A CONDITION-BASED DYNAMIC MAINTENANCE POLICY FOR AN EXTENDED GAMMA PROCESS

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A standard gamma process is widely used for cumulative deterioration modeling purpose and to make prediction over the system future behavior. However, this process may be restrictive within an applicative context since its variance-to-mean ratio is constant over time. Extended gamma process, which was introduced by Cinlar (1980), seems to be a good candidate to overcome the latter restriction. The aim of this paper is to investigate benefits of using an extended gamma process for modeling the system evolution instead of a standard gamma process, from a reliability point of view. With that goal, we propose a condition-based dynamic maintenance policy and evaluate its performance on a finite planning horizon. Numerical experiments are illustrated based on simulated data.

Keywords: Process with non-stationary independent increments, Standard gamma process, Extended gamma process, Condition-based maintenance policy, Dynamic inspections.

1. Introduction

Numerous models for time- and condition-based maintenance policies have been proposed in the literature under Standard Gamma Process (SGP) modeling assumption [8]. Condition-Based Maintenance (CBM) policy seems to be a very promising approach to shorten the system downtime and minimize the maintenance cost of a system subject to SGP deterioration

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(e.g. see [5,8]). Let us recall that a standard (non-homogeneous) gamma process $\boldsymbol{Y} = (Y_t)_{t\geq 0} \sim \Gamma_0(A(.), b_0)$, with an increasing and right-continuous function $A(.)$ as shape function and a positive (constant) b_0 as scale parameter, is a stochastic process with independent, non-negative and gamma distributed increments such that $Y_0 = 0$ almost surely. The probability density function of an increment $Y_t - Y_s$ (with $0 < s < t$) is given by

$$f(x) = \frac{b_0^{A(t)-A(s)}}{\Gamma(A(t) - A(s))} x^{A(t)-A(s)-1} \exp(-b_0 x), \forall x \geq 0 \quad (1)$$

(e.g. see [1]).

The mean and variance of Y_t are given by $\mathbb{E}[Y_t] = \frac{A(t)}{b_0}$ and $\mathbb{V}[Y_t] = \frac{A(t)}{b_0^2}$ for all $t \geq 0$, so that the variance-to-mean ratio $\frac{\mathbb{V}[Y_t]}{\mathbb{E}[Y_t]} = b_0$ is constant over time. As a consequence, SGP is not a suitable choice to model the deterioration and to make prediction over the system behavior, when there is some "empirical evidence" against a constant variance-to-mean ratio [6]. A way to overcome this restriction is to consider an Extended Gamma Process (EGP), which was introduced by [4]. Though an EGP presents some technical difficulties (no exact simulation method, no explicit formula for the distribution function), the recent development of new technical and statistical tools [2,3,6] makes its practical use now possible, for application purpose. Up to our knowledge, EGP has not been much studied for cumulative degradation modeling, except in [6]. The aim of the present paper is to optimize a particular maintenance policy under EGP modeling assumption and to show the interest of extending SGPs. The proposed preventive CBM policy relies on two important decisions: when to inspect and when to make maintenance actions. The inspections times are dynamically determined in order to ensure some safety level up to the next inspection (as in [7]). Also, a preventive replacement takes place as soon as the deterioration level is beyond a maintenance threshold M (to be optimized). CBM policy is often assessed through a cost function on an infinite planning horizon. We here consider the cost function on finite time horizon which may be more adapted to real life situations.

The remainder of this paper is organized as follows. Section 2 briefly introduces the EGP. The maintenance policy is proposed in Section 3. Section 4 finally presents numerical experiments, which show the advantage of considering extended versions of standard gamma processes for application purpose.

