# Asymptotic stability of an Euler Bernouilli beam coupled to non-linear spring-damper systems ⋆

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**Abstract:** The stability of an undamped Euler Bernouilli beam connected to non-linear mass spring damper systems is addressed. It is shown that under mild assumptions on the local behaviour of the non-linear springs and dampers the solutions exist and the system is globally asymptotically stable.

*Keywords:* Boundary control systems, infinite-dimensional port Hamiltonian systems, asymptotic stability, non-linear control.

## 1. INTRODUCTION

In many physical applications, distributed parameter systems are controlled through their boundaries. It is the case for example of compliant mechanical structures, electrical circuits defined over grids, biological networks, tubular reactors and so on. The abstract system representation stemming for this class of controlled systems is called Boundary Control Systems (BCS). Due to the unboundedness of the related input mapping operators, the study of the existence and properties of solutions is not trivial. In the linear case, the semigroup theory is used to prove existence of solutions, and Lyapunov arguments and LaSalle's invariant principle are used to prove asymptotic stability (Luo et al., 1999; Curtain and Zwart, 2016).

In the last decade, it has been shown that the port Hamiltonian formulations (van der Schaft and Maschke, 2002) are of great interest to prove the existence of solutions in some quite general linear cases: systems described by high order differential operators (Le Gorrec et al., 2004, 2005) and multidimensional systems (Kurula and Zwart, 2015)). The main reason is that the port Hamiltonian representations encompass the physical properties of the system, and then lead to very natural and easy to check conditions (matrix conditions) to prove existence and convergence of solutions (Le Gorrec et al., 2005; Villegas, 2007; Jacob and Zwart, 2012). In the 1D case, systems described by first order differential operators stemming for example for the wave equation, Timoshenko beam model or convection systems for example have been recast in the port Hamiltonian framework and conditions for asymptotic and exponential stability have been derived (Villegas et al., 2005; Villegas et al., 2009). The case of systems described by second order differential operators stemming for example for the Euler Bernouilli beam equation has been studied in (Augner and Jacob, 2014). Recently these results have been extended to the case of first order differential operators systems connected at their boundary to linear or non-linear finite dimensional systems (Ramirez et al., 2014).

In this paper we consider the case of a system described by a second order linear differential operator connected to non-linear finite dimensional systems, focusing on the Euler Bernouilli beam model connected to non-linear mass spring systems. This application case is motivated by the control of compliant micro-mechanical systems (micrograppers) used for the manipulation of bio samples. These systems, constituted of a flexible silicon arm clamped at one side to a slide moving shuttle and attached at the other side to a bio sample are represented by an undamped Euler Bernouilli model connected to non-linear mass spring damper systems. The same kind of problem has already been studied in (Miletic et al., 2016) and in (Augner, 2017) from a theoretical point of view. In (Miletic et al., 2016) LaSalle's principle is used and precompactness of trajectories is first established, but asymptotic stability was only shown for a dense set of initial conditions. In (Augner, 2017) non-linear contraction semigroups are used leading to strong assumptions on the considered non-linearities. In this paper we use a direct approach based

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on perturbation theory to prove the convergence to zero for all initial conditions. Furthermore, the conditions we have on the non-linearities are weak.

The paper is organised as follows. We first present the application case and the associated port Hamiltonian model of the open loop and closed loop (beam+mass spring damper systems) system in Section 2. In Section 3 we prove the existence of solutions of the closed loop system. The study of the asymptotic stability properties is detailed in Section 4. The paper ends with Section 5 in which are given some concluding remarks and some perspectives to this work.

## 2. CONSIDERED SYSTEM

We consider the system of Fig. 1 composed of a flexible beam clamped at one side to a non-linear mass spring damper system connected to a slide link, and to a non-linear mass spring damper system at the other side.

