Periodic waves of the Lugiato-Lefever equation at the onset of Turing instability

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Abstract

We study the existence and the stability of periodic waves for a nonlinear model, the Lugiato-Lefever equation, arising in optics. We give a detailed description of the stability properties of constant solutions, and then focus on the periodic waves which bifurcate at the onset of Turing instability. Using a center manifold reduction, we analyze these Turing bifurcations, and prove the existence of periodic waves. This approach also allows to conclude on the nonlinear orbital stability of these waves for co-periodic perturbations, i.e., for periodic perturbations which have the same period as the wave. This stability result is completed by a spectral stability result for general bounded perturbations. In particular, this spectral analysis shows that instabilities are always due to co-periodic perturbations.

Running head: Periodic waves of the Lugiato-Lefever equation

Keywords: periodic waves, Lugiato-Lefever equation, Turing instability, bifurcations, spectral stability

1 Introduction

We consider the Lugiato-Lefever equation

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F, \tag{1.1}$$

in which the unknown ψ is a complex-valued function depending upon the temporal variable t and the spatial variable x, the parameters α and β are real, and F is a positive constant. This nonlinear Schrödinger type equation with damping, detuning and driving has been derived in nonlinear optics by Lugiato and Lefever [8]. In this context, α represents the detuning parameter, F the driving term, and β the dispersion parameter which may be positive (normal dispersion) or negative (anomalous dispersion). Notice that upon rescaling x, we may take $|\beta| = 1$, hence $\beta = 1$ in the case of normal dispersion, and $\beta = -1$ in the case of anomalous dispersion. While intensively studied in the physics

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literature (e.g., see [1] and the references therein), there are relatively few rigorous mathematical studies of this equation. Of particular interest for the physical problem, is the dynamical behavior of periodic and localized stationary waves. The underlying mathematical questions concern, in particular, the existence and the stability of these types of waves.

A first rigorous bifurcation analysis of stationary solutions of the Lugiato-Lefever equation (1.1) has been done in [10] in the case $\beta < 0$ and $\alpha < 2$. For these parameter values the Lugiato-Lefever equation possesses one constant solution which undergoes a Turing instability when its modulus is equal to 1, [8]. Local bifurcations are analyzed by taking as bifurcation parameters the physical parameter α and the square modulus of the constant solution. Two approaches are presented, for the bifurcations of periodic solutions, on the one hand, and for the bifurcations of localized solutions, on the other hand. For the analysis of periodic solutions, the Lugiato-Lefever equation (1.1) is treated as an infinite-dimensional dynamical system, in a phase space consisting of spatially periodic functions, and a center manifold reduction is used for the analysis of local bifurcations. Several bifurcations of periodic waves are studied at the onset of Turing instability and in the parameter regime where the Turing instability is fully developed. In particular, by analyzing the onset of Turing instability at $\alpha = 41/30$, the results in [10] recover the supercritical and the subcritical bifurcations of periodic solutions found in the physics literature for $\alpha < 41/30$ and $\alpha > 41/30$, respectively. Next, localized solutions are constructed using a spatial dynamics approach, in which the steady Lugiato-Lefever equation,

$$0 = -i\beta\frac{\partial^2\psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi|\psi|^2 + F, \tag{1.2}$$

is written as a four-dimensional dynamical system by taking the spatial variable x as evolutionary variable. Restricting to the same parameter values $\beta < 0$ and $\alpha < 2$, the existence of localized waves is proved close to the onset of Turing instability of the constant solution, for $\alpha > 41/30$.

A systematic study of local bifurcations for the steady equation (1.2) has been done later in [2, 3], for both cases $\beta < 0$ and $\beta > 0$. Starting from a formulation of the steady equation as a four-dimensional dynamical system, similar to the one used for localized waves in [10], but taking now the physical parameters α and F as bifurcation parameters, the local bifurcations have been classified in [2], and then studied in detail in [3]. Using normal forms and the center manifold reduction, the existence of various types of solutions has been proved, including periodic waves, localized waves which are either asymptotically constant or decay to small periodic waves at infinity, and quasiperiodic waves. These results provide a very good description of the set of bounded solutions, but their validity is restricted to parameter values which are close to the bifurcation points. Very recently, tools from global bifurcation theory have been used in [9] for the study of global bifurcations of periodic solutions of the steady Lugiato-Lefever equation (1.2). The existence of global branches of periodic solutions is shown using the classical bifurcation theorems of Crandall-Rabinowitz and Rabinowitz, and their shape and location in the parameter space are determined from a priori bounds for the steady equation and numerically.

The aim of the present work is to initiate a systematic study of the stability of nonlinear waves of the Lugiato-Lefever equation (1.1), and in particular of periodic waves. The question of stability of the steady solutions found in the works mentioned above is widely open. So far, the only stability result have been obtained for the periodic waves bifurcating at the onset of Turing instability in the case $\beta < 0$. The local bifurcation analysis in [10] shows that these waves are stable with respect to co-periodic perturbations if $\alpha < 41/30$, when the bifurcation is supercritical, and unstable if $41/30 < \alpha < 2$, when the bifurcation is subcritical. This result which holds for perturbations which are H^2, has been extended to L^2 perturbations in [11], using Strichartz estimates.

We focus here on the periodic waves which bifurcate locally at the onset of instability of constant solutions. Our approach of the existence problem is the same as the one used in [10] for the analysis of periodic waves in the case $\beta < 0$ and $\alpha < 2$. We treat the partial differential equation (1.1) as an infinite-dimensional dynamical system of the form

$$\frac{dU}{dt} = \mathcal{G}(U, \beta, \alpha, F), \tag{1.3}$$

in which $U(t)$ belongs to a phase space of spatially periodic functions, and use a center manifold reduction for the analysis of local bifurcations. In contrast to [10], we consider both the cases of anomalous dispersion, $\beta = -1$, and normal dispersion, $\beta = 1$, and systematically investigate the local bifurcations of periodic waves for all values of the physical parameters α and F.

