Linking hyperbolic and parabolic p.d.e.'s.

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***Abstract*— In this article we show that from the existence and uniqueness of solutions to a hyperbolic partial differential equation (p.d.e.) existence and uniqueness of parabolic and other hyperbolic p.d.e.'s can be derived. Among others, we show that starting with the (undamped) wave equation we obtain existence and uniqueness of solutions for the uniform elliptic p.d.e.'s and for the Schrödinger equation.**

I. Introduction

Studying control theory for partial differential equations (p.d.e.'s), the first question normally encountered is the question of existence and uniqueness of solutions for the (homogeneous) p.d.e. Since the p.d.e. is linear we have to show the existence of a strongly continuous semigroup. In many cases it is known from the physical problem formulation that any solution will not increase in norm (energy). This leads to the problem of showing that the operator associated to the p.d.e. generates a contraction semigroup. In this paper we show that knowing that one operator generates a contraction semigroup implies that many other operators generate a contraction semigroup as well. This goes much further than the well-known bounded perturbation result for semigroups. Among others, we show that the existence and uniqueness of the diffusion equation and of the Schrödinger equation can be obtained from the same wave equation.

II. Motivations and problem statement

Consider the p.d.e.

$$\dot{x}(t) = \left(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*\right)\left(\mathcal{H}x(t)\right), \tag{1}$$

where $\mathcal{J}$ is formally skew-adjoint, $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$, and S is non-negative and $\mathcal{H}$ is positive. Furthermore, $x(t)$ is for every t a function of the spatial variable $\zeta \in \Omega$ with Ω a subset of $\mathbb{R}^d$. In many p.d.e.'s we can recognize the form (1). For a hyperbolic p.d.e., S will be zero, and for a parabolic p.d.e. $\mathcal{J}$ will be zero. We illustrate this with a simple one-dimensional p.d.e.

Example 2.1: Consider the one-dimensional wave equation on the spatial domain $[a, b]$. One cause of damping is structural damping. Structural damping arises from internal friction in the material converting vibrational energy into heat. In this case the vibrating string is modeled by

$$\begin{aligned}\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] + \\ &\qquad k_s \frac{\partial^2}{\partial \zeta^2}\left[\frac{\partial w}{\partial t}(\zeta,t)\right]\end{aligned} \tag{2}$$

where $\rho(\zeta)$ is the linear mass density, $T(\zeta)$ is the elasticity modulus (taking values in a compact interval of $(0,\infty)$) and k_s is the (positive) structural damping coefficient.

Defining the state as $x = \begin{pmatrix} \rho\frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta}\end{pmatrix}$ the p.d.e. (2) may be written as the p.d.e. (1) with:

$$\mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\frac{\partial}{\partial\zeta}$$

and

$$\mathcal{G}_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{\partial}{\partial \zeta}, \quad \mathcal{G}_R^* = -\begin{pmatrix} 1 & 0 \end{pmatrix}\frac{\partial}{\partial\zeta}, \quad S = k_s.$$

In this example the perturbation term indeed corresponds to some physical dissipation of energy, and when $k_s = 0$, or equivalently when $S = 0$, we have a hyperbolic p.d.e. ■

Equation (1) can be seen as the linear control system

$$\dot{x}(t) = \mathcal{J}\mathcal{H}x(t) + \mathcal{G}_R u(t) \tag{3}$$
$$y(t) = \mathcal{G}_R^*\mathcal{H}x(t), \tag{4}$$

which has conjugated input and output in the sense that the input and output maps are defined by the adjoint operators $\mathcal{G}_R$ and $\mathcal{G}_R^*$. It defines a so-called port-Hamiltonian system, see [3]. The p.d.e. (1) may then be regarded as closing the loop of the linear control system (3)–(4) with $u(t) = -Sy(t)$. If the control system (3)–(4) is well-posed, then the p.d.e. (1) possesses a solution according to Staffans [2] and Weiss [4]. The precise definition of well-posedness is not so important here. However, it is important to state that well-posedness implies that $\mathcal{J}$ is the operator which is the most unbounded. Or putting it more simply, $\mathcal{J}$ will be the operator containing the highest spatial derivatives. As may be seen from the following example, this is too restrictive.