2. Definition and properties of an EGP

Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and right-continuous function with $A(0) = 0$ and let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable positive function such that:

$$\int_{(0,t]} \frac{dA(s)}{b(s)} < \infty, \quad \forall t > 0. \tag{2}$$

An EGP with $A(.)$ as shape function and $b(.)$ as scale function (written $\boldsymbol{X} = (X_t)_{t\geq 0} \sim \Gamma(A(.), b(.))$), is a non-decreasing process with independent increments. It can be constructed as a stochastic integral with respect to a SGP $\boldsymbol{Y} \sim \Gamma_0(A(.), 1)$:

$$X_t = \int_{(0,t]} \frac{dY_s}{b(s)}, \quad \forall t > 0 \tag{3}$$

and $X_0 = 0$ (see [4]). If $b(.)$ is constant and equal to b_0, the EGP simply reduces to a standard Gamma process $\Gamma_0(A(t), b_0)$. An EGP allows to model many possible behaviors for the system deterioration (see [3] for more precision). Finally, the mean and variance of an EGP are given by

$$\mathbb{E}(X_t) = \int_{(0,t]} \frac{dA(s)}{b(s)} \quad \text{and} \quad \mathbb{V}(X_t) = \int_{(0,t]} \frac{dA(s)}{b(s)^2}, \forall t. \tag{4}$$

3. The deterioration model and the CBM policy

3.1. *The deterioration model and assumptions*

A single-unit system is considered, which is subject to an accumulative deterioration modeled by an EGP $\boldsymbol{X} = (X_t)_{t\geq 0} \sim \Gamma(A(.), b(.))$. The system is considered as failed as soon as the degradation level exceeds a predetermined threshold L. The time at which the failure occurs hence is

$$\sigma_L = \inf\{t > 0 : X_t > L\}. \tag{5}$$

The cumulative distribution function and survival function of X_t are denoted by F_{X_t} and $\bar{F}_{X_t}$, respectively. In the present case, we can write

$$F_{\sigma_L}(t) = \mathbb{P}[\sigma_L \leq t] = \mathbb{P}[X_t > L] = \bar{F}_{X_t}(L) = 1 - F_{X_t}(L). \tag{6}$$

Assumptions

(A_1) When necessary, it is possible to instantaneously replace the system by a new one, which is assumed to be identical and stochastically independent from the previous one.

(A_2) Failures are unrevealed (no continuous monitoring). Once failed, the system hence remains down until it is replaced (which induces some unavailability cost).

(A_3) The deterioration level can be assessed through instantaneous and perfect inspections.

3.2. *The proposed CBM policy*

Let M be the maintenance threshold. The idea of the CBM policy is to inspect the system from time to time until it is observed to be either too degraded (level beyond M) or failed (level beyond L). Then, a preventive or corrective replacement is carried out, accordingly. After a replacement, a new sequence of inspections is initiated and the same CBM policy is applied. The inspection times $T_1 < T_2 < \ldots < T_n < \ldots$ are recursively determined in order to ensure the reliability to remain beyond a minimal fixed level up to the next inspection. The first inspection time T_1 is chosen as a quantile of the failure time σ_L:

$$\mathbb{P}(\sigma_L > T_1) = \mathbb{P}(X_{T_1} \leq L) = F_{X_{T_1}}(L) = 1 - \epsilon, \tag{7}$$

where $1 - \epsilon \in (0, 1)$ is the predetermined reliability (safety) level.

Now, assume $T_1 < \ldots < T_i$ to be constructed. At time T_i, there are three possibilities, according to the observed deterioration level:

- If $X_{T_i} \geq L$, the system is failed and it is correctively replaced.
- If $M \leq X_{T_i} < L$, the system is still functioning but it is too degraded. Hence it is preventively replaced.
- If $X_{T_i} < M$, the system is still in a "good" working state. Nothing is done apart from planning a new inspection. The time T_{i+1} of the next inspection is chosen as a conditional quantile of the failure time σ_L given the observed state:

$$\mathbb{P}(\sigma_L > T_{i+1} | X_{T_i} = x) = F_{X_{T_{i+1}} - X_{T_i}}(L - x) = 1 - \epsilon. \tag{8}$$

The previously described procedure ends by a (corrective or preventive) replacement. The duration up to the first replacement is denoted by S, with $S \in \{T_i, i \geq 1\}$. At time S, the system is renewed and a similar CBM procedure takes place.

3.3. *Evaluation of the maintenance policy*

To assess the performance of the CBM policy, we focus on the expected cost function over a finite planning horizon t_0, which takes into consideration the inspection cost c_I, the cost of preventive replacement c_P, the cost of corrective replacement c_F and the unavailability cost per unit time c_U.

