary control systems, port-Hamiltonian systems, nonlinear control, existence of solutions, stabilization.

1. Introduction

In many physical processes the effects produced by distributed phenomena cannot be neglected. This is for instance the case for transmission lines, flexible beams and plates, tubular and nuclear fusion reactors and wave propagation to cite a few. These processes are hence modelled using partial differential equations (PDE) in which state variables and parameters are time and spatial dependent. In many relevant applications the measurement and the actuation occurs on the spatial boundary of the system, hence what the controller actually imposes through the physical actuators are time varying boundary conditions. Formally this class of control systems are called boundary control systems (BCS).

In engineering applications BCS are often controlled using localized actuators which exhibit *nonlinear* behavior. These nonlinearities are for example related to large deformations of compliant structures (nonlinear springs) in mechanical systems or hysteresis behaviour of ferro and piezo electrical materials in electro mechanical systems. This is for instance the case of silicon made nanotweezers built up from beams which are controlled using electrostatic comb drives and attached through nonlinear silicon made suspensions (thin beams) (Boudaoud et al., 2012), nonlinear fluid structure interaction, such as in distributed control of vibro-acoustic systems through nonlinear loudspeakers (Collet et al., 2009) or the stability characterization of biomechanical processes such as the blood flow dynamics in bio-prosthetic heart valves (Borazjani, 2013) or the vocal cords dynamics (Ishizaka & Flanagan, 1972). The nonlinear components are generally associated to nonlinear constitutive laws of the driving forces, usually present in a potential energy term and to nonlinear damping phenomena related to nonlinear resistors and dampers, respectively.

In the linear case the existence of solutions, the stability and the design of stabilizing controllers can be tackled using linear semigroup theory and the associated abstract formulation based on unbounded input/output mappings (Curtain & Zwart, 1995). When asymptotic or exponential stability is concerned, the main difficulty remains in finding the appropriate Lyapunov function candidate to prove the stability. It is usually done on a case by case basis using physical considerations depending on the application field. When characterizing exponential stability, contrary to asymptotic stability, the conditions insuring the exponential convergence are quite rigid as the controller has to damp infinitely high frequency as well as all low frequency modes.

In the last decade an approach based on the extension of the Hamiltonian formulation to open distributed parameter systems (van der Schaft & Maschke, 2002) has been developed for modeling and control. It has been shown that distributed port-Hamiltonian systems encompass a large class of physical systems, including mechanical, electrical, electro-mechanical, hydraulic and chemical systems to mention some. See Duindam et al. (2009) for an extensive exposition and a large list of references. Regarding the extension of the Hamiltonian formulation to stabilizing control of BCS, in the 1D linear case it gave rise to the definition of boundary control port-Hamiltonian systems (BC-PHS) (Le Gorrec et al., 2004) and allowed to parametrize, by using simple matrix conditions, the boundary conditions that define a well-posed problem (Le Gorrec et al., 2005). Different variations around these first results can be found in (Villegas, 2007) and in (Jacob & Zwart, 2012). Well-posedness and stability have been investigated in open-loop and for static boundary feedback control in (Zwart et al., 2010) and (Villegas et al., 2005; Villegas et al., 2009) respectively, and linear dynamic boundary control has been studied in (Macchelli et al., 2017; Ramirez et al., 2014; Augner & Jacob, 2014; Villegas, 2007).

In this paper the results on existence of solution and stabilisation of linear dynamic boundary control of BC-PHS are generalized to the case of nonlinear boundary control. This class of systems is of real practical interest since the controllers are often implemented with actuators and sensors with nonlinear behavior, due for instance to large deformations, the use of smart materials or saturation phenomena. The same kind of problem has already been studied in (Miletić et al., 2016) and in (Augner, 2016) from a theoretical point of view. In (Miletić et al., 2016) LaSalle's invariance principle is used and precompactness of trajectories is established but asymptotic stability was only shown for a dense set of initial conditions. In Augner (2016) nonlinear contraction semigroups are used leading to quite strong assumptions on the class of considered nonlinearities. This approach differs from the methods that we use in this paper, which are based on nontrivial extensions of the asymptotic and exponential stability results presented in Zwart et al. (2016) and Ramirez et al. (2014), respectively, allowing to deal with very large class of nonlinearities. More precisely, a general class of passive boundary controllers, with nonlinear potential energy function and damping matrix is considered. This class of controllers encompasses mechanical, electrical and electromechanical systems among others. First it is shown that under natural assumptions on the nonlinear potential function and damping matrix the solutions of the PDE with this class of nonlinear dynamic boundary conditions exist globally. Then, it is shown that the most general form of this class of passive controllers globally asymptotically stabilizes the closed loop system (PDE + nonlinear ODE). Finally, it is shown that by restricting the nonlinear potential energy to functions with quasi quadratic bound and a full rank condition on the feedthrough term of the controller global exponential stability is achieved. The first part of this work, dealing with asymptotic stability, has been illustrated on the particular example of pure nonlinear damper in Zwart et al. (2016).

The paper is organized as follows. In Section 2 the definition and main properties of the considered class of PDE and nonlinear dynamic boundary controller are given. The existence and the uniqueness of the solutions of the PDE are established in Section 3. The asymptotic stability is studied in Section 4 while the exponential stability is addressed in Section 5. Finally some concluding remarks and comments to future work are given in Section 6.

2. Port-Hamiltonian systems with nonlinear boundary control

Throughout this article we assume that our distributed parameter system is modeled by a PDE of the following form

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right) + (P_0 - G_0)\mathcal{H}(\zeta)x(t,\zeta), \tag{1}$$

with $\zeta \in (a,b)$, $P_1 \in M_n(\mathbb{R})$[1] a nonsingular symmetric matrix, $P_0 = -P_0^\top \in M_n(\mathbb{R})$, $G_0 \in M_n(\mathbb{R})$ with $G_0 \geq 0$ and x taking values in $\mathbb{R}^n$. Furthermore, $\mathcal{H}(\cdot) \in L_\infty((a,b); M_n(\mathbb{R}))$ is a bounded and measurable, matrix-valued function satisfying for almost all $\zeta \in (a,b)$, $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^\top$ and $\mathcal{H}(\zeta) > mI$, with m independent from ζ.

