Knots, links and (informationally complete) quantum measurements

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Published work and goals

- The concept of a magic state (in **quantum computing**) and that of minimal informationally complete positive operator valued measure: IC-POVM (in **quantum smeasurements**) make a good team as shown recently [1].
- Most low dimensional IC-POVMs found are from subgroups of the modular group $\Gamma = PSL(2,\mathbb{Z})$. This can be understood from the picture of the trefoil knot and the related **3-manifolds** [2].

[1] M. Planat and R. Ul Haq, The magic of universal quantum computing with permutations, *Advances in mathematical physics* **217**, ID 5287862 (2017); M. Planat and Z. Gedik, Magic informationally complete POVMs with permutations, *R. Soc. open sci.* **4** 170387 (2017).

[2] M. Planat, The Poincaré half-plane for informationally complete POVMs, *Entropy* **20** 16 (2018); M. Planat, R. Ascheim, M. Amaral and L. Irwin, Universal quantum computing and three-manifolds (Preprint).

From quantum information to the trefoil knot 3-manifold (s)

- From permutation groups to magic states and IC-POVMs
- IC-POVMs, the modular group $PSL(2,\mathbb{Z})$ and the trefoil knot

The magic of universal quantum computing with permutations 1

- **From permutation groups to quantum gates**

  $X=\begin{pmatrix}0&1\\1&0\end{pmatrix}\equiv(2,1)$, $I\otimes X\equiv(2,1)(4,3)$ acting on qubits,

  CNOT$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\equiv(1,2)(4,3)$ acting on 2-qubits,

  CCNOT$\equiv(1,2,3,4,5,6)(8,7)$ acting on 3-qubits,

  $X=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\equiv(2,3,1)$ acting on qutrits.

- **Magic states for universal quantum computation**

  Fault tolerant quantum computing protocols based on stabilizer states have to be complemented by **magic states** to reach **quantum universality**. Two distillation protocols based on single qubit magic states $|H\rangle$ and $|T\rangle$ are with

  $|H\rangle=\cos(\frac{\pi}{8})\,|0\rangle+\sin(\frac{\pi}{8})\,|1\rangle$ and

  $|T\rangle=\cos(\beta)\,|0\rangle+\exp\left(\frac{i\pi}{4}\right)\sin(\beta)\,|1\rangle\,,\quad \cos(2\beta)=\frac{1}{\sqrt{3}}.$ [3]

[3] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, *Phys. Rev.* **A71** 022316 (2005).

The generalized Pauli group

- Later, **we construct IC-POVMs using the covariance with respect to the generalized d-dimensional Pauli group** that is generated by the shift and clock operators as follows

$$X|j\rangle = |j+1 \mod d\rangle$$
$$Z|j\rangle = \omega^j |j\rangle \qquad (1)$$

with $\omega = \exp(2i\pi/d)$ a d-th root of unity.

A general Pauli (also called Heisenberg-Weyl) operator is of the form

$$T_{(m,j)} = \begin{cases} i^{jm} Z^m X^j & \text{if } d = 2 \\ \omega^{-jm/2} Z^m X^j & \text{if } d \neq 2. \end{cases} \qquad (2)$$

where $(j,m) \in \mathbb{Z}_d \times \mathbb{Z}_d$. **For N particules**, one takes the Kronecker product of qudit elements N times.

Stabilizer states are defined as eigenstates of the Pauli group.

The discrete Wigner function

- Phase point operators on $\mathbb{Z}_d \times \mathbb{Z}_d$ (d a prime) are as (Wootters, 1987)

$$A_\alpha = \frac{1}{d}\sum_{j,m=0}^{d-1} \omega^{pj-qm+jm/2} X^j Z^m \quad \text{with} \quad \alpha = (q,p \quad \text{and}$$