Example 2.2 (Heat equation): Let Ω be bounded open connected set in $\mathbb{R}^3$ with smooth boundary. The heat equation on Ω is given by

$$\frac{\partial x}{\partial t}(\zeta,t) = \Delta x(\zeta,t), \quad \zeta\in\Omega,\ t\geq 0, \tag{5}$$

where Δ denotes the Laplacian, i.e., $\Delta x = \frac{\partial^2 x}{\partial\zeta_1^2} + \frac{\partial^2 x}{\partial\zeta_2^2} + \frac{\partial^2 x}{\partial\zeta_3^2}$. We write this Laplacian as

$$\Delta = \mathrm{div}\cdot\nabla \tag{6}$$

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with $\nabla x = \left(\frac{\partial x}{\partial \zeta_1}, \frac{\partial x}{\partial \zeta_2}, \frac{\partial x}{\partial \zeta_3} \right)^T$ and $\mathrm{div} f = \frac{\partial f_1}{\partial \zeta_1} + \frac{\partial f_2}{\partial \zeta_2} + \frac{\partial f_3}{\partial \zeta_3}$. It is well-known that $-\nabla$ is the (formal) adjoint of the divergence div, and so if we choose $\mathcal{J} = 0$, $\mathcal{H} = I$, $\mathcal{G}_R = \mathrm{div}$, and $S = I$, then (5) is in the form (1). ■

Thus this example shows that the closed-loop point of view is not the correct way of regarding the p.d.e. (1), and hence we shall not follow this idea. Instead of this, we decompose the right hand-side of equation (1) as the operator mapping $\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)$ to $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ defined by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} := \mathcal{J}_{\mathrm{ext}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad (7)$$

together with the closure relation

$$e_2 = S f_2. \quad (8)$$

Combining these equations it is easy to see that $f_1 = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) e_1$, and thus in this way we are able to build new p.d.e.'s even when $\mathcal{J} = 0$. As explained in [5] the signals appearing in the closed loop system form always a subset of the signals in the open loop system. However, in our closure this does not longer hold, as can be seen in e.g. Example 2.7 in which we transform a hyperbolic p.d.e. into a parabolic one.

It may be noted that in the decomposition (7)–(8), the formally skew-symmetric operator $\mathcal{J}_{\mathrm{ext}}$ appears. This operator is related to the extension of Hamiltonian systems defined on state spaces endowed with a Poisson bracket to controlled Hamiltonian systems (called *port-Hamiltonian systems*) defined on Dirac structures [3].

In this paper we study the relation between the p.d.e. (1) and the (extended) p.d.e., (*i.e.* the Hamiltonian system):

$$\dot{x}_{\mathrm{ext}}(t) = \mathcal{J}_{\mathrm{ext}} \mathcal{H}_{\mathrm{ext}} x_{\mathrm{ext}}(t). \quad (9)$$

where $\mathcal{H}_{\mathrm{ext}}$ is an appropriate positive valued matrix. This may be replaced by a coercive operator, but we don't need that generality in this paper. As stated in the beginning of this section, the aim is to show that (1) possesses a unique solution for any initial condition. For this we need boundary conditions to the p.d.e. and a space of initial condition. Putting it differently, we have to define operators associated to our p.d.e.'s. By doing so, $\mathcal{J}_{\mathrm{ext}}$ becomes an operator with a proper domain. Distinguishing between these cases, we change the notation and use A, A_{ext} for the operators. Furthermore, we assume that our linear spaces are complex valued. Thus we consider the following operator defined on the product space of two complex Hilbert spaces X_1 and X_2:

$$A_{\mathrm{ext}} = \begin{pmatrix} A_1 \\ A_{21} \;\; 0 \end{pmatrix} \quad (10)$$

with A_1 a linear operator defined on $X_1 \times X_2$ and A_{21} a linear operator defined on X_1. The domain of this operator is given by

$$D(A_{\mathrm{ext}}) = \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in D(A_{21}) \\ \text{and } (x_1, x_2) \in D(A_1)\}. \quad (11)$$

Furthermore, S is a bounded operator from X_2 to X_2. We make the following assumptions throughout the rest of the paper.