For simplicity $\mathcal{H}(\zeta)x(t,\zeta)$ will be denoted by $(\mathcal{H}x)(t,\zeta)$. For the above PDE we assume that some boundary conditions are homogeneous, whereas others are controlled. Thus we consider two matrices $W_{B,1}$ and $W_{B,2}$ of appropriate sizes such that

$$u(t) = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix} \tag{2}$$

and

$$0 = W_{B,2}\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix}. \tag{3}$$

Furthermore, the boundary output is given by

$$y(t) = W_C\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix}. \tag{4}$$

To study the existence and uniqueness of solution to the above controlled PDE, we follow the semigroup theory, see also (Le Gorrec et al. (2005); Jacob & Zwart (2012)). Therefore we define the state space $X = L_2((a,b);\mathbb{R}^n)$ with inner product $\langle x_1, x_2\rangle_{\mathcal{H}} = \langle x_1, \mathcal{H}x_2\rangle$ and norm $\|x\|_{\mathcal{H}} = \sqrt{\langle x, x\rangle_{\mathcal{H}}}$. Note that due to the assumptions on $\mathcal{H}$ this is a norm on X and equivalent to the L_2 norm. Hence X is a Hilbert space. The reason for selecting this space is that $\|\cdot\|_{\mathcal{H}}^2$ is related to the energy function of the system, i.e., the total energy of the system equals $E = \frac{1}{2}\|x\|_{\mathcal{H}}^2$. The Sobolev space of order p is denoted by $H^p((a,b),\mathbb{R}^n)$.

Associated to the (homogeneous) PDE, *i.e.,* to the case $u(t) = 0$, we define the operator $Ax = P_1\frac{d}{d\zeta}(\mathcal{H}x) + (P_0 - G_0)\mathcal{H}x$ with domain

$$D(A) = \left\{\mathcal{H}x \in H^1((a,b);\mathbb{R}^n) \,\middle|\, \begin{bmatrix}(\mathcal{H}x)(b)\\(\mathcal{H}x)(a)\end{bmatrix} \in \ker W_B\right\},$$

where $W_B = \left[\begin{smallmatrix}W_{B,1}\\W_{B,2}\end{smallmatrix}\right]$. For the rest of the paper we make the following hypothesis.

Assumption 1. *For the operator A and the pde (1)–(4) the following hold:*

1. *The matrix W_B is an $n \times 2n$ matrix of full rank;*
2. *For $x_0 \in D(A)$ we have $\langle Ax_0, x_0\rangle_{\mathcal{H}} \leq 0$.*

[1] $M_n(\mathbb{R})$ denote the space of real $n \times n$ matrices

3. *The number of inputs and outputs are the same, k, and for classical solutions of (1)–(4) there holds $\dot{E}(t) \leq u(t)^\top y(t)$ with $E(t) = \frac{1}{2}\|x(t)\|^2_{\mathcal{H}}$.*

It follows from Assumption 1, points 1 and 2, that the system (1)–(4) is a boundary control system (see Le Gorrec et al. (2005); Jacob & Zwart (2012); Jacob et al. (2015)), and so for $u \in C^2([0,\infty);\mathbb{R}^k)$, $\mathcal{H}x(0) \in H^1((a,b);\mathbb{R}^n)$, satisfying (2) and (3) (for $t = 0$), there exists a unique classical solution to (1)–(4), (Jacob & Zwart, 2012, Theorem 11.2). Thus for this dense (in X) set of initial conditions and inputs, point 3 of Assumption 1 makes sense. We remark that the internal damping operator G_0 will hardly play a role in the proof of the existence of solutions. In Jacob et al. (2015) it is shown that item 2 of Assumption 1 implies that the same inequality holds with $G_0 = 0$. When stability is concerned, the worst case scenario corresponds to $G_0 = 0$, being the case $G_0 > 0$ less restrictive.

There is a special class of systems for which Assumption 1 is directly satisfied. If $k = n$ and if $W_B = W_{B,1}$ and W_C satisfy

$$W_B\tilde{\Sigma}W_B^\top = W_C\tilde{\Sigma}W_C^\top = 0 \qquad W_B\tilde{\Sigma}W_C^\top = W_C\tilde{\Sigma}W_B^\top = I$$

with $\tilde{\Sigma} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & -P_1^{-1} \end{bmatrix}$, the change of energy of the system becomes (Le Gorrec et al., 2005; Jacob & Zwart, 2012)

$$\dot{E}(t) = u^\top(t)y(t) - \langle G_0(\mathcal{H}x)(t,\cdot), x(t,\cdot)\rangle_{\mathcal{H}}.$$

Since the input and output act and sense at the boundary of the spatial domain, in the absence of internal dissipation ($G_0 = 0$) the system only exchanges energy with the environment through the boundaries. In this case the BCS fullfils

$$\dot{E}(t) = u^\top(t)y(t). \tag{5}$$

Consider that the BCS is interconnected through its boundary with a nonlinear finite dimensional controller in a power preserving way i.e.,

$$\begin{aligned} u &= r - y_c, \\ y &= u_c, \end{aligned} \tag{6}$$

with $u_c \in \mathbb{R}^k$, $y_c \in \mathbb{R}^k$ the input and output of the controller, respectively, and $r \in \mathbb{R}^k$ the new input of the closed loop system. The feedback is illustrated in Figure 1. In what follows we consider the regulation problem and for a sake of clarity focus on $r = 0$.

Definition 2. *Consider a nonlinear control system given by the following state space representation*

$$\begin{cases} \dot{v}_1 &= K_2 v_2 \\ \dot{v}_2 &= -\frac{\partial\mathcal{P}}{\partial v_1}(v_1)^\top - R(K_2v_2) + B_c u_c \\ y_c &= B_c^\top K_2 v_2 + S_c u_c \end{cases} \tag{7}$$

where $v_1 \in \mathbb{R}^{n_c}$, $v_2 \in \mathbb{R}^{n_c}$, form the components of the state vector, $B_c \in M_{k,n_c}(\mathbb{R})$, $K_2 \in M_{n_c}(\mathbb{R})$, $K_2 = K_2^\top$, $K_2 > 0$, $S_c \in M_k(\mathbb{R})$ with $S_c = S_c^\top$ and $S_c \geq 0$. Furthermore, $\frac{\partial\mathcal{P}}{\partial v_1}$ is the (Fréchet) derivative of the scalar-valued function $\mathcal{P} : \mathbb{R}^{n_c} \mapsto [0,\infty)$, i.e., $\frac{\partial\mathcal{P}}{\partial v_1} : \mathbb{R}^{n_c} \mapsto M_{1,n_c}(\mathbb{R})$. We assume that R and $\frac{\partial\mathcal{P}}{\partial v_1}$ are locally Lipschitz continuous functions. The Hamiltonian (energy) associated to this system is given by

$$E_c(v_1, v_2) = \mathcal{P}(v_1) + \frac{1}{2}v_2^\top K_2 v_2. \tag{8}$$

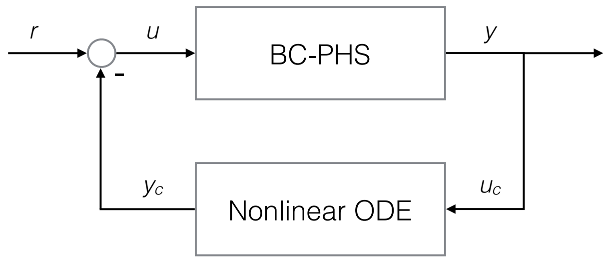

Figure 1: Power preserving interconnection

All along this paper we use the term controller to refer to the ensemble controller - sensors - actuators. In this context, the above class of nonlinear controllers encompasses for example mechanical actuators with nonlinear stiffness and/or damping, mechanical systems with saturations and electrical components with nonlinear capacitance. These type of models are frequently encountered in micro-mechanical systems, such as micro-grippers and controlled flexible structures, or fluid structure interaction processes.