The expected cost in one cycle of length S is given by

$$\mathbb{E}[C(S)] = c_I \mathbb{E}[N_I(S)] + c_P \mathbb{P}(\mathcal{E}(S)) + c_F(1 - \mathbb{P}(\mathcal{E}(S))) + c_U \mathbb{E}[d(S)] \quad (9)$$

where $N_I(S)$ is the number of inspections per cycle; $\mathbb{P}(\mathcal{E}(S)) = \mathbb{P}(\text{a cycle ends by a preventive replacement}) = \mathbb{P}(X_S < L)$ and $d(S) = (S - \sigma_L)^+$.

Note that, based on the complexity of the CBM policy, there is no hope to obtain a closed form expression of this cost.

The expected cost $\mathbb{E}[C(t_0)]$ over the planning horizon t_0 simply is the sum of costs of all completed cycles before t_0 added to the expected cost over the remaining time horizon. It can be expressed by

$$\begin{aligned} \mathbb{E}[C(t_0)] = \mathbb{E}\left[\sum_{j=1}^{N_R(t_0)} C^{(j)}(S^{(j)})\right] + c_I\, \mathbb{E}\left[N_I\left(t_0 - \sum_{j=1}^{N_R(t_0)} S^{(j)}\right)^+\right] \\ + c_U\, \mathbb{E}\left[\left(t_0 - \sum_{j=1}^{N_R(t_0)} S^{(j)} - \sigma_L^{(N_R(t_0)+1)}\right)^+\right] \end{aligned} \quad (10)$$

where $N_R(t_0)$ is the number of replacements on $[0, t_0]$ and where $S^{(j)}$, $C^{(j)}(S^{(j)})$ and $\sigma_L^{(j)}$ stand for the duration, cost and eventual failure time of the j-th cycle, respectively.

The CBM policy is optimized with respect to the maintenance threshold M and the reliability of the maintained system is then computed at time t_0 with $R(t_0) = \mathbb{P}(\tau > t_0)$, where τ is the lifetime of the maintained system.

In a real case application, the effective model is unknown. The most commonly used stochastic process to model cumulative deterioration is SGP. Sometimes it is employed without any checking of its matching with the data. We now come to numerical experiments, where the point is to highlight the possible consequences on the performance of the optimized CBM policy, if an erroneous SGP is used for modeling the system deterioration, instead of an EGP.

4. Numerical experiments

Two examples are here provided, where all the computations are made using Monte-Carlo simulation with $N = 1000$ repetitions.

Example 4.1. A system is considered, with deterioration described by an EGP, $A(t) = t^{1.5}$ and $b(t) = 1.1(t+1)^{-0.5}$. We also take

$L = 2, t_0 = 3, \epsilon = 0.05$, $c_I = 8$, $c_P = 90$, $c_F = 150$ and $c_U = 160$. The proposed CBM policy has been applied and optimized. The minimal cost is 304.8157, which is obtained for $M^\circ \simeq 0.8$ under the EGP model.