The starting point of our analysis is a detailed stability analysis of the constant solutions of (1.1), which are equilibria of the above dynamical system (see Section 2). Since these solutions do not depend upon the spatial variable x, they can be computed explicitly by solving an algebraic equation, and their stability can be determined using a standard Fourier analysis. We detect two types of instabilities, the Turing instability mentioned above, in which the instability of the constant solution is due to nontrivial periodic perturbations (nonzero Fourier modes), and a zero-mode instability in which the instability is due to constant perturbations (zero Fourier mode). While the expressions of the constant solutions and the types of instabilities do not depend upon β, the values of the parameters α and F at which these instabilities occur are different in the two cases $\beta = -1$ and $\beta = 1$. In our presentation we focus on the first case (see Sections 2–4), and then only outline the differences which occur in the second case (see Section 5).

Next, we restrict to the onset of instability and analyze the local bifurcations of steady periodic waves (see Section 3). Notice that these periodic waves are also equilibria of the dynamical system (1.3), just as the constant solutions. Relying upon a center manifold reduction, we prove the existence of steady periodic waves in the case of the Turing instability. For $\beta = -1$, we recover the results known in the physical literature, and in particular the qualitative change of the type of the bifurcation (super- or sub-critical) which occurs at $\alpha = 41/30$ and which was analyzed in [10]. In contrast, for $\beta = 1$ we find a subcritical bifurcation, only. In the case of the zero-mode instability, steady periodic waves exist as well [3], but their existence cannot be obtained using the present approach (see Section 6). These periodic waves have periods which tend to infinity, as the parameters approach the bifurcation points, and such solutions are not captured by the dynamical formulation (1.3) in which the phase space is restricted to periodic solutions with fixed wavelengths.

The local bifurcation result showing the existence of periodic waves, allows to also conclude on the nonlinear stability, or instability, of these waves for perturbations which belong to the phase space of the dynamical system (1.3), hence for co-periodic perturbations which have the same period as the wave. The waves found in the supercritical bifurcations are stable, whereas the ones found in the subcritical bifurcations are unstable. By replacing this phase space with a space of periodic functions with periods which are integer multiples of the period of the wave, and using a center manifold reduction again, we can extend this result and conclude on stability, or instability, for subharmonic perturbations, i.e., periodic perturbations with periods equal to an integer multiple of the period of the wave. However, the local center manifolds found in these phase spaces shrink to a point, as the integer multiples tend to infinity, so that we cannot conclude on stability of a given periodic wave for all subharmonic perturbations (see Section 6).

For the stability analysis, we restrict to the question of spectral stability, but will consider general bounded, including localized, perturbations of the periodic wave (see Section 4). For such types of perturbations, the more difficult question of nonlinear stability, or instability, remains open. The spectral stability is determined by the location in the complex plane of the spectrum of the linear operator obtained by linearizing the dynamical system (1.3) at a steady periodic wave, and the choice of the function space depends upon the type of the perturbations. For bounded perturbations the spectrum is continuous, and we use a Bloch-Floquet decomposition in order to reduce the question of finding this continuous spectrum to the simpler question of finding isolated eigenvalues of a family of operators with compact resolvent. Then we locate the potentially unstable eigenvalues using perturbations arguments for linear operators. Our main result shows that the periodic waves found in the supercritical bifurcations are also spectrally stable with respect to general bounded perturbations, whereas the ones found in the subcritical bifurcations are unstable, the instability being a direct consequence of their instability with respect to co-periodic perturbations.

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2 Spectral stability of constant solutions

In this section we recall the stability properties of the constant solutions of the Lugiato-Lefever equation (1.1) in the case of anomalous dispersion ($\beta = -1$).

Constant solutions $\psi \in \mathbb{C}$ of the equation (1.1) satisfy the algebraic equation

$$(1 + i\alpha)\psi - i\psi|\psi|^2 = F.$$

Upon decomposing into real and imaginary parts, $\psi = \psi_r + i\psi_i$, and setting $\rho = |\psi|^2 = \psi_r^2 + \psi_i^2$, we obtain

$$\psi_r = \frac{\rho}{F}, \qquad \psi_i = \frac{\rho(\rho - \alpha)}{F}, \qquad \rho\left((\rho - \alpha)^2 + 1\right) = F^2. \tag{2.1}$$

For $\rho > 0$, the cubic polynomial in the left hand side of the last equation above is monotonously increasing, when $\alpha \leqslant \sqrt{3}$, and has two positive critical points $\rho_+(\alpha) < \rho_-(\alpha)$, when $\alpha > \sqrt{3}$. Consequently, the Lugiato-Lefever equation possesses precisely one constant solution when $\alpha \leqslant \sqrt{3}$,

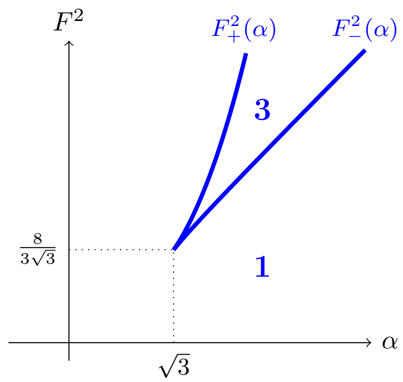

Figure 2.1: Number of constant solutions of the Lugiato-Lefever equation (1.1): three solutions in the region between the curves $F^2 = F_\pm^2(\alpha)$, two solutions along the curves, and one solution otherwise.

for any $F > 0$. For $\alpha > \sqrt{3}$, there exist two values $F_-(\alpha) < F_+(\alpha)$,

$$F_\pm^2(\alpha) = \rho_\pm(\alpha)\left((\rho_\pm(\alpha) - \alpha)^2 + 1\right), \qquad \rho_\pm(\alpha) = \frac{1}{3}\left(2\alpha \mp \sqrt{\alpha^2 - 3}\right),$$

such that the Lugiato-Lefever equation possesses three constant solutions with $\rho = \rho_j$, $j = 1, 2, 3$,

$$\rho_1 < \rho_+(\alpha) < \rho_2 < \rho_-(\alpha) < \rho_3,$$

when $F_-(\alpha) < F < F_+(\alpha)$, two distinct constant solutions when $F = F_\pm(\alpha)$, and one constant solution when $F < F_-(\alpha)$ or $F_+(\alpha) < F$. This result is summarized in Figure 2.1 (see also [3]).