Since the nonlinear terms in the differential equation (7) are locally Lipschitz continuous, it possesses for every initial condition a unique (local) solution. Furthermore, the change of energy along solutions satisfies

$$\dot{E}_c(t) = u_c(t)^\top y_c(t) - v_2(t)^\top K_2 R(K_2 v_2(t)) - u_c(t)^\top S_c u_c(t). \tag{9}$$

For the two systems being interconnected in the power preserving manner (6), the closed-loop energy function E_{tot} is given by

$$E_{\text{tot}}(t) = E(t) + E_c(t). \tag{10}$$

The closed-loop system obtained by applying (6) can be written as the abstract nonlinear differential equation

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}f(\tilde{x}) \tag{11}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ v_1 \\ v_2 \end{bmatrix},$$

the linear part equals

$$\tilde{A}\tilde{x} = \begin{bmatrix} P_1\frac{d}{d\zeta}(\mathcal{H}x) + (P_0 - G_0)\mathcal{H}x \\ K_2v_2 \\ -Iv_1 + B_cW_C\begin{bmatrix}(\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix} \end{bmatrix}$$

with domain

$$D(\tilde{A}) = \left\{\mathcal{H}x \in H^1(a,b;\mathbb{R}^n), v_1, v_2 \in \mathbb{R}^{n_c} \,\middle|\, \begin{bmatrix}(\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \\ v_2\end{bmatrix} \in \ker \widetilde{W}_D \right\},$$

with

$$\widetilde{W}_D = \begin{bmatrix} W_{B,1} + S_c W_C & B_c^\top K_2 \\ W_{B,2} & 0 \end{bmatrix},$$

$\tilde{B} = \begin{bmatrix} 0 & 0 & I \end{bmatrix}^\top$, and

$$f(\tilde{x}) = v_1 - \frac{\partial \mathcal{P}}{\partial v_1}(v_1)^\top - R(K_2 v_2). \tag{12}$$

As state space we choose $\tilde{X} = X \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ with inner product $\langle \tilde{x}_1, \tilde{x}_2 \rangle_{\tilde{X}} = \langle x_1, \mathcal{H} x_2 \rangle + \langle v_{1_1}, v_{1_2} \rangle + \langle v_{2_1}, K_2 v_{2_2} \rangle$ and norm $\|\tilde{x}\|^2 = \langle \tilde{x}, \tilde{x} \rangle_{\tilde{X}}$. Using similar arguments as in Ramirez et al. (2014) and in (Villegas, 2007, Chapter 5) the following is quickly shown.

Lemma 3. *The linear operator $\tilde{A}$ with its domain generates a contraction semigroup on $\tilde{X}$. Moreover, $\tilde{A}$ has a compact resolvent.*

3. Existence of solutions

In this section it is shown that the closed-loop system is well posed, i.e., that the closed-loop solutions exist locally. Under some mild assumptions on the nonlinear potential energy function and damping matrix of the controller we show the global existence of the solutions.

Assumption 4. *The potential energy function $\mathcal{P}$ has a unique minimum at $v_1 = 0$, i.e., $\mathcal{P}(v_1) > \mathcal{P}(0) = 0$ for $v_1 \neq 0$. Furthermore, $\mathcal{P}$ is radially unbounded. Thus if $\|v_1\| \to \infty$, then $\mathcal{P}(v_1) \to \infty$.*

That R represents damping is assumed next.

Assumption 5. *The function R is a function of v_2 and for all v_2 it satisfies*

$$v_2^\top K_2 R(K_2 v_2) \geq 0.$$

Remark 6. *Notice that since $K_2 = K_2^\top > 0$, Assumption 5 is equivalent to*

$$\tilde{v}_2^\top R(\tilde{v}_2) \geq 0, \quad \text{for all } \tilde{v}_2.$$

Theorem 7. *The system (11) satisfying Assumption 1 with the nonlinear term (12) satisfying Assumptions 4 and 5 possesses for every initial condition a unique mild solution which is uniformly bounded. Furthermore,*

$$E_{\text{tot}}(t) \leq E_{\text{tot}}(0) - \int_0^t \begin{bmatrix} (\mathcal{H}x)(\tau, b) \\ (\mathcal{H}x)(\tau, a) \end{bmatrix}^\top W_C^\top S_c W_C \begin{bmatrix} (\mathcal{H}x)(\tau, b) \\ (\mathcal{H}x)(\tau, a) \end{bmatrix} d\tau - \int_0^t v_2^\top(\tau) K_2 R(K_2 v_2(\tau)) d\tau. \tag{13}$$

Proof. Since f is a locally Lipschitz continuous function on $\tilde{X}$, and since $\tilde{B}$ is a bounded linear mapping, it follows from e.g. (Pazy, 1983, Chapter 6, Theorem 1.4) that for every initial condition, the closed-loop equation possesses a unique mild solution on some time interval $[0, t_{\max})$. If the initial condition is in the domain of $\tilde{A}$, then this mild solution is classical, see (Zheng, 2004, Theorem 2.5.4).

Consider the total energy E_{tot} of the system as given in (10), then along classical solutions it holds

$$\begin{aligned} \dot{E}_{\text{tot}}(t) &= \dot{E}(t) + \dot{E}_c(t) \\ &\leq u(t)^\top y(t) + u_c(t)^\top y_c(t) - v_2(t)^\top K_2 R(K_2(v_2(t))) \\ &\quad - y(t) S_c y(t), \end{aligned} \tag{14}$$

where we have used (5), (9) and (6). Integrating this expression and using (4) we obtain (13). Since the domain of $\tilde{A}$ forms a dense set of the state space $\tilde{X}$, and since the solution depends continuously on the initial condition, see (Zheng, 2004, Theorem 2.5.1 and 2.5.4), we see that the above equality holds for all initial conditions. So (13) is shown.

From the uniform boundedness of $E_{\text{tot}}(t)$, we see that $E(t)$, $\mathcal{P}(v_1(t))$ and $v_2(t)^\top K_2 v_2(t)$ are uniformly bounded. Since $K_2 > 0$, we have that $\|v_2(t)\|$ is bounded. Furthermore, since $\sqrt{2E(t)}$ equals the norm, see Assumption 1, the norm of the state x is uniformly bounded. To conclude about the norm of the first state of the finite dimensional controller, $\|v_1\|^2$, we observe that by Assumption 4 we have that $\mathcal{P}(v_1(t))$ bounded implies $\|v_1(t)\|^2$ bounded as well. Now (Pazy, 1983, Chapter 6, Theorem 1.4) gives that $t_{\max} = \infty$, and so we have global existence and the solution is uniformly bounded. ■

4. Asymptotic stability

In the previous section we have shown that under mild conditions we have global existence of solutions. To prove asymptotic stability we need to impose a stronger condition on the damping term R.