(i) A_α is Hermitian, **(ii)** $\mathrm{tr}(A_\alpha A_\beta) = d\delta_{\alpha\beta}$, **(iii)**Taking any complete set of d parallel lines (called a striation), construct the average $P_\lambda = \frac{1}{d}\sum_{\alpha\in\lambda} A_\alpha$ on each line λ. The d operators P_λ form a set of mutually orthogonal projectors whose sum is I. The d^2 (linearly independent) phase point operators A_α form a basis of the d-dimensional Hilbert space so that

$$\rho = \sum_{q,p} W_\rho(q,p)A(q,p), \quad W_\rho(q,p) = \frac{1}{d}\mathrm{tr}[\rho A(q,p)].$$

- with the (real) coefficients given by the Wootters discrete Wigner function. Unlike the continuous case, the discrete Wigner function is **a quasi probability distribution that may take negative values**. On a Hilbert space of odd dimension (Gross, 2007), the only pure states to possess a non-negative discrete Wigner function are stabilizer states.

The magic of universal quantum computing with permutations 2 (M. Planat and R. Ul Haq, 2017)

dim	magic state ρ	sum of negative entries W_ρ	Remark
2	$\|H\rangle$	$(1-\sqrt{2})/4 \sim -0.1035$	Bravyi
	$\|T\rangle$	positive	Bravyi
3	$(0,1,1)$	-1/3	Norrell
	$(0,1,-1)$	-1/3	strange
4	$(0,1,1,1)$	$-1/6$	A_4
	$(0,1,-\omega,\omega-1)$	$(2-3\sqrt{3})/12 \sim -0.266$	
5	$(0,1,1,1,1)$	$-\sqrt{5}/5 \sim -0.447$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
	$(0,1,-1,-1,1)$	$-2/5$	
	$(0,0,0,1,\pm 1)$	$-(\sqrt{5}+1)/10 \sim -0.324$	S_5
	$(0,0,1,1,1)$	$-(1+3\sqrt{5})/15 \sim -0.514$	
6	$(0,1,1,1,1,1)$	$-(3\sqrt{3}+7)/30 \sim -0.406$	A_5
	$(0,0,1,1,1,1)$	$-(\sqrt{3}+1)/6 \sim -0.455$	A_6
	$(0,0,1,-1,-1,1)$	$-(\sqrt{3}+4)/12 \sim -0.478$	
7	$(0,1,1,1,1,1,1)$	-0.499	$\mathbb{Z}_7 \rtimes \mathbb{Z}_6$
	$(0,0,0,0,1,1,1)$	-0.504	$PSL(2,7)$
	$(0,0,0,0,0,1,\pm 1)$	-0.321	

Informationally complete POVMs: IC-POVMs (M. Planat and Z. Gedik, 2017)

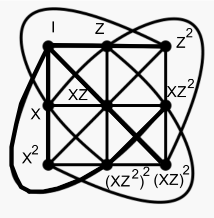
- Using **permutation groups**, we discover **minimal IC-POVMs** (i.e. whose rank of the Gram matrix is d^2) and with Hermitian angles $|\langle\psi_i|\psi_j\rangle|_{i\neq j} \in A = \{a_1,\ldots,a_l\}$, a discrete set of values of small cardinality l. A SIC is equiangular with $|A| = 1$ and $a_1 = \frac{1}{\sqrt{d+1}}$.
- The states encountered below are considered to live in a **cyclotomic field** $\mathbb{F} = \mathbb{Q}[\exp(\frac{2i\pi}{n})]$, with $n = \text{GCD}(d,r)$, the greatest common divisor of d and r, for some r. The Hermitian angle is defined as $|\langle\psi_i|\psi_j\rangle|_{i\neq j} = \|(\psi_i,\psi_j)\|^{\frac{1}{\deg}}$, where $\|.\|$ means the field norm [4] of the pair (ψ_i,ψ_j) in $\mathbb{F}$ and deg is the degree of the extension $\mathbb{F}$ over the rational field $\mathbb{Q}$.
- For the IC-POVMs under consideration below, in dimensions $d = 3$, 4, 5, 6 and 7, one has to choose $n = 3$, 12, 20, 6 and 21 respectively, in order to be able **to compute the action of the Pauli group**. Calculations are performed with **Magma**.