The spectral stability of these constant solutions is determined by the location in the complex plane of the spectrum of the linear operator obtained by linearizing the Lugiato-Lefever equation (1.1) at such a constant solution. For a constant solution $\psi^* = \psi_r^* + i\psi_i^*$ as above, with modulus square ρ^*, the right hand side of the linearized Lugiato-Lefever equation

$$\frac{dV}{dt} = \mathcal{A}^* V,$$

defines a linear operator of the form

$$\mathcal{A}^* = -\mathbb{I} + \mathcal{J}\mathcal{L}^*, \tag{2.2}$$

where $\mathbb{I}$ represents the 2×2 identity matrix, and

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{L}^* = \begin{pmatrix} \partial_x^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & \partial_x^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

Since $\mathcal{A}^*$ is an operator with constant coefficients, with the Fourier Ansatz $(u(x), v(x)) = e^{ikx}(u_k, v_k)$, we find that its spectrum $\sigma(\mathcal{A}^*)$, in both the Hilbert space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ of square integrable functions and the Banach space $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ of uniformly continuous functions, is given by

$$\sigma(\mathcal{A}^*) = \bigcup_{k \in \mathbb{R}} \sigma(\mathcal{A}^*(k)),$$

where $\mathcal{A}^*(k)$ are the 2×2 matrices

$$\mathcal{A}^*(k) = -\mathbb{I} + \mathcal{J}\mathcal{L}^*(k), \qquad \mathcal{L}^*(k) = \begin{pmatrix} -k^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & -k^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

A direct calculation then gives

$$\sigma(\mathcal{A}^*) = \left\{\lambda \in \mathbb{C} \,;\; \lambda^2 + 2\lambda + a(k) = 0,\; k \in \mathbb{R}\right\},$$

in which

$$a(k) = k^4 + 2(\alpha - 2\rho^*)k^2 + \alpha^2 - 4\alpha\rho^* + 3\rho^{*2} + 1.$$

For any $k \in \mathbb{R}$, the sum of the two eigenvalues of $\mathcal{A}^*(k)$ is equal to -2, so that their location in the complex plane depends upon the value of $a(k)$. For $a(k) > 1$, the two eigenvalues are complex conjugated with real parts equal to -1, and -1 is a double eigenvalue when $a(k) = 1$. For $a(k) < 1$, the two eigenvalues are real and symmetric with respect to the line $\operatorname{Re}\lambda = -1$: both eigenvalues are negative when $0 < a(k) < 1$, 0 and -2 are eigenvalues when $a(k) = 0$, and one eigenvalue is negative and the other one positive, when $a(k) < 0$. Consequently, the constant solution changes its stability when $a(k)$ becomes positive for some values of k. A direct calculation shows that this occurs in the following two cases:

(i) for $\alpha < 2$ and $\rho^* = 1$, when $a(k)$ is nonnegative and vanishes for precisely two values $k = \pm\sqrt{2-\alpha} \neq 0$ (Turing instability); in the parameter plane (α, F^2) this occurs along the parabola of equation $F^2 = (1-\alpha)^2 + 1$;

(ii) for $\alpha \geqslant 2$ and $\rho_* = \rho_+(\alpha)$, when $a(k)$ is nonnegative and vanishes at $k = 0$ (zero-mode instability); in the parameter plane (α, F^2) this occurs along the half curve $F^2 = F_+^2(\alpha)$ (see Figure 2.1).

In Figure 2.2 we represent the shape of the largest real eigenvalue $\lambda(k)$ in these two cases, and in Figure 2.3 we summarize the stability properties of the constant solutions of the Lugiato-Lefever equation (1.1). Notice that the onset of Turing instability moves from the parameter region where the equation has one constant solution to the parameter region where the equation has three constant solutions as α is increased above the value 7/4. At $\alpha = 2$ the Turing instability reaches the double constant solution $\rho_+(\alpha)$ and becomes a zero-mode instability for $\alpha > 2$.

Remark 2.1 *The symmetry of the spectrum with respect to the vertical line* $\operatorname{Re}\lambda = -1$ *is a consequence of the particular structure of the linear operator* $\mathcal{A}^*$ *in* (2.2)*, in which the operators* $\mathcal{J}$ *and* $\mathcal{L}^*$ *are skew-symmetric and symmetric, respectively. This property implies that the spectrum of the product operator* $\mathcal{J}\mathcal{L}^*$ *is symmetric with respect to the imaginary axis (e.g., see [5]), so that the spectrum of* $\mathcal{A}^*$ *is symmetric to the vertical line* $\operatorname{Re}\lambda = -1$.

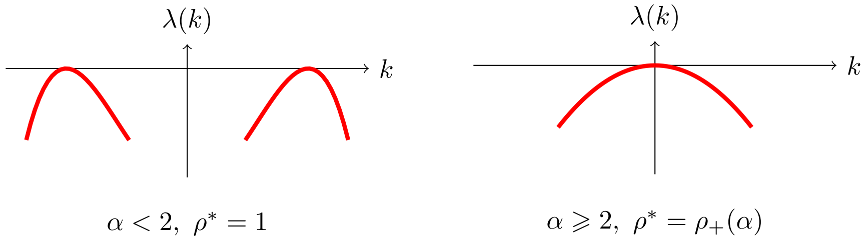

Figure 2.2: Shape of the largest eigenvalue $\lambda(k)$ for k close to $\pm\sqrt{2-\alpha}$ in the case of the Turing instability (left plot) and for k close to 0 in the case of the zero-mode instability (right plot).

3 Bifurcations of periodic waves

In this section we analyze the Turing bifurcation which occurs for $\alpha < 2$ and $F^2 = (1-\alpha)^2 + 1$, when $\rho^* = 1$. We fix $\alpha < 2$ and take as bifurcation parameter $F^2 = F_1^2 + \mu$, with $F_1^2 = (1-\alpha)^2 + 1$ and small μ.