Assumption 8. *For the damping we assume that there exist positive constants δ, α, γ such that $\tilde{v}_2^\top R(\tilde{v}_2) \geq \alpha \|\tilde{v}_2\|^2$ when $\|\tilde{v}_2\| < \delta$ and $\tilde{v}_2^\top R(\tilde{v}_2) \geq \gamma$ when $\|\tilde{v}_2\| \geq \delta$ (sector condition near the origin).*

For mechanical systems this means that for small velocities the damping acts linearly and for large velocity the damping force cannot go to zero. Hence it allows for saturation of the damping force.

For asymptotic stability we also need that the derivative of the potential energy, i.e., the force, is differentiable and its derivative is bounded on bounded sets.

Assumption 9. *Define the function $g_1 : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ as $g_1(v_1) = \frac{d\mathcal{P}}{dv_1}(v_1)^\top$. We assume that $\frac{dg_1}{dv_1}$ exists and maps bounded sets on bounded sets.*

Note that if $\frac{dg_1}{dv_1}$ is (locally) Lipschitz continuous, then the assumption is satisfied.

Theorem 10. *Consider the closed-loop system (11) and assume that zero is the only equilibrium point of this equation for which $v_2 = 0$. If the system $\Sigma(\tilde{A}, \tilde{B}, \tilde{B}^*, 0)$ is approximately controllable or approximately observable on infinite time, and Assumptions 1, 4, 5, 8, and 9 hold, then the system is globally asymptotically stable.*

Before we prove this result we make some remarks. The five references to previous assumptions are generally satisfied, in the sense that they are in accordance with common physical nonlinearities known in the field of mechanical and electromechanical systems. Hence they will pose no real restrictions on the class of systems considered. The observability assumption will strongly depend on the system at hand. Given our system this condition can be rewritten as: The only mild solution of

$$\frac{\partial x}{\partial t}(t,\zeta) = P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right) + (P_0 - G_0)\mathcal{H}(\zeta)x(t,\zeta), \quad (15)$$

satisfying

$$0 = W_{B,1}\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix} + S_c W_C\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix}, \quad (16)$$

$$0 = W_{B,2}\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix} \quad (17)$$

and

$$B_c W_C\begin{bmatrix}(\mathcal{H}x)(t,b)\\(\mathcal{H}x)(t,a)\end{bmatrix} \quad (18)$$

constant, is the zero solution. From this it is easy to see that if the uncontrolled system (1)–(4) is not observable, then so is the system $\Sigma(\tilde{A}, \tilde{B}, \tilde{B}^*, 0)$. In general the other implication will hold as well.

Next we prove Theorem 10.

4.1. Proof of Theorem 10

For the proof of this theorem, we show that all the conditions of Theorem 22 from Appendix A are satisfied, and so by that theorem the result follows. For that purpose we consider that $\Sigma(A, B, C)$ is in Theorem 22 what is $\Sigma(\tilde{A}, \tilde{B}, \tilde{C})$ in what follows.

First by the weighted inner product on $\tilde{X}$ we have that

$$\tilde{B}^* = \begin{bmatrix}0 & 0 & K_2\end{bmatrix}.$$

We define $\tilde{C} = \begin{bmatrix}0 & I & 0\end{bmatrix}$, and with this we write f of (12) as

$$f(\tilde{x}) = -R(K_2 v_2) + v_1 - \frac{\partial \mathcal{P}}{\partial v_1}(v_1)^\top = f_0(B^*\tilde{x}) + g(\tilde{C}\tilde{x}). \quad (19)$$

Secondly we show that $f_0(B^*\tilde{x})$ and $B^*\tilde{x}$ are square integrable functions.

Lemma 11. *Under the conditions of Theorem 10 the functions $f_0(B^*\tilde{x})$ and $B^*\tilde{x}$ are square integrable.*

Proof. Since $E_{\text{tot}}(t)$ is always positive, we conclude from (13) that

$$\int_0^\infty v_2(t)^\top K_2 R(K_2 v_2(t))dt < \infty. \quad (20)$$

Let $\Omega_1 := \{t \in [0,\infty) : \|K_2 v_2(t)\| > \delta\}$ and $\Omega_2 := \{t \in [0,\infty) \mid \|K_2 v_2(t)\| \le \delta\}$. So by the assumptions of R, see Assumption 8, we obtain

$$\int_{\Omega_1} v_2(t)^\top K_2 R(K_2 v_2(t))dt \ge \gamma\mu(\Omega_1),$$

and so (20) implies that Ω_1 has finite measure. Moreover,

$$\infty > \int_{\Omega_2} v_2(t)^\top K_2 R(K_2 v_2(t))dt \ge \alpha \int_{\Omega_2} \|K_2 v_2(t)\|^2 dt.$$

Thus

$$\int_0^\infty \|K_2 v_2(t)\|^2 dt = \left(\int_{\Omega_1} + \int_{\Omega_2}\right)\|K_2 v_2(t)\|^2 dt < \infty.$$

Since $K_2 v_2(t)$ is bounded (see (13)) and R is (locally) Lipschitz, we find that $R(K_2 v_2(t))$ is bounded. Combining this with the fact that the measure of Ω_1 is finite, we have

$$\int_{\Omega_1} \|R(K_2 v_2(s))\|^2 ds < \infty.$$

For $s \in \Omega_2$ we have $\|K_2 v_2(s)\| \le \delta$ and so

$$\int_{\Omega_2} \|R(K_2 v_2(s))\|^2 ds \le L(\delta)^2 \int_{\Omega_2} \|K_2 v_2(s)\|^2 ds < \infty,$$

where $L(\delta)$ is the Lipschitz constant for elements in the ball with radius δ. Combining the above inequalities gives that $R(K_2 v_2(\cdot))$ and hence $f_0(K_2 v_2(\cdot))$ is a square integrable function. ■

Since $\tilde{C}\tilde{x} = v_1$, and since $\dot{v}_1 = K_2 v_2$, see (11), we have that v_1 is absolutely continuous with a square integrable derivative, see Lemma 11. Furthermore, by Assumption 9 and (19) we have that g satisfies the corresponding conditions in Theorem 22.

The final property which we have to show is that the set V, see (A.2) contains only zero. The conditions in (A.2) precisely gives that x_∞ is an equilibrium solution on (11) which satisfies $v_2 = 0$. By assumption, $x_\infty = 0$. Now all conditions of Theorem 22 are satisfied, and so Theorem 10 is shown.