[4] H. Cohen, A course in computational algebraic number theory (Springer, New York, 1996, p. 162).

The single qutrit (Hesse) SIC-POVM from permutations: 1

- The symmetric group S_3 contains the **permutation matrices** I, X and X^2 of the Pauli group, where $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \equiv (2,3,1)$ and three **extra permutations** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \equiv (2,3)$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \equiv (1,3)$ and $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv (1,2)$, that do not lie in the Pauli group but are parts of the Clifford group.
- Taking the **eigensystem of the latter matrices**, it is not difficult check that there exists two types of qutrit magic states of the form $(0,1,\pm 1) \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle \pm |2\rangle)$. Then, taking the action of the nine qutrit Pauli matrices, one arrives at the well known **Hesse SIC** (Bengtsson, 2010,Tabia, 2013, Hughston, 2007).

The single qutrit (Hesse) SIC-POVM from permutations: 2



Magic qutrit POVM's
(0,1,1) or (0,1,-1)

(a)

- The Hesse configuration resulting from the qutrit POVM. The lines of the configuration correspond to traces of triple products of the corresponding projectors equal to $\frac{1}{8}$ [for the state $(0,1,-1)$] and $\pm\frac{1}{8}$ [for the state $(0,1,1)$]. Bold lines are for commuting operator pairs.

IC-POVMs in dimensions 2 to 12

dim	magic state	$\lvert\langle\psi_i\rvert\psi_j\rangle\rvert^2_{i\neq j}$	Geometry
2	$\lvert T\rangle$	$1/3$	tetrahedron
3	$(0,1,\pm 1)$	$1/4$	Hesse SIC
4	$(0,1,-\omega_6,\omega_6-1)$	$\{1/3,1/3^2\}$	**Mermin square***
5	$(0,1,-1,-1,1)$	$1/4^2$	Petersen graph
	$(0,1,i,-i,-1)$		
	$(0,1,1,1,1)$	$\{1/3^2,(2/3)^2\}$	
6	$(0,1,\omega_6-1,0,-\omega_6,0)$	$\{1/3,1/3^2\}$	Borromean rings
7	$(1,-\omega_3-1,-\omega_3,\omega_3,\omega_3+1,-1,0)$	$1/6^2$	unknown
8	$(-1\pm i,1,1,1,1,1,1,1)$	$1/9$	$[63_3]$ **Hoggar SIC***
9	$(1,1,0,0,0,0,-1,0,-1)$	$\{1/4,1/4^2\}$	$[9_3]$ **Pappus conf.***
12	$(0,1,\omega_6-1,\omega_6-1,1,1,$ $\omega_6-1,-\omega_6,-\omega_6,0,-\omega_6,0)$	8 values	Fig. 6

- Magic states of IC-POVMs in dimensions 2 to 12. *In dimensions 4, 8 and 9, a proof of the two-qubit, two-qutrit and three-qubit Kochen-Specker theorem follows from the IC-POVM.

A two-qubit IC-POVM from permutations and the Mermin square: 1

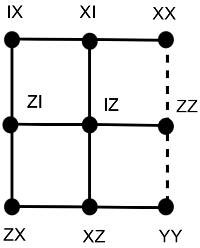
- From now we restrict to a **magic groups** (of gates showing one entry of 1 on their main diagonals). This only happens for a group isomorphic to the alternating group

$$A_4 \cong \left\langle \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix} \right\rangle .$$

  One finds magic states of type $(0,1,1,1)$ and $(0,1,-\omega_6,\omega_6-1)$, with $\omega_6=\exp(\frac{2i\pi}{6})$.

