SIMULATION TECHNIQUES AND APPROXIMATED DISTRIBUTION OF AN EXTENDED GAMMA PROCESS

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Many research deals with the standard Gamma process to model the evolution of the cumulative deterioration of a system with time. When the variance-to-mean ratio of the system deterioration level varies with time, the standard Gamma process is not convenient anymore because it provides a constant ratio. A way to overcome this restriction is to consider the extended version of a Gamma process proposed by Cinlar [1980]. However, based on its technicity, the use of such a process for applicative purpose requires the preliminary development of technical tools. In this paper, we propose methods to simulate its paths and approximate its cumulative distribution function, as a first step.

Keywords: Degradation, Process with independent increments, Series representation, Inverse Laplace transform, Post-Widder formula.

1. Introduction

Standard Gamma processes are widely used to model the phenomena of cumulative degradation (see Van Noortwijk [10]). However, a notable restriction of a standard Gamma process is that its variance-to-mean ratio is constant with time. In order to overcome this restriction, we propose to use the extended version of a Gamma process introduced by Cinlar [1]. A standard Gamma process is characterized by a shape function and a constant scale parameter. For an Extended Gamma Process (EGP), the scale parameter may vary with time. This allows for more flexibility than its standard version, for modelling purpose. However, there is a cost and the use of an EGP presents some technical difficulties. Firstly, except for specific cases, there is no explicit formula for the probability distribution. Secondly, the exact simulation of such a process is generally impossible. These technical difficulties have lead Guida et al. [4] to use a discrete version of an EGP. We here propose to deal with the original continuous time

version.

This paper presents methods to simulate approximate paths of an EGP and to numerically assess its probability density function (pdf) and cumulative distribution function (cdf).

2. Definition of an EGP and first properties

Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and continuous function, and let $b > 0$. Recall that a standard Gamma process $\Gamma_0(a(t), b)$ with $a\,(t)$ as shape function and b as (constant) scale parameter is a stochastic process with independent, non-negative and gamma distributed increments. Its pdf at time t is given by

$$f_t(x) = \frac{b^{a(t)}}{\Gamma(a(t))} x^{a(t)-1} \exp(-bx), \forall x \in \mathbb{R}_+.$$

Now, let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\int_0^t \frac{1}{b(s)} da(s) < \infty$, for all $t \in \mathbb{R}_+$. Following [1,2], the process $X = (X_t)_{t\in\mathbb{R}_+}$ is said to be an EGP $\Gamma(a(t), b(t))$ with shape function $a(t)$ and scale function $b(t)$ if it can be represented as a stochastic integral with respect to a standard Gamma process:

$$X_t = \int_0^t \frac{dY_s}{b(s)} \tag{1}$$

where $(Y_t)_{t\geq 0}$ is a standard Gamma process $\Gamma_0(a(t), 1)$.

If $b(t)$ is constant, we obtain a standard Gamma process. An EGP can be proved to have independent increments and its distribution to be infinitely divisible. Also, an explicit formula is available for its Laplace transform, with

$$\mathcal{L}_{X_t}(\lambda) := \mathbb{E}\left(e^{-\lambda X_t}\right) = \exp\left(-\int_0^t \ln\left(1 + \frac{\lambda}{b(s)}\right) da(s)\right), \tag{2}$$

for all $t \in \mathbb{R}_+$ and all $\lambda \geq 0$. The mean and variance of an EGP are given by $\mathbb{E}(X_t) = \int_0^t \frac{da(s)}{b(s)}$ and $\mathbb{V}(X_t) = \int_0^t \frac{da(s)}{b(s)^2}$. In all the following, $(X_t)_{t\in\mathbb{R}_+}$ stands for an EGP $\Gamma(a(t), b(t))$, without any further notification.

3. Extensions of known results

3.1. *Series representations*

Series representations of an EGP are here presented, which can be used to approximately simulate its paths. The four series representations proposed

by Rosinski [9] for standard Gamma processes can easily be extended to the case of EGPs. The four methods have been studied and tested, but we only provide two of them here, which have been selected for their performance (in terms of computing time and precision).