Dynamical system

For $F^2 = F_1^2 + \mu$, we denote by $\psi_\mu^* = \psi_{r\mu}^* + i\psi_{i\mu}^*$ and $\rho_\mu^* = |\psi_\mu^*|^2$ the constant solution of the Lugiato-Lefever equation (1.1) and its square modulus, respectively, given by (2.1). At $\mu = 0$ we have the constant solution with modulus $\rho_0^* = 1$, at which the Turing instability occurs, and according to the linear stability analysis in Section 2, $\lambda = 0$ is an eigenvalue of the corresponding linearized operator $\mathcal{A}^*$, with eigenmodes $(u(x), v(x)) = e^{ikx}(u_k, v_k)$, $k = \pm\sqrt{2-\alpha}$. We therefore expect periodic bifurcating solutions to have wavenumbers $\pm\sqrt{2-\alpha}$, and hence look for solutions of the Lugiato-Lefever equation (1.1) close to the branch of constant solutions ψ_μ^* of the form

$$\psi(x,t) = \psi_\mu^* + (u+iv)(y,t), \qquad y = \sqrt{2-\alpha}\,x,$$

with u and v real-valued, 2π-periodic functions in y. The resulting equation is a system for the couple $U = (u,v)^T$ of the form

$$\frac{dU}{dt} = \mathcal{A}_\mu^* U + \mathcal{F}(U,\mu), \tag{3.1}$$

in which $\mathcal{A}_\mu^*$ is the linear operator

$$\mathcal{A}_\mu^* = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu^*,$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{L}_\mu^* = \begin{pmatrix} (2-\alpha)\partial_y^2 - \alpha + 3{\psi_{r\mu}^*}^2 + {\psi_{i\mu}^*}^2 & 2\psi_{r\mu}^*\psi_{i\mu}^* \\ 2\psi_{r\mu}^*\psi_{i\mu}^* & (2-\alpha)\partial_y^2 - \alpha + {\psi_{r\mu}^*}^2 + 3{\psi_{i\mu}^*}^2 \end{pmatrix},$$

and the nonlinear terms $\mathcal{F}(U,\mu)$ are given by

$$\mathcal{F}(U,\mu) = \mathcal{J}\left(\mathcal{R}_2(U,U,\mu) + \mathcal{R}_3(U,U,U)\right),$$

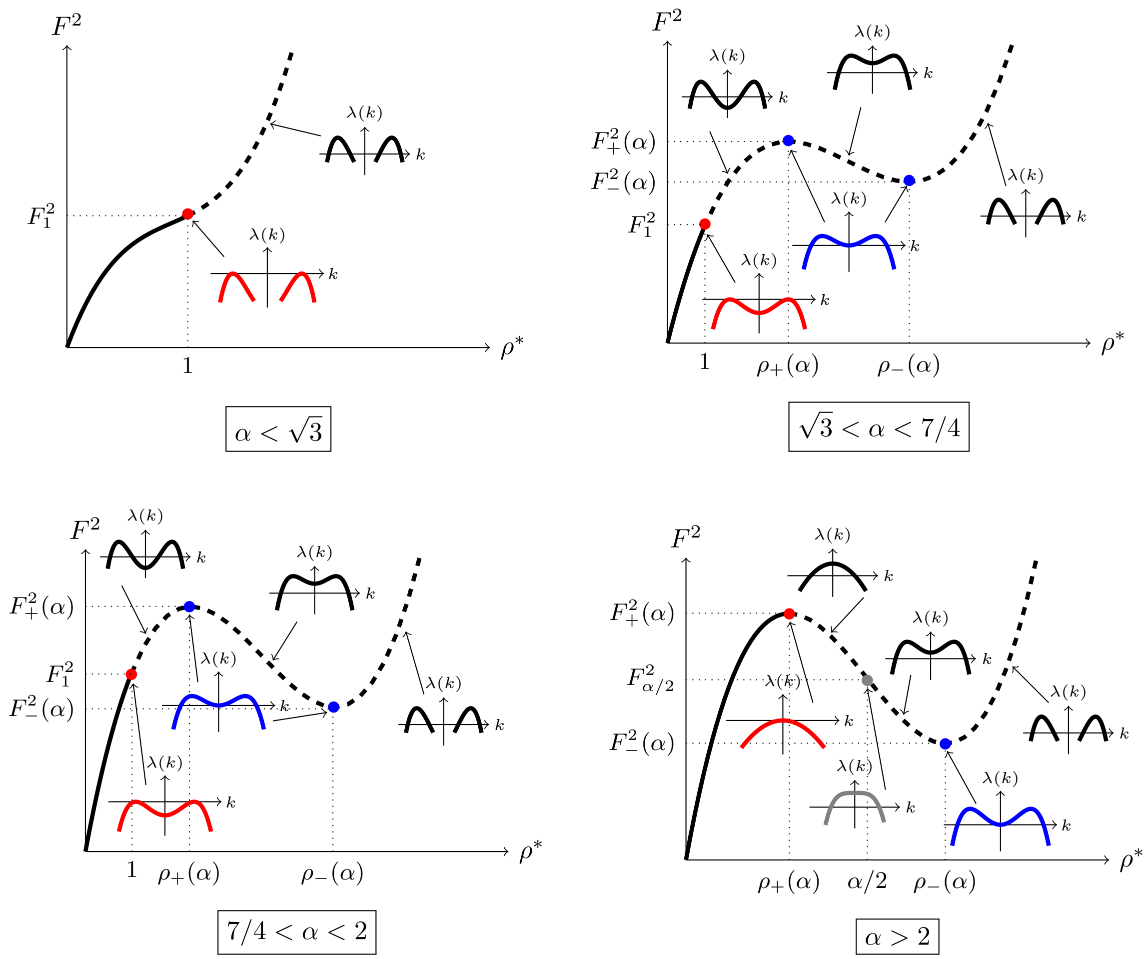

Figure 2.3: Stability of constant solutions of the Lugiato-Lefever equation in the case of anomalous dispersion ($\beta = -1$). Stable branches are represented by solid lines, unstable branches by dashed lines, and we set $F_1^2 = (1-\alpha)^2 + 1$ and $F_{\alpha/2}^2 = \alpha(4+\alpha^2)/8$. The shape of the largest eigenvalue $\lambda(k)$ is given at the points where the solutions loose their stability and for the unstable solutions.