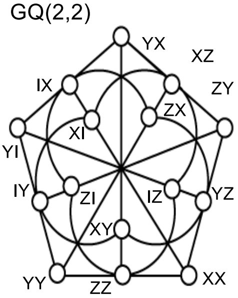
- Taking the action of the 2**QB Pauli group on the latter type of state**, the corresponding pure projectors sum to 4 times the identity (to form a **POVM**) and are independent, with the pairwise distinct products satisfying the dichotomic relation $\mathrm{tr}(\Pi_i\Pi_j)_{i\neq j}=|\langle\psi_i|\,\psi_j\rangle|^2_{i\neq j}\in\{\frac{1}{3},\frac{1}{3^2}\}$. Thus the 16 projectors Π_i build an **asymmetric informationally complete** measurement not discovered so far.

A two-qubit IC-POVM from permutations and the Mermin square: 2



2QB IC-POVM : Mermin square

(b)



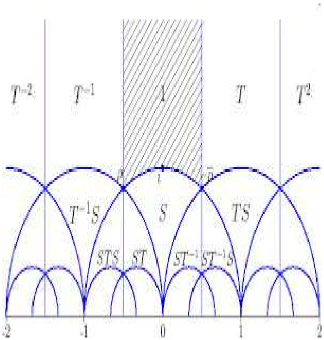
(c)

- The organization of triple products of projectors leads to the generalized quadrangle $GQ(2,2)$ pictured in (c) whose subset is Mermin square (b). Traces of triple products for rows (resp. columns) of Mermin square equal $-\frac{1}{27}$ (resp. $\frac{1}{27}$).

The modular group $\Gamma = PSL(2,\mathbb{Z})$

- The modular group $\Gamma = PSL(2,\mathbb{Z})$ acts on the **Poincaré hyperbolic plane** $\mathbb{H} = \{x, y \in \mathbb{R} | y > 0\}$ as a discrete subgroup of real Möbius transformations $z \to \frac{az+b}{cz+d}$ of $PSL(2,\mathbb{R})$ acting on $\mathbb{H}$.
- Important mathematical objects are the moduli space of elliptic curves, which is the quotient space $\mathbb{H}/\Gamma$, and modular forms that map pair of points of $\mathbb{H}$ up to a weight factor and elliptic curves (via the 1995 modularity theorem) (Diamond,2005).
- The modular group Γ acts **discontinuously** on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \infty$. Γ tesselates $\mathbb{H}^*$ with ∞ many copies of a fundamental domain $\mathcal{F} = \{z \in \mathbb{H}$ with $|z| > 1, \Re(z) < \frac{1}{2}$. The modular group Γ is generated by two transformations $S_\Gamma : z \to -\frac{1}{z}$ and $T_\Gamma : z \to z + 1$. It can also be represented as the two-generator free group $G = \langle e, v | e^2 = v^3 = 1 \rangle$ using the variable change $e = S_\Gamma$ and $v = S_\Gamma T_\Gamma$.

Tessellation of the upper half-plane with the modular group $\Gamma = PSL(2, \mathbb{Z})$.



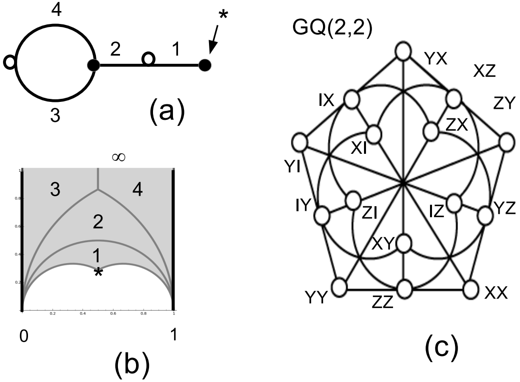
Congruence subgroups of Γ

- Some finite index subgroups of Γ, called congruence subgroups, are obtained by fixing congruence relations on the entries of elements of Γ. The principal congruence subgroup of level N of Γ is the normal subgroup $\Gamma(N) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a, d = \pm 1 \mod N$ and $b, c = 0 \mod N\}$ whose index is $n^3 \Pi_{p|N}(1 - \frac{1}{p^2})$, p a prime number. Another important subgroup of Γ is the **congruence subgroup** $\Gamma_0(N)$ of level N defined as the subgroup of upper triangular matrices with entries defined modulo N. The index of $\Gamma_0(N)$ is the **Dedekind psi function** $\psi(N)$.
- References to the conversion from permutation groups to subgroups of Γ and vice versa are [5] [6]

[5]Chris A. Kurth and Ling Long, Computations with finite index subgroups of $PSL(2,\mathbb{Z})$ using Farey symbols, in *Advances in Algebra and Combinatorics* edited by K. P. Shum et al (World Scientific, 2008), pp 225-242.