Let $T \geq 0$ and let $\{U_n\}_{n\geq 1}$ be the points of a homogeneous Poisson process M with parameter $a(T)$. Let also $\{V_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables (r.v.) with distribution $H(dv) = \frac{da(v)}{a(T)}\mathbb{1}_{[0,T]}(v)$, independent of M. We have the two following representation results of $(X_t)_{0\leq t\leq T}$, where $\stackrel{\mathcal{D}}{=}$ means "is identically distributed as":

- Bondesson's series representation:

$$X_t \stackrel{\mathcal{D}}{=} \sum_{n\geq 1} \frac{1}{b(V_n)} \exp(-U_n)\ W_n \mathbb{1}_{[0,t]}(V_n),\ \text{for } 0 \leq t \leq T,$$

  where $\{W_n\}_{n\geq 1}$ is a sequence of i.i.d exponential r.v. with mean 1, independent of the Poisson process M and of the V_n's,
- Rejection's series representation:

$$X_t \stackrel{\mathcal{D}}{=} \sum_{n\geq 1} \frac{1}{b(V_n)} \frac{1}{\exp(U_n) - 1} \mathbb{1}_{[0,t]}(V_n) \mathbb{1}_E(U_n, R_n),\ \text{for}\ \ 0 \leq t \leq T,$$

  where $E(u,r) = \{\frac{\exp(u)}{\exp(u)-1} \exp(-(\exp(u)-1)^{-1}) \geq r\}$ and where $\{R_n\}_{n\geq 1}$ is a sequence of i.i.d uniform r.v. on $[0,1]$, independent of M and of the V_n's.

Approximate simulation of $(X_t)_{0\leq t\leq T}$ is done by truncating the series and by selecting only the points of the Poisson process which belong to a compact set $[0, B]$. This allows to have some control on the truncation error (see [5] for details).

3.2. *Post-Widder formulas*

As already noted, the distribution of an EGP is infinitely divisible and its Laplace transform is available in full form (see (2)). This allows to use the method from [11] to compute the pdf and cdf of an EGP, by inverting its Laplace transform through the Post-Widder formula. We get:

$$F_{X_t}(x) = \lim_{N\to\infty} \sum_{k=1}^{N} \left(-\frac{N}{x}\right)^k \frac{1}{k} \sum_{j=0}^{k-1} (-1)^{k-j} \frac{\mathcal{L}^{(j)}_{X_t}(\frac{N}{x})}{j!} \int_0^t \frac{da(s)}{g_{N,x}(s)^{k-j}}, \quad (3)$$

$$f_{X_t}(x) = \lim_{N\to\infty} \frac{(-N)^N}{x^{N+1}} \sum_{j=0}^{N-1} \frac{(-1)^{N-j} \mathcal{L}^{(j)}_{X_t}(\frac{N}{x})}{j!} \int_0^t \frac{da(s)}{g_{N,x}(s)^{N-j}}, \quad (4)$$

for all $x \in \mathbb{R}_+$, where $g_{N,x}(s) = (b(s) + \frac{N}{x})$ and $\mathcal{L}_{X_t}^{(j)}$ stands for the j-th derivative of $\mathcal{L}_{X_t}$.

4. A new approach

In case of a piecewise constant scale function $b(t)$, the process $(X_t)_{t\in\mathbb{R}_+}$ can be seen as the sum of standard Gamma processes. The simulation of its paths is hence immediate. Also, the random variable X_t simply is the sum of standard Gamma variables, and different tools are available in the literature to compute both its pdf and cdf (see [8] for a review). Based on this, we propose to approximate an EGP with a general scale function by another EGP with a piecewise constant scale function.