Assumption 2.3: We assume that with the domain (11), A_{ext} generates a contraction semigroup on $X_1 \times X_2$. Furthermore, S satisfies

$$\mathrm{Re}\langle S x_2, x_2 \rangle \geq 0. \quad (12)$$

We recall that the operator A generates a contraction semigroup on the Hilbert space X if and only if A is *dissipative*, i.e., $\mathrm{Re}\langle Ax, x\rangle \leq 0$ for all x in the domain of A, and the range of $A - I$ equals X. This result is known as the *Lumer-Phillips theorem*.

With A_{ext} and S we define the operator A_S on X_1 as

$$A_S x_1 = A_1 \begin{pmatrix} x_1 \\ S A_{21} x_1 \end{pmatrix} \quad (13)$$

with domain

$$D(A_S) = \left\{ x_1 \in D(A_{21}) \,\middle|\, \begin{pmatrix} x_1 \\ S A_{21} x_1 \end{pmatrix} \in D(A_{\mathrm{ext}}) \right\}. \quad (14)$$

This A_S is the operator associated to $\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*$, see also Examples 2.7 and 2.10. In the class of p.d.e.'s (1), the operator $\mathcal{H}$ corresponds to the definition of the energy of the system and the dissipativity of the physical system is naturally expressed with respect to the norm induced by the energy. Although this energy characterizes an essential physical property, we show in the following lemma that for the proofs of the existence of a contraction semigroup, we may assume that $\mathcal{H} = I$ without loss of generality.

Note that the operator $\mathcal{H}$ is *coercive* if it is bounded, self-adjoint, and satisfies $\langle x, \mathcal{H}x\rangle \geq \varepsilon \|x\|^2$ for all x and some $\varepsilon > 0$.

Lemma 2.4: Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{H}$ be a coercive operator on X. With this operator we define the new inner product

$$\langle x_1, x_2 \rangle_{\mathcal{H}} := \langle x_1, \mathcal{H} x_2 \rangle. \quad (15)$$

Then the following holds

1) The norms induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ are equivalent.
2) The operator A with domain $D(A)$ generates a contraction semigroup on X with respect to the norm $\|\cdot\|$ if and only if the operator $A\mathcal{H}$ with domain $D(A\mathcal{H}) = \{x \in X \mid \mathcal{H}x \in D(A)\}$ generates a contraction semigroup on X with respect to the norm $\|\cdot\|_{\mathcal{H}}$

In the sequel, we shall derive conditions, such that A_S generates a contraction semigroup on X_1. The above lemma implies that we may prove this under the assumption that $\mathcal{H} = I$. We begin by proving that A_S is dissipative.

Lemma 2.5: Let A_{ext} be a dissipative operator and let S satisfy (12). The operator A_S as defined by (13) and (14) is dissipative.

Proof: Since $\langle x, y\rangle + \langle y, x\rangle = 2\mathrm{Re}\,\langle x, y\rangle$, we only have to estimate the real part of $\langle A_S x_1, x_1\rangle$. Using its definition, we find for $x_1 \in D(A_S)$:

$$\begin{aligned}\mathrm{Re}\langle A_S x_1, x_1\rangle &= \mathrm{Re}\left\langle A_{\rm ext}\begin{pmatrix} x_1 \\ SA_{21}x_1\end{pmatrix}, \begin{pmatrix} x_1 \\ 0\end{pmatrix}\right\rangle \\ &= \mathrm{Re}\left\langle A_{\rm ext}\begin{pmatrix} x_1 \\ SA_{21}x_1\end{pmatrix}, \begin{pmatrix} x_1 \\ SA_{21}x_1\end{pmatrix}\right\rangle \\ &\quad - \mathrm{Re}\,\langle A_{21}x_1, SA_{21}x_1\rangle \\ &\le 0 + 0,\end{aligned}$$

where we have used that $A_{\rm ext}$ is a dissipative operator, and that S satisfies (12). ∎

The following theorem shows that A_S generates a contraction semigroup for dissipation terms S with $S+S^*$ coercive.

Theorem 2.6: If $A_{\rm ext}$ is the generator of a contraction semigroup, and if S satisfies $\mathrm{Re}\langle Sx, x\rangle \ge \varepsilon\|x\|^2$ for some $\varepsilon > 0$, independent of x, then A_S generates a contraction semigroup.