where $\mathcal{R}_2(\cdot,\cdot,\mu)$ is the bilinear map defined through

$$\mathcal{R}_2(U_1, U_2, \mu) = \begin{pmatrix} \psi_{r\mu}^*(3u_1u_2 + v_1v_2) + \psi_{i\mu}^*(u_1v_1 + u_2v_2) \\ \psi_{i\mu}^*(u_1u_2 + 3v_1v_2) + \psi_{r\mu}^*(u_1v_1 + u_2v_2) \end{pmatrix},$$

for $U_j = (u_j, v_j)^T$, $j = 1, 2$, and $\mathcal{R}_3$ is the trilinear map satisfying

$$\mathcal{R}_3(U, U, U) = \begin{pmatrix} u(u^2 + v^2) \\ v(u^2 + v^2) \end{pmatrix}.$$

As phase-space for the dynamical system (3.1) we choose the space of 2π-periodic, square-integrable functions $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$. In this space, the linear operator $\mathcal{A}_\mu^*$ is closed with domain $\mathcal{Y} = H^2(0, 2\pi) \times H^2(0, 2\pi)$, the linear operators $\mathcal{J}$ and $\mathcal{L}_\mu^*$ are skew- and self-adjoint, respectively, and the nonlinear map $\mathcal{F}(\cdot, \mu)$ is smooth in $\mathcal{Y}$.

The dynamical system (3.1) possesses one discrete and one continuous symmetry which will play an important role in our analysis. As a consequence of the invariance of the Lugiato-Lefever equation (1.1) under the reflection $x \mapsto -x$ and under spatial translations $x \mapsto x+a$, $a \in \mathbb{R}$, the dynamical system (3.1) is equivariant under the action of the reflection operator $\mathcal{T}$ and of the translation operators $\mathcal{T}_a$ defined through

$$(\mathcal{T}U)(y) = U(-y), \qquad (\mathcal{T}_a U)(y) = U(y+a), \quad y \in \mathbb{R}, \tag{3.2}$$

i.e., both $\mathcal{A}_\mu^*$ and $\mathcal{F}(\cdot,\mu)$ commute with $\mathcal{T}$ and $\mathcal{T}_a$, for any μ.

Center manifold reduction

With the choice of the phase space above, the linear operator $\mathcal{A}_\mu^*$ has compact resolvent, since its domain $\mathcal{Y}$ is compactly embedded in $\mathcal{X}$. Consequently, $\mathcal{A}_\mu^*$ has discret spectrum, and the calculations in Section 2 imply that at $\mu = 0$ the spectrum of $\mathcal{A}_0^*$ is given by

$$\sigma(\mathcal{A}_0^*) = \left\{ \lambda_\pm(n) = -1 \pm \sqrt{1-(2-\alpha)^2(n^2-1)^2},\ n \in \mathbb{Z} \right\}.$$

The eigenvalues $\lambda_\pm(n)$ are either negative or have negative real parts when $n \neq \pm 1$, and $\lambda_+(\pm 1) = 0$. We can therefore decompose $\sigma(\mathcal{A}_0^*)$ into a stable and a central spectrum,

$$\sigma(\mathcal{A}_0^*) = \sigma_s(\mathcal{A}_0^*) \cup \sigma_c(\mathcal{A}_0^*), \tag{3.3}$$

with

$$\sigma_c(\mathcal{A}_0^*) = \{0\}, \qquad \sigma_s(\mathcal{A}_0^*) = \{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\,\lambda < -\delta\},$$

for some $\delta > 0$. Here 0 is a double semi-simple eigenvalue with associated eigenvectors

$$\zeta = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} e^{iy}, \qquad \overline{\zeta} = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} e^{-iy}.$$

In order to apply the center manifold theorem, we rewrite the dynamical system (3.1) in the form

$$\frac{dU}{dt} = \mathcal{A}_0^* U + \mathcal{G}(U,\mu), \tag{3.4}$$

with

$$\mathcal{G}(U,\mu) = \mathcal{J}\left(\mathcal{R}_1(U,\mu) + \mathcal{R}_2(U,U,\mu) + \mathcal{R}_3(U,U,U)\right),$$

where $\mathcal{R}_1(\cdot,\mu)$ is the linear map

$$\mathcal{R}_1(U,\mu) = (\mathcal{A}_\mu^* - \mathcal{A}_0^*)U,$$

and J, $\mathcal{R}_2$, $\mathcal{R}_3$ are defined as above. Upon checking the hypotheses of the center manifold theorem [6, Chapter 2, Theorem 3.3], we conclude that the dynamical system (3.4) possesses a two-dimensional center manifold,

$$\mathcal{M}_c(\mu) = \{U \in \mathcal{Y}\ ;\ U = A\zeta + \overline{A\zeta} + \Psi(A,\overline{A},\mu),\ A \in \mathbb{C}\},$$

which contains all sufficiently small bounded solutions of (3.4), for any μ sufficiently small. Here Ψ is a map of class C^k, for any arbitrary but fixed $k \geqslant 2$, defined in a neighborhood of 0 in $\mathbb{C}\times\overline{\mathbb{C}}\times\mathbb{R}$, where $\mathbb{C}\times\overline{\mathbb{C}} = \{(A,\overline{A})\ ;\ A \in \mathbb{C}\}$, and taking values in the spectral subspace $\mathcal{X}_s$ associated to the stable spectrum $\sigma_s(\mathcal{A}_0^*)$ of the operator $\mathcal{A}_0^*$.

Reduced equation

The dynamics on the center manifold is governed by the reduced equation

$$\frac{dA}{dt} = f(A, \overline{A}, \mu), \tag{3.5}$$

in which f is a complex-valued map obtained by inserting the Ansatz

$$U = A\zeta + \overline{A\zeta} + \Psi(A, \overline{A}, \mu),$$

into the dynamical system (3.4) and then projecting on the eigenvector ζ. This reduced equation captures the qualitative changes which occur in the dynamics of the full system (3.4) at the bifurcation point $\mu = 0$. We summarize in the lemma below the properties of the reduced vector field f which are needed in our bifurcation analysis.

Lemma 3.1 *For any μ sufficiently small, the vector field f in (3.5) has the following properties.*

(i) The complex-valued map f is of class C^k in a neighborhood of 0 in $\mathbb{C}\times\overline{\mathbb{C}}\times\mathbb{R}$, for any arbitrary but fixed $k \geqslant 2$.

(ii) There exists a real-valued map g of class C^{k-1} defined in a neighborhood of 0 in $\mathbb{R}^2$ such that

$$f(A, \overline{A}, \mu) = Ag(|A|^2, \mu),$$

for any A sufficiently small.