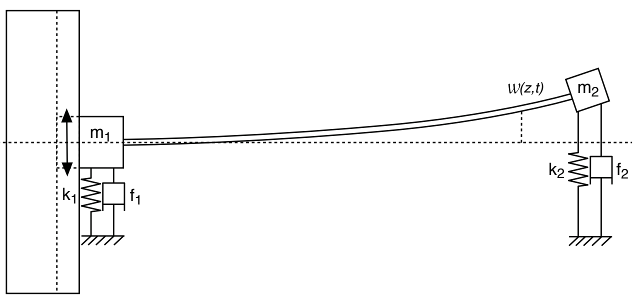


Fig. 1. Flexible beam + mass spring damper systems.

This system reflects the dynamic behaviour of a boundary actuated compliant structure connected to a non-linear finite dimensional dynamic system as it is the case for example with nanotweezers used for the manipulation of non-linear bio samples (Boudaoud et al., 2012). For the sake of simplicity and without any restriction, we assume the following.

*Assumption 1.* The equilibrium position is the horizontal line *i.e.* the considered stiffnesses are related.

The proposed results can be extended to the general setting by using as state variables the deviation around a steady state profile defined by additional external boundary force.

### *2.1 Infinite dimensional system*

The out of plane deflection of the beam, noted $\omega(z,t)$, is a function of time and space ($z \in [a,b]\,,\ t>0$). When small deformations are considered the Euler-Bernouilli assumptions (Timoshenko (1953)) apply and the shear stress is neglected. The model of the beam is derived from the balance equation on the kinetic momentum $\rho\frac{\partial\omega(z,t)}{\partial t}$ *i.e.*:

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial\omega(z,t)}{\partial t}\right) = -\frac{\partial}{\partial z}F_s + F_l\ ,$$

where $F_s$ is the shear force and $F_l$ the external longitudinal force applied to the beam. Here $F_l = 0$. Following the Euler-Bernouilli assumptions, the shear force is derived from the bending moment that is a linear function of the curvature $\frac{\partial^2\omega(z,t)}{\partial z^2}$ leading to:

$$F_s = -\frac{\partial}{\partial z}\left(-EI\frac{\partial^2\omega(z,t)}{\partial z^2}\right)\ ,$$

where $EI > 0$ is the flexure rigidity of the beam. This lead to the Euler- Bernouilli model of the beam:

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial\omega(z,t)}{\partial t}\right) = -\frac{\partial^2}{\partial z^2}\left(EI\frac{\partial^2\omega(z,t)}{\partial z^2}\right)\ ,$$

Using as state variable the curvature $x_1 = \frac{\partial^2\omega}{\partial z^2}$ and the kinetic momentum $x_2 = \rho\frac{\partial\omega}{\partial t}$ this system can be written as a first order system in time and a second order system in space:

$$\frac{\partial}{\partial t}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & \frac{\partial^2}{\partial z^2}\\ -\frac{\partial^2}{\partial z^2} & 0\end{pmatrix}\begin{pmatrix}EI & 0\\ 0 & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{1}$$

The total energy of the infinite dimensional system is given by the integral of the sum of kinetic energy and elastic potential energy

$$\begin{aligned}\mathcal{H}(t) &= \frac{1}{2}\int_a^b\left(\rho\left(\frac{\partial\omega(z,t)}{\partial t}\right)^2 + EI\left(\frac{\partial^2\omega(z,t)}{\partial z^2}\right)^2\right)dz\\ &= \frac{1}{2}\int_a^b\left(EIx_1^2 + \frac{1}{\rho}x_2^2\right)dz \quad (2)\end{aligned}$$