Proof: By Lemma 2.4, we know that A_S is dissipative. By the Lumer-Phillips theorem it remains to show that $I - A_S$ is surjective.

Since S satisfies $\mathrm{Re}\langle Sx, x\rangle \ge \varepsilon\|x\|^2$, we see that $\mathrm{Re}\langle y, S^{-1}y\rangle \ge \varepsilon\|S^{-1}y\|^2 \ge \frac{\varepsilon}{\|S\|^2}\|y\|^2$. So there exists a $\delta \in (0,1)$ is such that $\mathrm{Re}\langle S^{-1}x, x\rangle \ge \delta\|x\|^2$.

Let P be defined as

$$P = \begin{pmatrix} (1-\delta)I & 0 \\ 0 & S^{-1} - \delta I\end{pmatrix}.$$

By the choice of δ we see that $\mathrm{Re}\langle Px, x\rangle \ge 0$. Thus the bounded perturbation of $A_{\rm ext}$ given by $A_{\rm ext} - P$ generates a contraction semigroup. By the Lumer-Phillips Theorem this implies that for all $f \in X_1$ there exists a $(x_1, x_2) \in D(A_{\rm ext})$ such that

$$\begin{pmatrix} f \\ 0\end{pmatrix} = [\delta I - A_{\rm ext} + P]\begin{pmatrix} x_1 \\ x_2\end{pmatrix}. \tag{16}$$

Hence

$$f = x_1 - A_1\left(\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right) \tag{17}$$

$$0 = \delta x_2 - A_{21}x_1 + S^{-1}x_2 - \delta x_2 \tag{18}$$

From equation (10) we see that $x_2 = SA_{21}x_1$ and thus $x_1 \in D(A_S)$. Combining this with equation (9), we find

$$f = x_1 - A_1\left(\begin{smallmatrix} x_1 \\ SA_{21}x_1\end{smallmatrix}\right) = (I - A_S)x_1. \tag{19}$$

Thus $I - A_S$ has full range, and so we conclude that A_S generates a contraction semigroup. ∎

We apply this result on uniformly elliptic p.d.e.'s

Example 2.7: Let Ω be bounded open connected set in $\mathbb{R}^3$ with smooth boundary. From Example 2.2, we see that the choice for $A_{\rm ext}$ is

$$A_{\rm ext} = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0\end{pmatrix}.$$

As domain we choose

$$D(A_{\rm ext}) = \Big\{\begin{pmatrix} e_1 \\ e_2\end{pmatrix} \in L^2(\Omega)\times L^2(\Omega;\mathbb{C}^3) \mid e_2 \in H_{\rm div}(\Omega), \\ e_1 \in H^1(\Omega) \text{ and } e_1 = 0 \text{ on } \partial\Omega\Big\}.$$

Since the adjoint of the operator ∇ with domain $H_0^1(\Omega)$ equals $-\mathrm{div}$ with domain $H_{\rm div}(\Omega) = \{f \in L^2(\Omega;\mathbb{C}^3) \mid \mathrm{div} f \in L^2(\Omega)\}$, we have that $A_{\rm ext}$ generates a unitary group. We remark that this operator is associated to the three dimensional wave equation, which is hold still at the boundary.

Let $Q(\zeta) \in L^\infty(\Omega;\mathbb{C}^{3\times 3})$ be a matrix valued function such that there exists an $\varepsilon > 0$

$$\mathrm{Re}\langle z, Q(\zeta)z\rangle \ge \varepsilon\|z\|^2, \qquad z \in \mathbb{C}^3, \zeta \in \Omega \tag{20}$$

With this function we associate the operator from $L^2(\Omega;\mathbb{C}^3)$ to $L^2(\Omega;\mathbb{C}^3)$ defined as

$$(Sf)(\zeta) = Q(\zeta)f(\zeta). \tag{21}$$

The operator A_S becomes, see (13),

$$(A_S e_1)(\zeta) = \sum_{k=1}^3 \frac{\partial}{\partial\zeta_k}\left(\sum_{\ell=1}^3 q_{k\ell}(\zeta)\frac{\partial e_1}{\partial\zeta_\ell}(\zeta)\right). \tag{22}$$

with domain

$$D(A_S) = \{e_1 \in H^1(\Omega) \mid S\nabla e_1 \in H_{\rm div}(\Omega) \\ \text{and } e_1 = 0 \text{ on } \partial\Omega\}.$$

By condition (20) we see that S is coercive, and so by Theorem 2.6 A_S generates a contraction semigroup on $L^2(\Omega)$. The operator A_S with S satisfying (20) is known to be a *uniformly elliptic* operator written in *divergence form*, see e.g. [1]. We remark that for $Q(\zeta) \equiv I_3$, we obtain the heat equation of Example 2.2.