5. Exponential stability

In this section we characterize the conditions for exponential stability of the closed-loop system. Before presenting the main theorem of this section we derive some input/output properties of the controller. We shall now consider stronger assumptions on the finite dimensional control system. Specifically, we shall consider some quasi-quadratic bounds of the energy related to the nonlinear potential energy and the dissipation matrix.

Assumption 12. *There exist constants $\delta_1, \delta_2 > 0$ such that for all $v_1 \in \mathbb{R}^n$ holds*

$$v_1^T \frac{\partial \mathcal{P}}{\partial v_1}(v_1) \ge \delta_1 \mathcal{P}(v_1) \ge \delta_2 \|v_1\|^2.$$

Assumption 13. *There exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that for all $\tilde{v}_2 \in \mathbb{R}^n$ holds*

$$\tilde{v}_2^T R(\tilde{v}_2) \ge \varepsilon_1 \|\tilde{v}_2\|^2 \ge \varepsilon_2 \|R(\tilde{v}_2)\|^2.$$

We also need, for the exponential stability proof, assumptions on the number of actuated inputs and outputs and on the strict positivity of the feedthrough term of the controller in order to cope with high frequencies.

Assumption 14. *The k input/output of the system are chosen such that*

$$\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{H}x(t,b)\|^2$$
$$\left(or \quad \|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{H}x(t,a)\|^2\right)$$

Assumption 15. *The controller is strictly input passive. The feedthrough term of the controller is strictly positive i.e.* $S_c > 0$.

Assumptions 12 and 13 refer to the class of admissible nonlinearities. We observe however that the class of nonlinearities is still very general and encompasses a large class of nonlinear mechanical and electro-mechanical actuators, including saturations actuators with saturation. The other assumptions refer to dissipation properties of the infinite dimensional system and of the finite dimensional controller. These are standard assumptions, and are moreover the same that are required for the exponential stabilization of BC-PHS with linear dynamic boundary control (Ramirez et al., 2014). The first one comes from the fact that a part of the boundary port variables of the infinite dimensional system can be set to zero (and hence not used for the interconnection). Hence Assumption 14 imposes that the energy flowing through any of the boundaries is bounded by the energy flowing in/out through the inputs/outputs. Assumption 15 on other hand establishes that the finite dimensional controller is strictly input passive. These assumptions are not necessary for the asymptotic stability but are necessary for the exponential stability since the controller has to damp infinitely high frequency as well as all low frequency modes, which represents a strong constraint from a control perspective.

5.1. Some properties of the controller

The following inequalities for $v, w \in \mathbb{R}^n$ and $\alpha > 0$ shall be used frequently

$$\begin{aligned} -\alpha^2\|v\|^2 - \frac{1}{\alpha^2}\|w\|^2 &\leq v^\top w + w^\top v \\ &\leq \alpha^2\|v\|^2 + \frac{1}{\alpha^2}\|w\|^2. \end{aligned} \tag{21}$$

Notice that the previous relations hold since $\|\alpha v \pm \frac{1}{\alpha} w\|^2 \geq 0$. The following lemmas follow from Definition 2 and Assumption 12.

Lemma 16. *For the function*

$$V(v) := E_c + \gamma v_1^\top v_2 = \mathcal{P}(v_1) + \frac{1}{2} v_2^\top K_2 v_2 + \gamma v_1^\top v_2 \tag{22}$$

there exists a constant $\gamma_0 > 0$ and constants $0 < q_1 < q_2$, which may depend on γ_0, such that for all $\gamma \in (0, \gamma_0)$ there holds

$$q_1 V \leq E_c \leq q_2 V. \tag{23}$$

Proof. using (21) the cross term in (22) can be bounded as

$$\gamma v_1^\top v_2 \leq \frac{1}{2}\left(\gamma^2\|v_1\|^2 + \|v_2\|^2\right).$$

Hence

$$\begin{aligned} V(v) &\leq \mathcal{P}(v_1) + \frac{1}{2}\gamma^2\|v_1\|^2 + \frac{1}{2}v_2^\top K_2 v_2 + \frac{1}{2}\|v_2\|^2 \\ &\leq \left[1 + \frac{1}{2}\gamma^2\frac{\delta_1}{\delta_2}\right]\mathcal{P}(v_1) + \frac{1}{2}\left[1 + \|K_2\|^{-1}\right] v_2^\top K_2 v_2, \end{aligned}$$

where we have used Assumption 12 and that $K_2 > 0$. Hence there exists a $\tilde{q}_1 > 0$ such that for all $\gamma \in (0, \gamma_0)$

$$V \leq \tilde{q}_1 E_c.$$

For the other implication, we use that

$$\gamma v_1^\top v_2 \geq -\frac{1}{2}\left(\gamma^2\|v_1\|^2 + \|v_2\|^2\right).$$

Similarly, as above we find

$$V(v) \geq \left[1 - \frac{1}{2}\gamma^2\frac{\delta_1}{\delta_2}\right]\mathcal{P}(v_1) + \frac{1}{2}\left[1 - \|K_2\|^{-1}\right] v_2^\top K_2 v_2,$$

Hence there exists a $\tilde{q}_2 > 0$ such that for all $\gamma \in (0, \gamma_0)$

$$V \leq \tilde{q}_2 E_c.$$

Combining these results gives (23). ■

Lemma 17. *There exist positive constants κ_2, κ_4 and κ_3 such that for all $\tau > 0$ the energy of (7) satisfies:*

$$E_c(\tau) \leq \kappa_1(\tau) E_c(0) + \kappa_3 \int_0^\tau \|u_c(t)\|^2 dt \tag{24}$$

where $\kappa_1(\tau) = \kappa_4 e^{-\kappa_2 \tau}$. Furthermore, there exist positive constants ξ_1 and ξ_2 such for all $\tau > 0$ the energy of (7) satisfies

$$\int_0^\tau E_c(t)dt \leq \xi_1 E_c(0) + \xi_2 \int_0^\tau \|u_c(t)\|^2 dt \tag{25}$$

Proof. Consider the function V from Lemma 16, where we assume that $\gamma \in (0, \gamma_0)$. Taking the time derivative of V and using that $K_2 = K_2^\top$, one has

$$\begin{aligned} \dot{V} &= \frac{\partial \mathcal{P}}{\partial v_1}^\top \dot{v}_1 + v_2^\top K_2^\top \dot{v}_2 + \gamma \dot{v}_1^\top v_2 + \gamma v_1^\top \dot{v}_2 \\ &= \frac{\partial \mathcal{P}}{\partial v_1}^\top K_2 v_2 + v_2^\top K_2\left(-\frac{\partial \mathcal{P}}{\partial v_1} - R(K_2 v_2) + B_c u_c\right) \\ &\quad + \gamma v_2^\top K_2 v_2 + \gamma v_1^\top\left(-\frac{\partial \mathcal{P}}{\partial v_1} - R(K_2 v_2) + B_c u_c\right) \\ &= -v_2^T K_2 R(K_2 v_2) + v_2^T K_2 B_c u_c + \gamma v_2^T K_2 v_2 \\ &\quad - \gamma v_1^T \frac{\partial \mathcal{P}}{\partial v_1} - \gamma v_1^T R(K_2 v_2) + \gamma v_1^T B_c u_c \end{aligned}$$