*Proposition 1.* The infinite dimensional system defined by:

$$\dot{x}(z,t) = A\left(\mathcal{L}x(z,t)\right), \tag{3}$$

with :

$$A = \begin{pmatrix}0 & \frac{\partial^2}{\partial z^2}\\ -\frac{\partial^2}{\partial z^2} & 0\end{pmatrix},\ \mathcal{L} = \begin{pmatrix}EI & 0\\ 0 & \frac{1}{\rho}\end{pmatrix} > 0\ , \tag{4}$$

and

$$u(t) = \begin{pmatrix}\frac{1}{\rho}x_2(a,t)\\ \frac{1}{\rho}x_2(b,t)\\ \frac{\partial}{\partial z}\left(\frac{1}{\rho}x_2\right)(b,t)\end{pmatrix},\ \frac{\partial}{\partial z}\left(\frac{1}{\rho}x_2\right)(a,t) = 0 \tag{5}$$

is a boundary control system. As a consequence $A$ with

$$D(A) = \left\{\mathcal{L}x \in H^2([a,b]\,;\mathbb{R}^2)\middle|u=0\right\}$$

generates a contraction semigroup and for any $u \in C^2(\left[0,\infty;\mathbb{R}^3\right),\ \mathcal{L}x(0) \in H^2([a,b]\,;\mathbb{R}^2)$ satisfying (5) (for $t=0$) there exists a unique classical solution to (3)–(5). Furthermore, with $y(t)$ given by

$$y(t) = \begin{pmatrix}\frac{\partial}{\partial z}\left(EIx_1\right)(a,t)\\ \frac{\partial}{\partial z}\left(EIx_1\right)(b,t)\\ EIx_1(b,t)\end{pmatrix}, \tag{6}$$

the balance equation on the energy equals

$$\frac{d\mathcal{H}(t)}{dt} = y^T(t)u(t) \tag{7}$$

**Proof.** The proof is a direct application of Lemma 4.5 of (Le Gorrec et al., 2005). Equation (1) is of the form:

$$\frac{\partial}{\partial t}x(z,t) = P_2 \frac{\partial^2}{\partial z^2}\left(\mathcal{L}x(z,t)\right), \text{ with } P_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and from (Le Gorrec et al., 2005) we define as input and output

$$u_{ext}(t) = \begin{pmatrix} \frac{\partial \omega(a,t)}{\partial t} \\ \frac{\partial^2 \omega(a,t)}{\partial z \partial t} \\ \frac{\partial \omega(b,t)}{\partial t} \\ \frac{\partial^2 \omega(b,t)}{\partial z \partial t} \end{pmatrix}, \quad y_{ext}(t) = \begin{pmatrix} \frac{\partial}{\partial z}\left(EI\frac{\partial^2 \omega}{\partial z^2}\right)(a,t) \\ EI\frac{\partial^2 \omega(a,t)}{\partial z^2} \\ \frac{\partial}{\partial z}\left(EI\frac{\partial^2 \omega}{\partial z^2}\right)(b,t) \\ EI\frac{\partial^2 \omega(b,t)}{\partial z^2} \end{pmatrix} \tag{8}$$

such that

$$\frac{d\mathcal{H}(t)}{dt} = y_{ext}^T(t)u_{ext}(t)$$

By choosing as input and output the interconnection variables $u$ and $y$ defined by (5) and (6) associated with the clamping condition $\frac{\partial}{\partial z}\left(\frac{1}{\rho}x_2\right)(a,t) = 0$ we can conclude we have a boundary control system with balance equation given by (7).

### 2.2 Finite dimensional systems

We are now considering the motion of finite dimensional systems to which the beam is attached. The motion of the left non-linear mass spring system along the slide link is described through its port Hamiltonian formulation as:

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} p_1 \\ m_1 v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k_{10}p_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ k_{10}p_1 - k_1(p_1) - f_1(v_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1 \\ y_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ v_1 \end{pmatrix}, \end{cases}$$

where $p_1$ and $v_1$ are the position and the velocity of the inertia $m_1$, respectively, $u_1$ is the longitudinal force applied to the system, $y_1$ the resulting longitudinal velocity. $k_{10} > 0$ is an additional linear stiffness and $k_1(p_1)$ and $f_1(v_1)$ the non-linear stiffness and damping functions.