So for $S + S^* \ge \varepsilon I > 0$, the operator A_S generates a contraction semigroup. The following example shows that this does not hold when $S + S^* = 0$

Example 2.8: Let A_0 be a bounded, injective, positive, self-adjoint operator on the Hilbert space X_0, and assume further that the (algebraic) inverse of A_0 is unbounded. Let this operator define $X_1 = X_2 = X_0 \oplus X_0$,

$$A_{12} = \begin{pmatrix} 0 & A_0 \\ A_0^{-1} & 0\end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & -A_0^{-1} \\ -A_0 & 0\end{pmatrix}. \tag{23}$$

It is easy to see that $A_{\rm ext} := \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0\end{smallmatrix}\right)$ is skew-adjoint, and hence it generates a unitary group.

For S we take the operator

$$S = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}. \tag{24}$$

Calculating A_S gives

$$\begin{aligned} A_S &= A_{12}SA_{21} \\ &= \begin{pmatrix} 0 & A_0 \\ A_0^{-1} & 0\end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} 0 & -A_0^{-1} \\ -A_0 & 0\end{pmatrix} \\ &= \begin{pmatrix} -A_0 & 0 \\ 0 & A_0^{-1}\end{pmatrix}\begin{pmatrix} 0 & -A_0^{-1} \\ -A_0 & 0\end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}.\end{aligned}$$

Hence it is a bounded operator. However, by definition, the domain of A_S is a subset of the domain of A_{21}. The domain is dense, but unequal to $X_1 \oplus X_2$. Hence the operator A_S is not closed and therefore cannot be the generator of a semigroup. ■

So if $S + S^* \geq 0$, then Theorem 2.6 does not need to hold. However, we still have the following result.

Theorem 2.9: Let $A_{\text{ext}} = \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right)$ with domain $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$ generate a contraction semigroup, then $A_S := -iA_{12}A_{21}$ with domain $D(A_S) = \{x_1 \in D(A_{21}) \mid A_{21}x_1 \in D(A_{12})\}$ generates a group on X_1.

We apply the above result on the Schrödinger equation.

Example 2.10: Let Ω be bounded open connected set in $\mathbb{R}^3$ with smooth boundary. The A_{ext} of Example 2.7 satisfies the condition of Theorem 2.9. Choosing $S = iI$ we the associated equation given by

$$A_S = i\Delta,$$

with domain

$$D(A_S) = \{e_1 \in H^1(\Omega) \mid \nabla e_1 \in H_{\text{div}}(\Omega) \\ \text{and } e_1 = 0 \text{ on } \partial\Omega\}.$$

By Theorem 2.9 we know that this generates a unitary group on $L^2(\Omega)$. Since positive constants will not effect this, the Schrödinger equation on Ω for a free particle given by

$$\frac{\partial x}{\partial t}(\zeta, t) = i\frac{\hbar}{2m}\Delta x(\zeta, t), \quad \zeta \in \Omega,\ t \geq 0,\ x|_{\partial\Omega} = 0, \tag{25}$$

where $\hbar$ is the reduced Planck constant, m the mass of the particle, has a unique solution with constant $L^2(\Omega)$-norm. This corresponds to a particle trapped in a potential well.

III. Conclusion

In this paper we have presented a new idea for proving existence and uniqueness of p.d.e.'s. We showed that starting from the same wave equation all uniformly elliptic p.d.e.'s and the Schrödinger equation can be recovered. However, much more is possible, starting from two Schrödinger equations the double Laplacian $-\Delta^2 = i\Delta \cdot I \cdot i\Delta$ can be constructed. Furthermore, the characterization of all boundary conditions for which a hyperbolic p.d.e.'s generates a contraction semigroup can be obtained.

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