(iii) The coefficients of the leading order terms in the Taylor expansion of f,

$$f(A, \overline{A}, \mu) = c_{11}A\mu + c_{30}A|A|^2 + \mathcal{O}(|A|(|\mu|^2 + |A|^4)), \tag{3.6}$$

are given by

$$c_{11} = \frac{1}{(2-\alpha)^2}, \qquad c_{30} = \frac{4F_1^2(30\alpha - 41)}{9(2-\alpha)^2}.$$

Proof. *(i)* This property follows from the fact that the map Ψ provided by the center manifold theorem is of class C^k.

(ii) Recall that the dynamical system (3.1) is equivariant under the action of the operators $\mathcal{T}$ and $\mathcal{T}_a$, $a \in \mathbb{R}$. According to [6, Section 2.3.3], these equivariances are inherited by the reduced equation (3.5), so that f satisfies

$$\overline{f}(A, \overline{A}, \mu) = f(\overline{A}, A, \mu), \qquad f(e^{ia}A, e^{-ia}\overline{A}, \mu) = e^{ia}f(A, \overline{A}, \mu),$$

for any $a \in \mathbb{R}$ and A sufficiently small. Applying the result in [6, Chapter 1, Lemma 2.4], the second equality above implies that f is of the form (3.6) with g a complex-valued function, and then from the first equality above we conclude that g is a real-valued function. This proves the second property.

(iii) We compute the coefficient c_{11} from the equality

$$\frac{\partial f}{\partial A}(0,0,\mu) = \lambda_1(\mu), \tag{3.7}$$

in which $\lambda_1(\mu)$ is the continuation of the eigenvalue 0 of the operator $\mathcal{A}_0^*$ for small μ, i.e., $\lambda_1(\mu)$ is the largest eigenvalue of $\mathcal{A}_\mu^*$ (e.g., see [6, Chapter 2, Exercise 3.5]). According to the linear stability analysis in Section 2, $\lambda_1(\mu)$ is a double eigenvalue and it is the largest root of the polynomial

$$\lambda^2 + 2\lambda + a_\mu(\sqrt{2-\alpha}) = 0, \tag{3.8}$$

in which

$$a_\mu(\sqrt{2-\alpha}) = (2-\alpha)^2 + 2(\alpha - 2\rho_\mu^*)(2-\alpha) + \alpha^2 - 4\alpha\rho_\mu^* + 3{\rho_\mu^*}^2 + 1 = 3{\rho_\mu^*}^2 - 8\rho_\mu^* + 5.$$

Differentiating (3.7) with respect to μ and taking $\mu = 0$ we find

$$c_{11} = \frac{d\lambda_1}{d\mu}(0).$$

The right hand side in this equation can be computed from (3.8), and after some elementary calculations we find the formula of the coefficient c_{11} in *(iii)*.

For the computation of the coefficient c_{30}, we may set $\mu = 0$ in the following calculations. Inserting the Taylor expansion of the reduction function Ψ,

$$\Psi(A,\overline{A},0) = \Psi_{20}A^2 + \Psi_{11}A\overline{A} + \overline{\Psi}_{20}\overline{A}^2 + \Psi_{30}A^3 + \Psi_{21}A^2\overline{A} + \overline{\Psi}_{21}A\overline{A}^2 + \overline{\Psi}_{30}\overline{A}^3 + \mathcal{O}(|A|^4),$$

in the system (3.4), for $\mu = 0$, and collecting successively the terms of orders $\mathcal{O}(A^2)$, $\mathcal{O}(A\overline{A})$, and $\mathcal{O}(A^2\overline{A})$ we obtain the equalities

$$\begin{aligned} 0 &= \mathcal{A}_0^*\Psi_{20} + \mathcal{J}\mathcal{R}_2(\zeta,\zeta,0), \\ 0 &= \mathcal{A}_0^*\Psi_{11} + 2\mathcal{J}\mathcal{R}_2(\zeta,\overline{\zeta},0), \\ c_{30}\zeta &= \mathcal{A}_0^*\Psi_{21} + \mathcal{J}\left(2\mathcal{R}_2(\zeta,\Psi_{11},0) + 2\mathcal{R}_2(\overline{\zeta},\Psi_{20},0) + 3\mathcal{R}_3(\zeta,\zeta,\overline{\zeta})\right). \end{aligned}$$

Solving the first two linear equations we obtain

$$\Psi_{20} = \frac{2F_1}{9(2-\alpha)^2}\begin{pmatrix} 6+4\alpha-3\alpha^2 \\ 2(5-3\alpha)(2-\alpha)\end{pmatrix} e^{2iy}, \qquad \Psi_{11} = \frac{4F_1}{(2-\alpha)^2}\begin{pmatrix} \alpha^2-2 \\ -2(1-\alpha)(2-\alpha)\end{pmatrix},$$

and by taking the scalar product of the third equality with an eigenvector ζ^{ad} in the kernel of the adjoint operator $(\mathcal{A}_0^*)^{ad}$ satisfying $\langle \zeta, \zeta^{ad}\rangle \neq 0$, we obtain

$$c_{30} = \frac{1}{\langle\zeta,\zeta^{ad}\rangle}\left\langle \mathcal{J}\left(2\mathcal{R}_2(\zeta,\Psi_{11}) + 2\mathcal{R}_2(\overline{\zeta},\Psi_{20}) + 3\mathcal{R}_3(\zeta,\zeta,\overline{\zeta})\right), \zeta^{ad}\right\rangle.$$