*Assumption 2.* There exists a function $\mathcal{P}_1 : \mathbb{R} \mapsto [0, \infty)$ which has a unique minimum at $p_1 = 0$, *i.e.*, $\mathcal{P}_1(p_1) > \mathcal{P}_1(0) = 0$ for $p_1 \neq 0$, and $\frac{d\mathcal{P}_1}{dp_1}(p_1) = k_1(p_1)$. Furthermore, $\mathcal{P}_1(p_1)$ is radially unbounded. Thus if $|p_1| \to \infty$, then $\mathcal{P}(p_1) \to \infty$.

*Assumption 3.* We assume that $f_1$ is a function of $v_1$ and that for all $v_1$ it satisfies

$$v_1 f_1(v_1) \geq 0$$

The energy of the system $H_1 = \mathcal{P}_1(p_1) + \frac{1}{2}\left(m_1 v_1^2\right)$ satisfies the following balance equation:

$$\frac{dH_1}{dt} = y_1 u_1 - v_1 f_1(v_1).$$

The motion of the inertia at the right side of the beam is described by the following equations:

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} p_2 \\ m_2 v_2 \\ J\dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} k_{20}p_2 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_{20}p_2 - k_2(p_2) - f_2(v_2) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} u_2 \\ y_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_2 \\ v_2 \\ \dot{\theta}_2 \end{pmatrix} \end{cases}$$

where $p_2$ denotes the position of the inertia $m_2$, $v_2$ the velocity, $\theta_2$ the angular position of the load, $\dot{\theta}_2$ the angular velocity. $u_2$ is composed by longitudinal force and torque applied to the system, $y_2$ is composed by resulting longitudinal velocity and angular velocity. $k_{20} > 0$ is an additional linear stiffness and $k_2(p_2)$ and $f_2(v_2)$ the non-linear stiffness and damping functions.

*Assumption 4.* There exists a function $\mathcal{P}_2 : \mathbb{R} \mapsto [0, \infty)$ which has a unique minimum at $p_2 = 0$, *i.e.*, $\mathcal{P}_2(p_2) > \mathcal{P}_2(0) = 0$ for $p_2 \neq 0$, and $\frac{d\mathcal{P}_2}{dp_2}(p_2) = k_2(p_2)$. Furthermore, $\mathcal{P}_2(p_2)$ is radially unbounded. Thus if $|p_2| \to \infty$, then $\mathcal{P}_2(p_2) \to \infty$.

*Assumption 5.* We assume that $f_2$ is a function of $v_2$ and that for all $v_2$ it satisfies

$$v_2 f_2(v_2) \geq 0$$

The energy associated to this system if given by:

$$H_2 = \mathcal{P}_2(p_2) + \frac{1}{2}\left(m_2 v_2^2 + J\left(\dot{\theta}_2\right)^2\right)$$

and satisfies the following balance equation

$$\frac{dH_2(t)}{dt} = y_2^T u_2 - v_2 f_2(v_2).$$

### 2.3 Closed loop system

We consider the power preserving interconnection

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} y = -\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} u = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and the boundary condition (clamping)

$$\frac{\partial}{\partial z}\left(\frac{\partial \omega(a,t)}{\partial t}\right) = 0$$

Denoting $\tilde{x} = \begin{pmatrix} \frac{\partial^2 \omega}{\partial z^2} & \rho\frac{\partial \omega}{\partial t} & p_1 & m_1 v_1 & p_2 & m_2 v_2 & J\dot{\theta}_2 \end{pmatrix}$ the closed loop system can be written:

$$\dot{\tilde{x}} = \underbrace{\begin{pmatrix} 0 & \frac{\partial^2}{\partial z^2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial z^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial .}{\partial z}\Big|_a & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\partial .}{\partial z}\Big|_b & 0 & 0 & 0 & -1 & 0 & 0 \\ \cdot|_b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\tilde{\mathcal{A}}}\left(\tilde{\mathcal{L}}\tilde{x}\right) + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\tilde{\mathcal{B}}}\tilde{f}(\tilde{x}) \tag{9}$$

with

$$\tilde{\mathcal{L}} = \text{diag}\left(EI \;\; \frac{1}{\rho} \;\; k_{10} \;\; \frac{1}{m_1} \;\; k_{20} \;\; \frac{1}{m_2} \;\; \frac{1}{J}\right)$$

and

$$\tilde{f}(\tilde{x}) = \begin{pmatrix} k_{10}p_1 - k_1(p_1) - f_1(v_1) \\ k_{20}p_2 - k_2(p_2) - f_2(v_2) \end{pmatrix} \tag{10}$$