4.1. *Construction of the approximate process* $X^{(\epsilon)}$

Let $\epsilon > 0$ and $T > 0$. We first construct a piecewise constant $\frac{1}{b^{(\epsilon)}(t)}$ approximation of $\frac{1}{b(t)}$ on $[0, T]$ as below:

$$\forall t \in [0,T],\ \frac{1}{b^{(\epsilon)}(t)} = \sum_{i=0}^{n_t(\epsilon)} \frac{1}{b_i} \mathbb{1}_{[l_i, l_{i+1}[}(t)$$

where $n_t(\epsilon)$ is such that $l_{n_t(\epsilon)} \leq t < l_{n_t(\epsilon)+1}$, $l_0 = 0$ and the l_i's are defined recursively by

$$l_{i+1} = \sup\left\{ l \in]l_i; T] :\ \text{for all } l' \in [l_i; l], \left| \frac{1}{b(l_i)} - \frac{1}{b(l')} \right| < \epsilon \right\}$$

and

$$\frac{1}{b_i} = \frac{1}{l_{i+1} - l_i} \int_{l_i}^{l_{i+1}} \frac{1}{b(s)} ds.$$

Note that, based on the definition of an EGP (1) where we divide by $b(t)$, we construct an approximation of $\frac{1}{b(t)}$ instead of $b\,(t)$. Also, by construction, $\left| \frac{1}{b(t)} - \frac{1}{b^{(\epsilon)}(t)} \right| < \varepsilon$, for all $t \in [0, T]$.

Two other piecewise approximations of $\frac{1}{b(t)}$ are used in Subsection 4.3: $\frac{1}{b^{(\epsilon,-)}(t)} = \sum_{i=0}^{n_t(\epsilon)} \frac{1}{b_i^-} \mathbb{1}_{[l_i, l_{i+1}[}(t)$ and $\frac{1}{b^{(\epsilon,+)}(t)} = \sum_{i=0}^{n_t(\epsilon)} \frac{1}{b_i^+} \mathbb{1}_{[l_i, l_{i+1}[}(t)$ where $\frac{1}{b_i^-} = \inf\left\{ \frac{1}{b(l)}, l \in [l_i, l_{i+1}[\right\}$ and $\frac{1}{b_i^+} = \sup\left\{ \frac{1}{b(l)}, l \in [l_i, l_{i+1}[\right\}$. This provides

$$\frac{1}{b^{(\epsilon,-)}(t)} \leq \frac{1}{b(t)} \leq \frac{1}{b^{(\epsilon,+)}(t)},\ \text{for all } 0 \leq t \leq T. \tag{5}$$

In the following, we set $X^{(\epsilon)}$ ($X^{(\epsilon,-)}$, $X^{(\epsilon,+)}$) to be EGPs with scale functions $b^{(\epsilon)}(t)$ ($b^{(\epsilon,-)}(t)$, $b^{(\epsilon,+)}(t)$), and the same shape function $a(t)$. Let us note that, under mild assumptions, one can check that $X^{(\epsilon)}(X^{(\epsilon,-)}, X^{(\epsilon,+)}) \xrightarrow[n\to\infty]{} X$ in distribution, in norm L^2 and almost surely when $\varepsilon \to 0^+$.

4.2. *Approximation of the pdf and cdf of $X_t^{(\epsilon)}$ (and of X_t)*

We use a representation in terms of an infinite integral of the pdf and cdf of a finite sum of independent Gamma r.v.s [3] to compute the pdf and the cdf of $X_t^{(\epsilon)}$. We obtain:

$$f_{X_t^{(\epsilon)}}(x) = \lim_{K\to\infty} f^{(K)}_{X_t^{(\epsilon)}}(x), \qquad F_{X_t^{(\epsilon)}}(x) = \lim_{K\to\infty} F^{(K)}_{X_t^{(\epsilon)}}(x),$$

for all $x \in \mathbb{R}_+$, where

$$f^{(K)}_{X_t^{(\epsilon)}}(x) = \frac{1}{\pi}\int_0^K \frac{\cos(\sum_{p=1}^{n_t(\epsilon)} \alpha_p \arctan(u/b_p) - xu)}{\prod_{p=1}^{n_t(\epsilon)}(1+(u/b_p)^2)^{\alpha_p/2}} du, \tag{6}$$