We may slightly simplify the computation of the coefficient c_{30} by taking $\zeta^{ad} = \mathcal{J}\zeta_2$, so that

$$c_{30} = \frac{1}{\langle\zeta,\mathcal{J}\zeta_2\rangle}\left\langle 2\mathcal{R}_2(\zeta,\Psi_{11}) + 2\mathcal{R}_2(\overline{\zeta},\Psi_{20}) + 3\mathcal{R}_3(\zeta,\zeta,\overline{\zeta}), \zeta_2\right\rangle,$$

and observing that

$$\mathcal{A}_0^* \zeta_2 = -2\zeta_2. \tag{3.9}$$

Indeed, since $\mathcal{J}$ and $\mathcal{L}_0^*$ are skew- and self-adjoint operators, respectively, and $\mathcal{J}^2 = -\mathbb{I}$, we have that

$$0 = (\mathcal{A}_0^*)^{ad}(\mathcal{J}\zeta_2) = -(\mathbb{I} + \mathcal{L}_0^*\mathcal{J})(\mathcal{J}\zeta_2) = -\mathcal{J}(\mathbb{I} + \mathcal{J}\mathcal{L}_0^*)\zeta_2 = -\mathcal{J}(2\mathbb{I} + \mathcal{A}_0^*)\zeta_2,$$

which implies (3.9). Then a direct calculation gives

$$\zeta_2 = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} e^{iy},$$

which together with the expressions of ζ, Ψ_{11}, and Ψ_{20} gives the formula for c_{30} in *(iii)*, and completes the proof of the lemma. ∎

Steady bifurcation with $O(2)$ symmetry

The properties of the reduced vector field f in Lemma 3.1 show that for the reduced system (3.5) a steady bifurcation with $O(2)$ symmetry occurs at $\mu = 0$, provided $c_{30} \neq 0$ (e.g., see [6, Section 1.2.4]). This bifurcation can be easily studied using polar coordinates $A = re^{i\theta}$, in which the system decouples

$$\begin{aligned} \frac{dr}{dt} &= c_{11}\mu r + c_{30}r^3 + \mathcal{O}(r(|\mu|^2 + r^4)), \\ \frac{d\theta}{dt} &= 0. \end{aligned}$$

The second equation implies that the angle variable θ is always constant, and the first equation shows a pitchfork bifurcation for the radial variable r. This bifurcation is supercritical when $c_{30} < 0$ and subcritical when $c_{30} > 0$. For the reduced system (3.5) we obtain the following result (see also Figure 3.1).

Theorem 1 *Consider the Lugiato-Lefever equation* (1.1) *in the case $\beta = -1$ of anomalous dispersion. Assume that $\alpha < 2$, $\alpha \neq 41/30$, and $F^2 = F_1^2 + \mu$. Then the reduced system* (3.5) *undergoes a steady bifurcation with $O(2)$ symmetry at $\mu = 0$. The bifurcation is supercritical when $\alpha < 41/30$, and subcritical when $\alpha > 41/30$. More precisely, the following properties hold in a neighborhood of 0 in $\mathbb{C}$, for μ sufficiently small.*

- *(i) If $\alpha < 41/30$ and $\mu < 0$, then the reduced equation possesses a unique equilibrium $A = 0$ which is stable.*
- *(ii) If $\alpha < 41/30$ and $\mu > 0$, then the reduced equation possesses the unstable equilibrium $A = 0$ and a circle of stable equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium.*
- *(iii) If $\alpha > 41/30$ and $\mu < 0$, then the reduced equation possesses the stable equilibrium $A = 0$ and a circle of unstable equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$, for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium.*

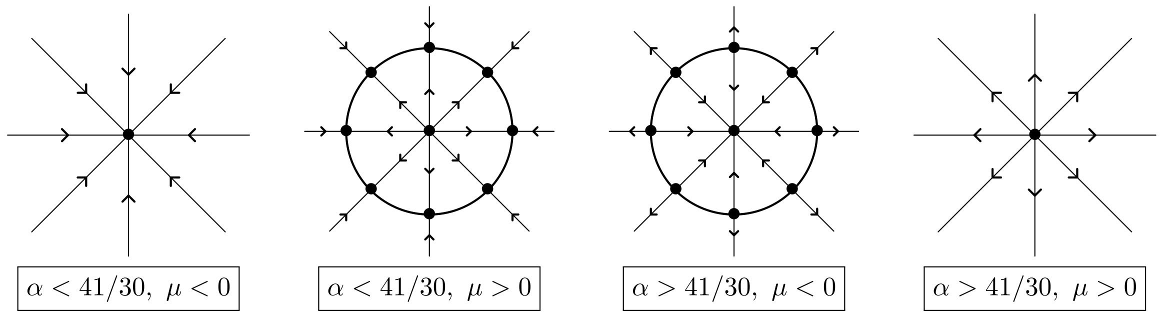

Figure 3.1: Phase portraits in the A-plane of the reduced system (3.5).

(iv) If $\alpha > 41/30$ and $\mu > 0$, then the reduced equation possesses a unique equilibrium $A = 0$ which is unstable.

In the cases (ii) and (iii) we have

$$r_\mu = \frac{3}{2F_1|41 - 30\alpha|^{1/2}}\,|\mu|^{1/2} + \mathcal{O}(|\mu|).$$

Going back to the Lugiato-Lefever equation (1.1), the equilibrium $A = 0$ of the reduced equation gives the constant solution ψ_μ^* in (2.1), whereas the circle of nontrivial equilibria $A_\mu(\phi) = r_\mu e^{i\phi}$ corresponds to a family of periodic solutions in x. The positive solution $A_\mu(0) = r_\mu$ gives an even periodic solution of the Lugiato-Lefever equation (1.1), with Taylor expansion

$$\psi_\mu(x) = \psi_0^* + \frac{3(\alpha + i(2-\alpha))}{F_1|41 - 30\alpha|^{1/2}}\,\cos(\sqrt{2-\alpha}\,x)\,|\mu|^{1/2} + \mathcal{O}(|\mu|), \tag{3.10}$$

and, since the rotation invariance of the reduced system (3.5) is inherited from the translation invariance of the Lugiato-Lefever equation (1.1), the other equilibria on the circle correspond to translations in x of this even periodic solution.

Remark 3.2 *The local bifurcation in Theorem 1 is known in the physics literature as a pitchfork bifurcation, which is indeed the bifurcation obtained when restricting this bifurcation analysis to solutions of the Lugiato-Lefever equation which are even in x. The case $\alpha = 41/30$, when the coefficient c_{30} vanishes, has been analyzed in [10].*

4 Spectral stability analysis

In this section, we study the spectral stability with respect to localized, or bounded, perturbations of the periodic solutions found in Section 3. The bifurcation result in Theorem 1 implies that the periodic solutions found for $\alpha < 41/30$ (supercritical bifurcation) are stable, whereas those found for $\alpha > 41/30$ (subcritical bifurcation) are unstable, for perturbations which belong to the space $\mathcal{Y}$, i.e., for co-periodic perturbations (see Section 6). In particular, this implies that, for perturbations which are bounded or localized, the solutions found for $\alpha > 41/30$ are also unstable, but leaves open the question of stability of the solutions found for $\alpha < 41/30$.