The domain of the differential operator is given by:

$$D(\tilde{A}) = \Big\{\tilde{x} \in L^2\left([a,b], \mathbb{R}^2\right) \times \mathbb{R}^2 \times \mathbb{R}^3 \Big| \left( m_1 x_2(a) \;\; m_2 x_2(b) \;\; \frac{\partial x_2}{\partial z}(a) \;\; \frac{\partial x_2}{\partial z}(b) \;\; \rho x_4 \;\; \rho x_6 \;\; x_7 \right)^T \in \ker W_D, \Big\} \quad (11)$$

where

$$W_D = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

The total energy of the system is:

$$H_{tot} = \mathcal{H} + H_1 + H_2$$

and satisfies

$$\frac{dH_{tot}(t)}{dt} = -v_1 f_2(v_1) - v_2 f_2(v_2) \leq 0 \quad (12)$$

*Lemma 2.* The linear operator $\tilde{A}$ with its domain (11) generates a contraction semigroup on $\tilde{X}$. Moreover $\tilde{A}$ has a compact resolvant.

## 3. EXISTENCE OF SOLUTIONS

In this section we show the closed loop system is well posed, *i.e.* closed loop solutions exist locally. Then from Assumptions 2, 3, 4, 5 we show the global existence of solutions. Since $\tilde{f}$ is locally Lischitz continuous on $\tilde{X}$ and $\tilde{X}$ bounded it follows from (Pazy (1983), Chapter 6, Theorem 1.5) that for any initial condition, the closed loop equation possesses a unique mild solution on the time interval $[0,\ t_{max}\ )$. We consider now the total energy $H_{tot}$ of the closed loop system. By integration of (12) we obtain (for every initial condition since classical solutions form a dense set)

$$H_{\mathrm{tot}}(t) \leq H_{\mathrm{tot}}(0) - \int_0^t v_1(\tau) f_1(v_1(\tau)) d\tau - \int_0^t v_2(\tau) f_2(v_2(\tau)) d\tau. \quad (13)$$

From the uniform boundedness of $H_{tot}$ we deduce the uniform boundedness of $\mathcal{H}(t), \mathcal{P}_1(p_1(t)), \mathcal{P}_2(p_2(t)), v_1(t) f_1(v_1(t))$ and $v_2(t) f_2(v_2(t))$. Since $\mathcal{H}(t)$ is bounded, $\|x(t)\|$ is bounded. Since $m_1 > 0$ and $m_2 > 0$, $\|m_1 v_1(t)\|$ and $\|m_2 v_2(t)\|$ are bounded. The energy $\mathcal{H}(t)$ equals half of the norm and then the norm of the distributed state is uniformly bounded. From the fact that $\mathcal{P}_1(p_1(t))$ and $\mathcal{P}_2(p_2(t))$ are bounded combined with Assumptions 2 and 4, we can conclude that $\|p_1\|$ and $\|p_2\|$ are bounded. From (Pazy (1983), Chapter 6, Theorem 1.4) we have that $t_{max} = \infty$ and so we have global existence of solutions. This result is sumarized in Theorem 3.

*Theorem 3.* The system (9) with the non-linearity (10) satisfying Assumptions 2, 3, 4 and 5 possesses for every initial condition a unique mild solution which is uniformly bounded. Furthermore inequality (13) holds.