$$F^{(K)}_{X_t^{(\epsilon)}}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^K \frac{\sin(\sum_{p=1}^{n_t(\epsilon)} \alpha_p \arctan(u/b_p) - xu)}{u\prod_{p=1}^{n_t(\epsilon)}(1+(u/b_p)^2)^{\alpha_p/2}} du \tag{7}$$

with $\alpha_p = a(l_p) - a(l_{p-1})$.
Taking K large enough, $(6-7)$ provide approximations of the cdf and of the pdf of $X_t^{(\epsilon)}$, and consequently of X_t. Note that an upper bound is available for the truncation error, with:

$$E^{(K)}_{X_t^{(\epsilon)}}(x) = |F_{X_t^{(\epsilon)}}(x) - F^{(K)}_{X_t^{(\epsilon)}}(x)| \le \frac{\prod_{p=1}^{n_t(\epsilon)} b_p^{\sum_{p=1}^{n_t(\epsilon)}(a(l_p)-a(l_{p-1}))}}{\pi a\left(l_{n_t(\epsilon)}\right) K^{\sum_{p=1}^{n_t(\epsilon)}(a(l_p)-a(l_{p-1}))}}, K \ge 0. \tag{8}$$

Similar results can be written for $X_t^{(\epsilon,-)}$ and $X_t^{(\epsilon,+)}$.

4.3. *Bounds for the cdf of X_t*

Based on (5), we easily derive bounds for X_t and for F_{X_t}:

$$\begin{aligned} \forall t, X_t^{(\epsilon,-)} \le X_t \le X_t^{(\epsilon,+)}, \\ F_{X_t^{(\epsilon,+)}}(x) \le F_{X_t}(x) \le F_{X_t^{(\epsilon,-)}}(x). \end{aligned} \tag{9}$$

Applying (8) and (9) we obtain:

$$F^{(K)}_{X_t^{(\epsilon,+)}}(x) - E^{(K)}_{X_t^{(\epsilon,+)}}(x) \le F_{X_t}(x) \le F^{(K)}_{X_t^{(\epsilon,-)}}(x) + E^{(K)}_{X_t^{(\epsilon,-)}}(x)$$

Table 1. Relative errors for the different methods of simulation of X_t for $t = 1$

	Approximation	*Bondesson*	*Rejection*
Relative error on mean	2.799×10^{-3}	2.784×10^{-3}	2.755×10^{-3}
Relative error on variance	8.778×10^{-3}	8.678×10^{-3}	8.762×10^{-3}
Relative error on Laplace	2.191×10^{-3}	2.256×10^{-3}	2.232×10^{-3}

and next:

$$\left| F_{X_t}(x) - \frac{m_t(x,\epsilon,K) + M_t(x,\epsilon,K)}{2} \right| \leq \frac{M_t(x,\epsilon,K) - m_t(x,\epsilon,K)}{2}, \quad (10)$$

with

$$m_t(x,\epsilon,K) = F^{(K)}_{X_t^{(\epsilon,+)}}(x) - E^{(K)}_{X_t^{(\epsilon,+)}}(x),$$
$$M_t(x,\epsilon,K) = F^{(K)}_{X_t^{(\epsilon,-)}}(x) + E^{(K)}_{X_t^{(\epsilon,-)}}(x).$$

This method provides us with computable bounds for the cdf F_{X_t} of an EGP, where the bounds can be made as tight as necessary, taking ε small enough. This method hence provides a way to numerically assess the cdf F_{X_t} at a known precision. This is used in Section 5 to get reference results.

5. Numerical Application

Here, we compare the methods presented before for an EGP $\Gamma(t,(t+1)^2)$. Firstly, we generate 10^5 approximate paths on $[0,1]$ by each of the three proposed simulation methods: the two series representations from Subsection 3.1 and the approximate process $X^{(\epsilon)}$ from Subsection 4.1. The control of the quality of the approximate simulation methods is made through the matching of empirical and theoretical mean, variance and average Laplace transform (mean on a grid of the domain). Table 1 gives the relative error for these three criteria and for the three methods. The results provided by the three methods are very close to one another.