The chosen methodology to explore the possible consequences of an erroneous SGP modeling is the following: As a first step, a SGP is fitted from feed-back data based on observations of the (unmaintained) EGP system. In order not to obtain results perturbed by a bad choice for a parametric form of the SGP shape function or by too large errors in the estimation procedure, a large data set is considered (10 000 trajectories with 15 observations each) and the semi-parametric estimation procedure from [9] is applied. This provides some estimates $\hat{A}_{SGP}(t)$ and $\hat{b}_{SGP}$. The mean, variance and variance-to-mean ratio are plotted in Figure 1 under both EGP and SGP assumptions. Even using a non-parametric form for the SGP shape function, we can see that the SGP is unable to reproduce a similar behavior as the effective underlying EGP model for the moments and variance-to-mean ratio. We now want to investigate the practical consequences of an erroneous SGP model on the optimized CBM policy. With that aim, we apply the CBM policy considering the SGP model $\mathbf{Y} \sim \Gamma_0(\hat{A}_{SGP}(t), \hat{b}_{SGP})$, which means that the inspection times are determined based on this SGP model. The cost function is then minimized, which provides $M_{\text{exp}} = 1.2$ as optimal value, where the subscript "exp" refers to the fact that 1.2 is *expected* to be the optimal value. The corresponding expected minimal cost is 161.8065. Now, the point is to compute the effective cost (and other indicators) if we apply this wrongly optimized CBM policy (optimized with respect to the erroneous SGP) to the EGP system. For that, trajectories are generated with the true EGP model, considering M_{exp} as maintenance threshold and computing the inter-inspection times with the supposed SGP model $\mathbf{Y}$. This provides an *effective* cost of 242.7640. The mean number of inspections per cycle along with the cost and reliability are presented in Table 1, according to the different modeling assumptions. The first column provides the model used for the CBM policy (computation of the optimal value of M and of the inter-inspection times) and the one used for the deterioration. The first line (EGP/EGP) hence corresponds to the true optimal results, the second line (SGP/SGP) to the expected ones under SGP assumption and the third line (SGP/EGP) to the effective ones obtained when wrongly considering a SGP instead of an EGP. In that table, we can see that the expected cost is lower for the wrongly optimized CBM policy (SGP/EGP case) than for the correctly optimized one (EGP/EGP case), which might be surprising at first sight. As a matter of fact, the correct optimization procedure is made under safety constraints on the inter-inspection times, with which the SGP/EGP case does not comply any more: For example, the reliability at the first inspection time T_1 is around 81.5% for the SGP/EGP scenario, which is much below the imposed safety level of 95%. This also lead to a much lower reliability at time t_0. An erroneous modeling may hence induce safety problems in an applicative context.

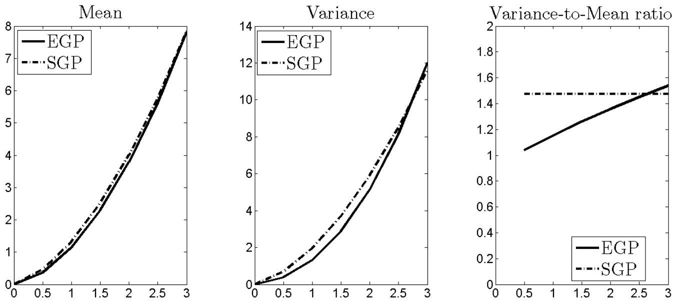

Figure 1. Mean, variance and variance-to-mean ratio for Example 4.2, EGP and SGP

Table 1. Results for Example 4.1.

Models for the CBM policy and for the deterioration	M	Mean number of inspections	$\mathbb{E}[C(t_0)]$	$R(t_0)$
EGP/EGP	$M^o = 0.8$	3.8288	304.8157	60.07%
SGP/SGP	$M_{\text{exp}} = 1.2$	2.1238	161.8065	61.81%
SGP/EGP	M_{exp}	1.6670	242.7640	29.99%

Example 4.2. The parameters of the EGP system now are $A(t) = t^{1.2}$ and $b(t) = (t+1)^{0.5}$. We also set $L = 7, t_0 = 10, \epsilon = 0.05, c_I = 6$, $c_P = 40$, $c_F = 80$ and $c_U = 100$. The same procedure is followed as in Example 4.1. Here, the required 95% safety level has been checked to be guaranteed for the SGP/EGP scenario (and it is even much beyond, around 99.98%). The results are displayed in Table 2, where we can see that both reliability and expected cost are higher for the SGP/EGP than for the EGP/EGP. Also, the wrong SGP assumption leads to too many inspections (and too many preventive replacements). Here, considering a SGP instead of an EGP leads to too high maintenance costs, when compared to those required to comply with the imposed safety constraints.

Table 2. Results for Example 4.2.

Models for the CBM policy and for the deterioration	M	Mean number of inspections	$\mathbb{E}[C(t_0)]$	$R(t_0)$
EGP/EGP	$M^o = 2.5$	1.0050	53.8326	95%
SGP/SGP	$M_{\text{exp}} = 4$	2.8952	347.1217	34.9%
SGP/EGP	M_{exp}	9.0713	158.5160	99.9%

As a conclusion, we can see that wrongly considering a SGP model instead of an EGP can lead to safety problems (overestimation of the reliability) or useless preventive maintenance actions and unnecessarily high maintenance costs. It is hence important to model the system deterioration as precisely as possible in an applicative context. This shows the interest to enlarge the modeling potential of existing models from the literature, such as EGPs do with respect to SGPs.

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