## 4. ASYMPTOTIC STABILITY

We use a direct approach based on non-linear perturbation of linear distributed parameter systems. It consists in expressing the dynamic system (9) on the form:

$$\dot{x}(t) = (\mathcal{A} - BB^*)\, x(t) + Bf(B^* x(t)) + Bg(Cx(t)) \quad (14)$$

and to use the general result from (Ramirez et al., 2017) recalled in Theorem 4.

*Theorem 4.* Let $Z$, $U$ and $Y$ be Hilbert spaces, $B \in \mathcal{L}(U,Z)$, $B^* \in \mathcal{L}(Z,Y)$ and $A$ the infinitesimal generator of a contraction $C_0$-semigroup. Assume that $A$ has compact resolvent, and that the state linear system $\Sigma(A,B,B^*,0)$ is approximately controllable or approximately observable on infinite time and $B$ is injective. Furthermore, assume that the (non-linear) functions $f$ and $g$ are (locally) Lipschitz continuous, with $f(0) = 0$, and $\frac{dg}{dy}$ is bounded on bounded sets.

Let $x(t)$ be a bounded solution of (14) such that $B^* x(\cdot), f(B^* x(\cdot)) \in L^2([0,\infty); U)$, $Cx(t)$ is absolutely continuous on $[0,\tau)$ for every $\tau > 0$ and its derivative lies in $L^2([0,\infty); U)$. Then the solution $x(t)$ converges to the set $V$ as $t \to \infty$. This set is given by

$$V = \{x_\infty \in D(A) \mid Ax_\infty + Bg(Cx_\infty) = 0 \text{ and } B^* x_\infty = 0\}. \quad (15)$$

In order to apply the aforementioned theorem to prove the asymptotic stability we have to show:

(1) The approximate controllability on infinite time of the linear system $(A,B,B^*,0)$.
(2) The square integrability of $B^* x(\cdot), f(B^* x(\cdot))$ where $x(t)$ is a bounded solution of (14).
(3) The square integrability of $Cx(t)$ and its derivative.
(4) The fact that $V$ defined by (15) reduces to $\{0\}$.

Before checking these conditions we need an additional assumption on the non-linear spring.

*Assumption 6.* We assume that the non-linear stiffnesses $k_1$ and $k_2$ defined in (10) are such that $k_1(0) = k_2(0) = 0$ and are (locally) Lipschitz continuous on $\mathbb{R}^4$.

We also need an additional assumption on the non-linear damping.

*Assumption 7.* For the damping we assume that there exist positive constants $\delta_1, \alpha_1, \gamma_1$ and $\delta_2, \alpha_2, \gamma_2$ such that $\tilde{v}_1^\top f_1(\tilde{v}_1) \geq \alpha_1 \|\tilde{v}_1\|^2$ and $\tilde{v}_2^\top f_2(\tilde{v}_2) \geq \alpha_2 \|\tilde{v}_2\|^2$ when $\|\tilde{v}_1\| < \delta_1$, $\|\tilde{v}_2\| < \delta_2$ and $\tilde{v}_1^\top f_1(\tilde{v}_1) \geq \gamma_1$, $\tilde{v}_2^\top f_2(\tilde{v}_2) \geq \gamma_2$ when $\|\tilde{v}_1\| \geq \delta_1$ and $\|\tilde{v}_2\| \geq \delta_2$ respectively.

Then we reformulate the original problem as system (14). The dynamic system (9) with non-linearity (10) is equivalent to the non-linear abstract differential equation

$$\dot{\tilde{x}}(t) = \left(\tilde{\mathcal{A}} - \tilde{B}\tilde{B}^*\right)\tilde{x}(t) + \tilde{B} f_0(\tilde{B}^* \tilde{x}(t)) + \tilde{B} g_0(\tilde{C}\tilde{x}(t)) \quad (16)$$

where $\tilde{B}^*$ is defined from the weighted inner product on $\tilde{X}$

$$\tilde{B}^* = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_2} & 0 \end{pmatrix}$$

and

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

with:

$$f_0(v_1, v_2) = \begin{pmatrix} -f_1(v_1) + v_1 \\ -f_2(v_2) + v_2 \end{pmatrix}, \text{ and } g_0(p_1, p_2) = \begin{pmatrix} k_{10} p_1 - k_1(p_1) \\ k_{20} p_2 - k_2(p_2) \end{pmatrix}$$

We now check the conditions of Theorem 4. We fist start with the approximate observability in infinite time condition as stated in Proposition 5.