We now compare the quality of the numerical assessment of the cdf of an EGP through both Post-Widder's (Subsection 3.2) and the approximation method provided by (7). Reference results are first computed for $F_{X_t}(x)$ according to Subsection 4.3 for $t = 2$, with a precision lower than 3.10^{-6}. The results are next provided in Table 2 for the mean absolute error on $F_{X_t}(x)$ for $x \in [0.01 : 0.1 : 10]$, for both the Post-Widder's and approximation methods with respect to the reference results. Both methods provide similar results with similar computation times. However, the approximation method may easily yield more accurate results (diminishing ε and increasing K) whereas the highest precision by Post-Widder's

Table 2. Mean absolute error on $F_{X_t}(x)$ for $t = 2$ and $x \in [0.01 : 0.1 : 10]$

Method	Mean absolute error	cpu time
Post-Widder ($N = 100$)	6.10^{-4}	4
Post-Widder ($N = 110$)	7.10^{-4}	4
Approximation ($\epsilon = 10^{-2}, K = 12$)	7.10^{-4}	3
Approximation ($\epsilon = 10^{-2}, K = 50$)	2.10^{-5}	7

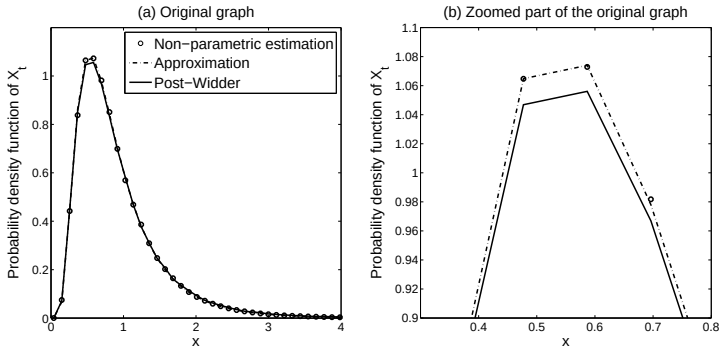

Figure 1. The pdf of X_t for $t = 10$ as a function of x

method is obtained for $N \simeq 100$ in the present example. This shortcut of Post-Widder's method had already been observed in [7].

6. Conclusion

Different tools have been presented here, firstly, for the approximate simulation of an EGP, secondly, for the numerical assessment of the cdf/pdf of an EGP. As for the simulation procedures, it seems that our approximate simulation scheme behaves as well as two of the most usual ones (Bondesson's and rejection methods), previously developed in the context of subordinators. Note that a specific simulation procedure for EGPs was proposed by [6]. It was not included here, because of the reduced size of the paper. However, we have tested it and it seems to be a little less performant than the three other ones presented here.

As for the numerical assessment of the cdf of an EGP, we could not find in the literature any procedures with a control on the precision. We here propose three discretization schemes of the rate function $1/b(t)$: one provides the best approximation among the three ($b^{(\varepsilon)}(t)$), the other two ($b^{(\varepsilon,+)}(t)$, $b^{(\varepsilon,-)}(t)$) provides bounds for X_t and for its cdf F_{X_t}. Beyond the fact that we have a control on the precision of the approximation of F_{X_t}, the results have been compared to those obtained by Post-Widder's

formula, at the advantage of our method.

As a consequence, the discretization method of the rate function seems to behave well, both for simulating paths and for the numerical assessment of the cdf of an EGP. Just as Post-Widder's method (4), the method also provides a way to compute an approximation of the pdf through (6). We illustrate it in Figure 1, where we plot a non-parametric estimation obtained from a sample of size 10^5, and the results by Post-Widder's and the present methods. The present method better agree with the non-parametric estimation than Post-Widder's method (see Figure 1b). Note, however, that contrary to the cdf, we do not have any control on the precision of the approximation of the pdf of an EGP.

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