Using (21), Assumption 12 and Assumption 13

$$
\begin{aligned}
\dot{V} \leq & -\varepsilon_1 \|K_2 v_2\|^2 + \frac{\alpha_1^2}{2} \|K_2 v_2\|^2 + \frac{1}{2\alpha_1^2} \|B_c u_c\|^2 \\
& + \gamma v_2^\top K_2 v_2 - \gamma \delta_1 \mathcal{P}(v_1) \\
& + \gamma \frac{\alpha_2^2}{2} \|v_1\|^2 + \gamma \frac{1}{2\alpha_2^2} \|R(K_2 v_2)\|^2 \\
& + \gamma \frac{\alpha_3^2}{2} \|v_1\|^2 + \gamma \frac{1}{2\alpha_3^2} \|B_c u_c\|^2 \\
\leq & \left(-\varepsilon_1 + \frac{\alpha_1^2}{2} + \gamma \left\| K_2^{-1} \right\| + \frac{\gamma \varepsilon_1}{2\alpha_2^2} \right) \|K_2 v_2\|^2 \\
& + \left(-\gamma \delta_1 + \gamma \frac{\left(\alpha_2^2 + \alpha_3^2\right)}{2} \frac{\delta_1}{\delta_2} \right) \mathcal{P}(v_1) \\
& + \left(\frac{1}{2\alpha_1^2} + \frac{\gamma}{2\alpha_3^2} \right) \|B_c u_c\|^2 .
\end{aligned}
$$

Considering $\alpha_1, \alpha_2, \alpha_3 \ll 1$, and $\gamma \ll 1$ the following inequality holds

$$
\dot{V} \leq -\kappa_2 V + \kappa_3 \|u_c\|^2 \tag{26}
$$

where κ_2, κ_3 are two positive constants. This implies that

$$
\frac{d}{dt} \left(e^{\kappa_2 t} V \right) \leq \kappa_3 e^{\kappa_2 t} \|u_c(t)\|^2. \tag{27}
$$

Integrating this relation over $t \in [0, \tau]$ and rearranging terms

$$
V(\tau) \leq e^{-\kappa_2 \tau} V(0) + \int_0^\tau \kappa_3 e^{\kappa_2 (t-\tau)} \|u_c(t)\|^2 dt. \tag{28}
$$

Using Lemma A.6.6 from (Curtain & Zwart, 1995, p. 638), we have that $\int_0^\tau \kappa_3 e^{\kappa_2(t-\tau)} \|u_c(t)\|^2 dt \leq \kappa_3 \int_0^\tau \|u_c(t)\|^2 dt$. Using once more the inequality (23), inequality (24) follows. For (25), integrate (26), to obtain

$$
\begin{aligned}
& V(\tau) - V(0) \leq -\kappa_2 \int_0^\tau V(t) dt + \kappa_3 \int_0^\tau \|u_c(t)\|^2 dt \\
& \Rightarrow \kappa_2 \int_0^\tau V(t) dt \leq V(0) - V(\tau) + \kappa_3 \int_0^\tau \|u_c(t)\|^2 dt \\
& \Rightarrow \int_0^\tau V(t) dt \leq \tfrac{1}{\kappa_2} V(0) + \tfrac{\kappa_3}{\kappa_2} \int_0^\tau \|u_c(t)\|^2 dt.
\end{aligned} \tag{29}
$$

By (23), inequality (25) follows. ∎

5.2. Exponential stability of the closed-loop system

Following (Villegas et al., 2009; Ramirez et al., 2014), the objective is to interconnect (7) at the boundaries with (1), as shown in Figure 1, such that the closed-loop system is exponentially stable. In Ramirez et al. (2014) it is shown that if the finite-dimensional control system is linear, strictly input-passive and exponentially stable, then the closed-loop system is exponentially stable. In the present case a nonlinear finite dimensional controller is considered, hence the arguments used in Ramirez et al. (2014), based on the existence of a contraction semi-group, cannot directly be applied.

To prove the main theorem some estimates and a technical lemma are derived. The estimates are presented in the following lemma.

Lemma 18. *The energy of the interconnected system satisfies*

$$
\dot{E}_{\text{tot}}(t) = -v_2^\top K_2 R(K_2 v_2) - u_c^\top S_c u_c, \tag{30}
$$

Furthermore, the output y_c satisfies for some real constant $\delta_2 > 0$,

$$
\|y_c\|^2 \leq \delta_2 \left[v_2^\top K_2 R(K_2 v_2) + \|u_c\|^2 \right]. \tag{31}
$$

Proof. Recalling that $E_{\text{tot}} = \frac{1}{2}\|x(t)\|_{\mathcal{H}}^2 + E_c$ and from (5), (7) and (8), we have

$$
\begin{aligned}
\dot{E}_{\text{tot}} &= u^\top y + \frac{\partial E_c}{\partial v}^\top (v) \dot{v} \\
&= u^\top y - v_2^\top K_2 R(K_2 v_2) + u_c^\top y_c - u_c^\top S_c u_c,
\end{aligned}
$$

Using the definition of the power preserving feedback (6) we obtain (30). The estimate (31) follows from the definition of y_c combined with (21). ∎

Lemma 18 is a measure of passivity of the interconnected system. It shows that the closed-loop solutions will be nonincreasing with respect to the total energy. The following lemma gives a bound on the total energy of the interconnected system.

Lemma 19. *(Ramirez et al., 2014) Consider a BCS defined by the interconnexion (6) of systems (1) and (7) with $r(t) = 0$, for all $t \geq 0$. Then, the energy of the system $E_{\text{tot}}(t) = \frac{1}{2}\|x(t)\|_{\mathcal{H}}^2 + E_c(t)$ satisfies for τ large enough*

$$
\begin{aligned}
E_{\text{tot}}(\tau) &\leq c(\tau) \int_0^\tau \|(\mathcal{H}x)(t,b)\|^2 dt + \tfrac{2c(\tau)}{c_1} \int_0^\tau E_c(t) dt, \\
E_{\text{tot}}(\tau) &\leq c(\tau) \int_0^\tau \|(\mathcal{H}x)(t,a)\|^2 dt + \tfrac{2c(\tau)}{c_1} \int_0^\tau E_c(t) dt,
\end{aligned} \tag{32}
$$

where c_1 is a positive constant and $c(\tau)$ is a positive function only depending on τ satisfying $c(\tau) \to 0$ for $\tau \to \infty$.