*Proposition 5.* The closed loop system (9) is approximately observable on infinite time.

**Proof.** We use Corollary 4.1.14 of ((Curtain and Zwart, 1995), Chapter 4, p.156) that states the system is observable in infinite time if the only solution to $B^*\tilde{x} = 0$ is $\{0\}$. If we assume $B^*\tilde{x} = 0$ we have $x_6(t) = 0 = x_4(t)$, which is equivalent to $v_1(t) = 0 = v_2(t)$. This implies that $p_1(t)$ and $p_2(t)$ are constant and $-k_{10}p_1(t) + \frac{\partial}{\partial z}x_1(a,t) = 0$, $-k_{20}p_2(t) + \frac{\partial}{\partial z}x_1(b,t) = 0$ and $\frac{\partial \omega}{\partial t}(a,t) = 0$, $\frac{\partial \omega}{\partial t}(b,t) = 0$. From the domain of $\tilde{A}$ we have $\frac{\partial}{\partial z}\left(\frac{\partial \omega}{\partial t}\right)(a,t) = 0$. It means that the only possible solution is the equilibrium position *i.e.* $\tilde{x} = 0$. We conclude the system is approximately observable on infinite time.

We are now considering the square integrability of the non-linear functions.

*Lemma 6.* The system being approximatively controllable on infinite time and under Assumptions 7 and 6 the function $f_0(\tilde{B}^*\tilde{x}(t))$ and $\tilde{B}^*\tilde{x}(t)$ are square integrable.

**Proof.** From boundedness of the energy we have

$$\int_0^\infty \left(v_1^\top(\tau) f_1(v_1(\tau)) + v_2^\top(\tau) f_2(v_2(\tau))\right) d\tau < \infty$$

Then from Assumption 7 it is possible to show that $f_1(v_1(.))$, $f_2(v_2(.))$, $v_1(.)$ and $v_2(.)$ are square integrable functions.

It remains now to prove the square integrability of $Cx(t)$ and its derivative, and to show that the only point contained in the set $V$ is 0. Since $\tilde{C}\tilde{x} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and since $\dot{p}_1 = v_1$, and $\dot{p}_2 = v_2$ we have from Lemma 6 that $p_1$ and $p_2$ are absolutely continuous with square integral derivative. Furthermore the solution of (16) which satisfies $m_1 v_1 = 0$ and $m_2 v_2 = 0$ is zero. All the conditions of the general Theorem hold and we have then proven the asymptotic stability of our closed loop system, as summarized in Theorem 7.

*Theorem 7.* Consider the closed-loop system (9) and assume that zero is the only equilibrium point of this equation for which $v_2 = 0$. If Assumptions 4, 2, 5, 3, 6, and 7 hold, then the system is globally asymptotically stable.

## 5. CONCLUSION

In this paper we consider the stability of a linear Euler Bernouilli beam connected to non-linear mass spring damper systems. It is shown that under mild assumptions on the non-linear constitutive equations the system is globally asymptotically stable. For that purpose we first prove global existence of solutions and then we show the asymptotic convergence to zero by using a general result on non-linear perturbation of linear infinite dimensional systems. This alternative approach to the use of the LaSalle's invariant principle save the difficult problem of proving precompactness of trajectories. The resulting assumptions n the non-linearities are weak and allow to consider the usual non-linearities encountered in physical applications (polynomial laws, sector conditions, saturations etc ...). Even if the paper focusses on a particular case, the use of the port Hamiltonian framework and of some general Theorems for the convergence proof makes the approach general and easy to extend to other scenari.

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