[6]William A. Stein et al. Sage Mathematics Software (Version 6.4.1), The Sage Development Team, 2014, http://www.sagemath.org.

The congruence subgroup $\Gamma_0(3)$ for the two-qubit IC-POVM



- Representation of $A_4 \cong \Gamma_0(3)$ as a dessin d'enfant (a) and as the tiling of the fundamental domain (b). The character * denotes the unique elliptic point (of order 3). The triple products of projectors leads to $GQ(2,2)$ pictured in (c).

The subgroups of Γ (congruence or not) leading to IC-POVMs

dim	sgs of $\Gamma = PSL(2,\mathbb{Z})$ -> IC-POVM	pp	geometry
3	$\Gamma_0(2)$	1	Hesse SIC
4	$\Gamma_0(3)$,$4A^0$ (under 2QB Pauli gr.)	2	$GQ(2,2)$
5	$5A^0$	1	Petersen graph
6	$\Gamma', \Gamma(2), 3C^0, \Gamma_0(4), \Gamma_0(5)$	2	Borromean ring
7	$7A^0$	2	Fig. 5b
	$\mathrm{NC}(0,6,1,1,[1^1 6^1])$	2	
9	$\mathrm{NC}(0,8,3,0,[1^1 8^1])$ (2QT)	2	(3×3)-grid, Pappus
	$\mathrm{NC}(0,9,1,3,[9^1])$ (2QT)	3	$[81_8, 216_3]$
10	$5C^0$	5	
11	$11A^0$	3	$[11_3]$
12	$10A^1$ (2QB-QT)	5	$K(3,3,3,3)$
	NC $(0,8,4,0,[4^1 8^1])$	5	Hesse ($\times$16)
	$\mathrm{NC}(0,8,4,0,[4^1 8^1])$	6	$[48_7, 112_3]$
12	under 12-dit Pauli group		
	$8A^1$, $\mathrm{NC}(0,8,4,0,[4^1 8^1])$	11,7	
13	$\mathrm{NC}(0,6,1,1,[1^1 6^2])$	4	
14	$7C^0$, $\mathrm{NC}(0,6,0,2,[1^1 6^2])$, $14A^1$	12,5,6	
15	$5E^0$, $\mathrm{NC}(0,6,3,0,[3^1 6^2])$, $15A^1$, $10B^1$	5,4,10,3	

When **non-congruence** the signature $\mathbf{NC}(g, N, \nu_2, \nu_3, [c_i^{W_i}])$ is made explicit.

Poincaré conjecture is the (deep) statement that every simply connected closed 3-manifold is homeomorphic to the 3-sphere S^3. Having in mind the correspondence between S^3 and the Bloch sphere that houses the qubits $\psi = a|0\rangle + b|1\rangle$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, one would desire a quantum translation of this statement.

Thurston's geometrization conjecture, from which Poincaré conjecture follows, dresses S^3 as a 3-manifold not homeomorphic to S^3. The wardrobe of 3-manifolds M^3 is huge but almost every dress is **hyperbolic** and W. Thurston [7] found the recipes for them.

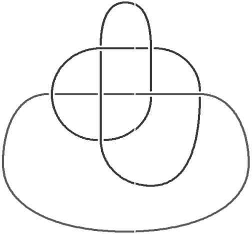
There exists a relationship between the **modular group** Γ and the (non hyperbolic) **trefoil knot** T_1 since the fundamental group $\pi_1(S^3 \setminus T_1)$ of the knot complement is the braid group B_3, the central extension of Γ.

[7] W. P. Thurston, Three-dimensional geometry and topology (vol. 1), (Princeton University Press, Princeton, 1997).