Proof. The proof in Ramirez et al. (2014) uses the contraction property of the semi-group generated by the interconnection of a BCS and a linear finite-dimensional controller to establish $E_{\text{tot}}(t_2) \leq E_{\text{tot}}(t_1)$. In the present case, since the controller is nonlinear, the interconnection does not define a semi-group in the sense of Ramirez et al. (2014). However, $E_{\text{tot}}(t_2) \leq E_{\text{tot}}(t_1)$ follows from Lemma 18, hence the proof follows identically to Ramirez et al. (2014) by taking this last point into consideration. ∎

The following theorem presents the main result of the section, namely the exponential stability of BCS subject to the class of nonlinear dynamic boundary controller of Definition 2. The proof of the theorem follows a similar reasoning to the proof of Theorem IV.2 in Ramirez et al. (2014). However, since a nonlinear controller is considered in the present case, lemmas 17, 18 and 19, are necessary to complete the proof.

Theorem 20. *Under the assumptions 12, 13, 14, and 15 the power preserving interconnection (6) of systems (1) and (7), with $r(t) = 0$, is exponentially stable.*

Proof. See Appendix B. ∎

6. Conclusion

The existence of solutions and stability properties of boundary controlled port-Hamiltonian systems (BC-PHS) defined on a 1D spatial domain with a class of nonlinear dynamic boundary control (conditions) have been characterized. The controller is assumed to be passive, with nonlinear (locally) Lipschitz continuous potential energy function and damping matrix. This definition of the finite dimensional dynamic controller encompasses a large class of nonlinear mechanical, electrical and electro-mechanical systems, which are moreover typical actuators in physical applications described by partial differential equations (PDE).

First it has been shown that the solutions of the BC-PHS with the nonlinear dynamic boundary conditions exist globally. Then under some nonrestrictive assumptions on the energy associated to the nonlinear potential energy function and damping matrix, which for instance allow for saturation of the damping force, it is shown that the controller globally asymptotically stabilizes the BC-PHS. Finally, exponential stability is established by assuming that the BC-PHS satisfies a standard passivity relation and the following properties on controller 1) the energy related to the nonlinear potential energy and the dissipation matrix possesses some quasi-quadratic bounds 2) there is a strictly positive feed-through term in order to cope with high frequencies.

The results of this paper are nontrivial extensions of the results presented in Zwart et al. (2016) and Ramirez et al. (2014). Indeed, regarding existence of solutions and exponential stability for the case of linear boundary control, neither the well-posedness nor the stability can be established by using linear semigroup theory nor LaSalle's invariance principle in the case of nonlinear dynamic boundary control.

Future work shall deal with dynamic boundary control of BC-PHS defined on higher dimensional spatial domains.

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Appendix A. General results

In this section we present a general result, which we need for the asymptotic stability of our controlled system. We begin by quoting a theorem from Oostveen (Oostveen, 2000, Chapter 2).

Theorem 21. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U,Z)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable on infinite time. Then*

a. for all $\kappa > 0$, the operator $A - \kappa BB^*$ generates a strongly stable semigroup, $T_{-\kappa BB^*}(t)$;

b. the closed-loop system $\Sigma(A-\kappa BB^*, B, B^*, 0)$ is input stable, i.e., for $u \in L_2((0,\infty); U)$

$$\| \int_0^\infty T_{-\kappa BB^*}(s)Bu(s)ds\|^2 \leq \frac{1}{2}\|u\|^2_{L_2((0,\infty);U)}.$$

c. for all $u \in L_2((0,\infty); U)$ we have

$$\int_0^t T_{-\kappa BB^*}(t-s)Bu(s)ds \to 0 \text{ as } t \to \infty.$$

Hence the above theorem gives that if we perturb the system $\dot{x}(t) = (A - BB^*)x(t)$ by a square integrable input, then the trajectory still converges to zero. This we apply to the following nonlinear abstract differential equation

$$\dot{x}(t) = (A - BB^*)x(t) + Bf(B^*x(t)) + Bg(Cx(t)), \quad x(0) = x_0. \tag{A.1}$$

Theorem 22. *Let Z, U and Y be Hilbert spaces, $B \in \mathcal{L}(U,Z)$, $C \in \mathcal{L}(Z,Y)$ and A the infinitesimal generator of a contraction C_0-semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable on infinite time and B is injective. Furthermore, assume that the (nonlinear) functions $f : U \mapsto U$ and $g : Y \mapsto U$ are locally Lipschitz continuous, with $f(0) = 0$, and $\frac{dg}{dy}$ is bounded on bounded sets.*

Let $x(t)$ be a bounded solution of (A.1) such that $B^*x(\cdot), f(B^*x(\cdot)) \in L^2([0,\infty); U)$, $Cx(t)$ is absolutely continuous on $[0,\tau)$ for every $\tau > 0$ and its derivative lies in $L^2([0,\infty); Y)$. Then the solution $x(t)$ converges to the set V, defined as

$$V = \{x_\infty \in D(A) \mid Ax_\infty + Bg(Cx_\infty) = 0 \text{ and } B^*x_\infty = 0\}, \tag{A.2}$$

as $t \to \infty$.

Proof. We know that the solution is given by

$$x(t) = T_{-BB^*}(t)x_0 + \int_0^t T_{-BB^*}(t-s)Bf(B^*x(s))ds + \int_0^t T_{-BB^*}(t-s)Bg(C(x(s))ds.$$

By the assumptions and our previous result we know that the first two terms converge to zero, and so we concentrate on the last term. We denote by $y(t)$ the signal $Cx(t)$. By integrating by parts we find

$$\begin{aligned}
&\int_0^t T_{-BB^*}(t-s)Bg(Cx(s))ds \\
&= \left[-(A-BB^*)^{-1}T_{-BB^*}(t-s)Bg(Cx(s))\right]_{s=0}^{s=t} + \\
&\quad (A-BB^*)^{-1}\int_0^t T_{-BB^*}(t-s)B\frac{dg}{dy}(y(s))\dot{y}(s)ds \\
&= -(A-BB^*)^{-1}Bg(Cx(t)) + \\
&\quad (A-BB^*)^{-1}T_{-BB^*}(t)Bg(Cx(0)) + \\
&\quad (A-BB^*)^{-1}\int_0^t T_{-BB^*}(t-s)B\frac{dg}{dy}(y(s))\dot{y}(s)ds.
\end{aligned} \tag{A.3}$$

By the boundedness of x, we have that $y(t)$ is bounded, and thus by the assumption on $\frac{dg}{dy}$ we see that

$$\tilde{u}(s) := \frac{dg}{dy}(y(s))\dot{y}(s)$$

lies in $L^2([0,\infty);U)$. So by Theorem 21.c the integral term in (A.3) converges to zero as $t \to \infty$. Combining this with the strong stability of $T_{-BB^*}(t)$, we see that for t large

$$\begin{aligned} x(t) &\approx \int_0^t T_{-BB^*}(t-s)Bg(y(s))ds \\ &\approx -(A-BB^*)^{-1}Bg(y(t)). \end{aligned} \tag{A.4}$$