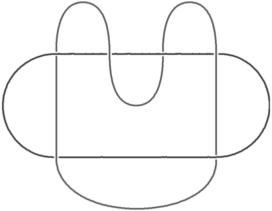
Coverings/subgroups of the fundamental group $\pi_1(T_1)$ of the trefoil knot T_1

d	ty	hom	cp	gens	CS	link	type in [?]
2	cyc	$\frac{1}{3}+1$	1	2	-1/6		D_4
3	irr	$1+1$	2	2	1/4	L7n1	$\Gamma_0(2)$, Hesse SIC
.	cyc	$\frac{1}{2}+\frac{1}{2}+1$	1	3	.		A_4
4	irr	$1+1$	2	2	1/6	L6a3	$\Gamma_0(3)$, 2QB IC
.	irr	$\frac{1}{2}+1$	1	3	.		$4A^0$, 2QB-IC
.	cyc	$\frac{1}{3}+1$	1	2	.		S_4
5	cyc	1	1	2	5/6		A_5
.	irr	$\frac{1}{3}+1$	1	3	.		$5A^0$, 5-dit IC
6	reg	$1+1+1$	3	3	0		Γ', 6-dit IC
.	irr	$1+1+1$	2	3	.	L6n1	$\Gamma(2)$, 6-dit IC
.	irr	$\frac{1}{2}+1+1$	2	3	.	L6n1	$\Gamma_0(4)$, 6-dit IC
.	irr	$\frac{1}{2}+1+1$	2	3	.		$3C^0$, 6-dit IC
.	irr	$\frac{1}{2}+1+1$	2	3	.		$\Gamma_0(5)$, 6-dit IC
.	cyc	$1+1+1$	1	3	.		
.	irr	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1$	1	4	.		
.	irr	$\frac{1}{3}+\frac{1}{3}+1$	1	3	.		
7	cyc	1	1	2	-5/6		
.	irr	$1+1$	2	3	.		NC 7-dit IC
.	irr	$\frac{1}{2}+\frac{1}{2}+1$	1	4	.		$7A^0$ 7-dit IC

The trefoil knot and links for the Hesse SIC and two-qubit IC.



(b)



(c)

- The trefoil knot $T_1 = 3_1$, (b) the link $L7n1$ associated to the Hesse SIC, (c) the link $L6a3$ associated to the two-qubit IC.

More on 3-manifolds: hyperbolic manifolds for the two-qubit IC-POVM [8]

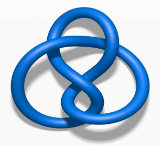


Figure-of-eigth knot K4a1

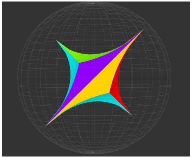
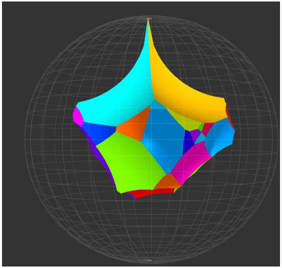
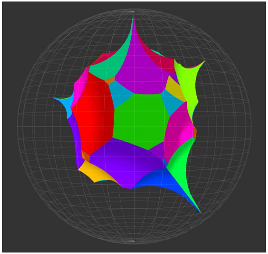


Figure-of-eight 3-manifold otet02_00001



Manifold otet08_00007

(a)



Manifold ooct04_00258

(b)

[8] M. Planat, R. Ascheim, M. Amaral and L. Irwin, Universal quantum computing and three-manifolds (Preprint).

Conclusion [9]

- magic state in uqc
- IC-POVMs in quantum measurements
- uqc and ICs on $\Gamma = PSL(2,\mathbb{Z})$
- uqc and ICs on 3 manifolds:
  * e.g the (non-hyperbolic) trefoil knot
  * or hyperbolic 3-manifolds.

[9] It is our task, both in science and in society at large, to prove the conventional wisdom wrong and to make our unpredictable dreams come true. Freeman Dyson