Let $t_n, n \in \mathbb{N}$ be an unbounded sequence in $[0,\infty)$. Since $y(t_n)$ is bounded, and $(A-BB^*)^{-1}$ is compact, we have that there exists a sub-sequence such that $-(A-BB^*)^{-1}Bg(y(t_n))$ converges along this sub-sequence. We denote this sub-sequence again by t_n. From (A.4), we see that $x(t_n)$ converges as $n \to \infty$. We denote this limit by x_∞. Since C is a bounded operator and g is continuous, we find by (A.4) that

$$x_\infty = -(A-BB^*)^{-1}Bg(Cx_\infty).$$

Hence $x_\infty \in D(A)$, and

$$0 = (A-BB^*)x_\infty + Bg(Cx_\infty). \tag{A.5}$$

Since we could have done the same argument with $A - BB^*$ replaced by $A-2BB^*$ and $f(B^*x)$ replaced by $f(B^*x)+B^*x$, we see that x_∞ also satisfies

$$0 = (A-2BB^*)x_\infty + Bg(Cx_\infty).$$

By the injectivity of B, this implies that $B^*x_\infty = 0$. Combining this with (A.5), we conclude that x_∞ lies in V. ■

Appendix B. Proof of Theorem 20

Proof. Let $\sigma > 0$ be such that $S_c \geq \sigma I$. By Lemma 18 the time derivative of the total energy satisfies

$$\begin{aligned} \dot{E}_{\text{tot}} &= -v_2^\top K_2 R(K_2 v_2) - u_c^\top S_c u_c \\ &\leq -v_2^\top K_2 R(K_2 v_2) - \sigma u_c^\top u_c, \qquad \text{since } S_c \geq \sigma I \\ &= -v_2^\top K_2 R(K_2 v_2) - \sigma\epsilon_1 u_c^\top u_c - \sigma\epsilon_2 u_c^\top u_c \\ &= -v_2^\top K_2 R(K_2 v_2) - \sigma\epsilon_1 \|u_c\|^2 - \sigma\epsilon_2 \|y\|^2 \\ &= -v_2^\top K_2 R(K_2 v_2) - \sigma\epsilon_1 \|u_c\|^2 + \sigma\epsilon_2 \|u\|^2 \\ &\quad - \sigma\epsilon_2\left(\|y\|^2 + \|u\|^2\right) \end{aligned}$$

with $\epsilon_1 + \epsilon_2 = 1$, $\epsilon_i > 0, i \in \{1,2\}$, and where we have used that $u_c = -y$. From Assumption 14 the following inequality holds

$$\|u(t)\|^2 + \|y(t)\|^2 \geq \epsilon \|\mathcal{H}x(t,b)\|^2$$

for some $\epsilon > 0$. Using this bound we have

$$\begin{aligned} \dot{E}_{\text{tot}} &\leq -v_2^\top K_2 R(K_2 v_2) \\ &\quad - \sigma\epsilon_1 \|u_c\|^2 - \sigma\epsilon_2\epsilon \|\mathcal{H}x(t,b)\|^2 + \sigma\epsilon_2 \|y_c\|^2. \end{aligned} \tag{B.1}$$

Integrating this equation from $t = 0$ to τ, with τ large enough such that Lemma 19 holds, we have

$$\begin{aligned} E_{\text{tot}}(\tau) - E_{\text{tot}}(0) &\leq -\int_0^\tau v_2(t)^\top K_2 R(K_2 v_2(t))dt + \\ &\int_0^\tau - \sigma\epsilon_1 \|u_c(t)\|^2 - \sigma\epsilon_2\epsilon \|\mathcal{H}x(t,b)\|^2 + \sigma\epsilon_2 \|y_c(t)\|^2 dt, \end{aligned}$$

and using Lemma 19

$$\begin{aligned} E_{\text{tot}}(\tau) - E_{\text{tot}}(0) &\leq -\int_0^\tau v_2(t)^\top K_2 R(K_2 v_2(t))dt + \sigma\epsilon_1 \|u_c\|^2 dt \\ &+ \frac{\sigma\epsilon_2\epsilon}{c(\tau)}\left(\frac{2c(\tau)}{c_1}\int_0^\tau E_c(t)dt - E_{\text{tot}}(\tau)\right) + \sigma\epsilon_2 \int_0^\tau \|y_c\|^2 dt. \end{aligned}$$

Grouping terms we have that

$$\begin{aligned} &E_{\text{tot}}(\tau)\left(1 + \frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right) - E_{\text{tot}}(0) \leq \\ &\quad -\int_0^\tau v_2(t)^\top K_2 R(K_2 v_2(t))dt - \sigma\epsilon_1 \int_0^\tau \|u_c(t)\|^2 dt \\ &\quad +\sigma\epsilon_2\left(\int_0^\tau \frac{2\epsilon}{c_1}E_c(t) + \|y_c(t)\|^2 dt\right). \end{aligned}$$

Defining $\delta_1 = \frac{2\epsilon}{c_1}$ and using Lemma 18, we have

$$\begin{aligned} &E_{\text{tot}}(\tau)\left(1 + \frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right) - E_{\text{tot}}(0) \leq \\ &(\sigma\epsilon_2\delta_2 - 1)\int_0^\tau v_2(t)^\top K_2 R(K_2 v_2(t))dt + \sigma\epsilon_2\delta_1 \int_0^\tau E_c(t)dt \\ &\qquad + \sigma(\epsilon_2\delta_2 - \epsilon_1)\int_0^\tau \|u_c(t)\|^2 dt. \end{aligned} \tag{B.2}$$

Now, using (25) from Lemma 17 we obtain

$$\begin{aligned} &E_{\text{tot}}(\tau)\left(1 + \frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right) - E_{\text{tot}}(0) \leq \\ &\quad (\sigma\epsilon_2\delta_2 - 1)\int_0^\tau v_2(t)^\top K_2 R(K_2 v_2(t))dt + \sigma\epsilon_2\delta_1\xi_1 E_c(0) \\ &\quad \sigma(\epsilon_2(\delta_2 + \delta_1\xi_2) - \epsilon_1)\int_0^\tau \|u_c(t)\|^2 dt. \end{aligned}$$

Since $E_c(0) \leq E_{\text{tot}}(0)$ and ϵ_2 may be chosen to be arbitrarily small, i.e, $\epsilon_2 \ll 1$ with $\epsilon_1 = 1 - \epsilon_2$, we finally have that

$$\left(1 + \frac{\sigma\epsilon_2\epsilon}{c(\tau)}\right)E_{\text{tot}}(\tau) \leq (1 + \sigma\epsilon_2\delta_1\xi_1)E_{\text{tot}}(0). \tag{B.3}$$

Since $c(\tau)$ converges to zero for $\tau \to \infty$, we can find a τ sufficiently large, such that $E_{\text{tot}}(\tau) \leq c_2 E_{\text{tot}}(0)$ with $c_2 < 1$, which proves the theorem. ■

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