Linearized problem and Bloch operators

Consider the even periodic solution $\psi_\mu(x)$ given by (3.10), for $\alpha < 2$, $F^2 = F_1^2 + \mu$, and μ sufficiently small as given in Theorem 1. As for the bifurcation analysis, it is more convenient to work with 2π-periodic functions, and replace x by $y = \sqrt{2-\alpha}\,x$. Linearizing the Lugiato-Lefever equation (1.1) at $\psi_\mu(x)$, and using the variable y instead of x, we obtain the linearized equation

$$\frac{dV}{dt} = \mathcal{A}_\mu V, \tag{4.1}$$

where $\mathcal{A}_\mu$ is the linear operator

$$\mathcal{A}_\mu = -\mathbb{I} + \mathcal{J}\mathcal{L}_\mu,$$

the operator $\mathcal{J}$ is defined as before, and

$$\mathcal{L}_\mu = \begin{pmatrix} (2-\alpha)\partial_y^2 - \alpha + 3\psi_{r\mu}^2(y) + \psi_{i\mu}^2(y) & 2\psi_{r\mu}(y)\psi_{i\mu}(y) \\ 2\psi_{r\mu}(y)\psi_{i\mu}(y) & (2-\alpha)\partial_y^2 - \alpha + \psi_{r\mu}^2(y) + 3\psi_{i\mu}^2(y) \end{pmatrix},$$

in which $\psi_{r\mu}(y)$ and $\psi_{i\mu}(y)$ represent the real and imaginary parts, respectively, of the periodic solution. According to the formula (3.10), we have the expansions

$$\psi_{r\mu}(y) = \psi_r^* + \alpha\,\psi_1\cos(y)\,|\mu|^{1/2} + \mathcal{O}(|\mu|),$$
$$\psi_{i\mu}(y) = \psi_i^* + (2-\alpha)\,\psi_1\cos(y)\,|\mu|^{1/2} + \mathcal{O}(|\mu|),$$

where ψ_r^* and ψ_i^* are the real and imaginary parts, respectively, of the constant solution ψ_0^*, and $\psi_1 = 3/F_1|41-30\alpha|^{1/2}$. The linear operator $\mathcal{A}_\mu$ is closed in both the Hilbert space $L^2(\mathbb{R})\times L^2(\mathbb{R})$ and in the Banach space $C_b(\mathbb{R})\times C_b(\mathbb{R})$, just as the operators $\mathcal{A}^*$ in Section 2, with dense domains $H^2(\mathbb{R})\times H^2(\mathbb{R})$ and $C_b^2(\mathbb{R})\times C_b^2(\mathbb{R})$, respectively. The space $L^2(\mathbb{R})\times L^2(\mathbb{R})$ corresponds to localized perturbations of the periodic wave and the space $C_b(\mathbb{R})\times C_b(\mathbb{R})$ to bounded perturbations.

In contrast to the operator $\mathcal{A}^*$ in Section 2 which has constant coefficients, the linear operator $\mathcal{A}_\mu$ is a differential operator with 2π-periodic coefficients. For the analysis of its spectrum, we therefore use a Floquet-Bloch decomposition, instead of Fourier analysis, which shows that its spectrum is the same in both spaces above, and that it is given by the union of the spectra of the Bloch operators,

$$\mathcal{A}_{\gamma,\mu} = -\mathbb{I} + \mathcal{J}\mathcal{L}_{\gamma,\mu},$$

where

$$\mathcal{L}_{\gamma,\mu} = \begin{pmatrix} (2-\alpha)(\partial_y + i\gamma)^2 - \alpha + 3\psi_{r\mu}^2(y) + \psi_{i\mu}^2(y) & 2\psi_{r\mu}(y)\psi_{i\mu}(y) \\ 2\psi_{r\mu}(y)\psi_{i\mu}(y) & (2-\alpha)(\partial_y + i\gamma)^2 - \alpha + \psi_{r\mu}^2(y) + 3\psi_{i\mu}^2(y) \end{pmatrix},$$

acting in $L^2(0,2\pi)\times L^2(0,2\pi)$ with domain the subspace of 2π-periodic functions $H^2_{\rm per}(0,2\pi)\times H^2_{\rm per}(0,2\pi)$, for $\gamma\in(-1/2,1/2]$ (e.g., see [4, 7]). The difference between the spectrum of the operator $\mathcal{A}_\mu$ when acting in $L^2(\mathbb{R})\times L^2(\mathbb{R})$ or in $C_b(\mathbb{R})\times C_b(\mathbb{R})$ is that it is purely essential spectrum in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and continuous purely point spectrum in $C_b(\mathbb{R}) \times C_b(\mathbb{R})$. In contrast, since the operators $\mathcal{A}_{\gamma,\mu}$ have compactly embedded domain, their resolvent is a compact operator, and therefore their spectrum is purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities. Our purpose now is to determine the spectrum of the Bloch operators $\mathcal{A}_{\gamma,\mu}$, for $\gamma \in (-1/2, 1/2]$ and μ sufficiently small.

Notice that the spectrum $\sigma(\mathcal{A}_{\gamma,\mu})$ of $\mathcal{A}_{\gamma,\mu}$ is symmetric with respect to the line $\operatorname{Re}\lambda = -1$ in the complex plane, just as the one of the operator $\mathcal{A}_*$ in Section 2, since $\mathcal{J}$ and $\mathcal{L}_{\gamma,\mu}$ are skew- and self-adjoint operators, respectively. In addition, the equalities

$$\overline{\mathcal{A}_{\gamma,\mu}} = \mathcal{A}_{-\gamma,\mu}, \qquad \mathcal{A}_{\gamma,\mu}\mathcal{T} = \mathcal{T}\mathcal{A}_{-\gamma,\mu}, \tag{4.2}$$

where the operator $\mathcal{T}$ in the last equality is the reflection operator defined in (3.2), imply that

$$\overline{\sigma(\mathcal{A}_{\gamma,\mu})} = \sigma(\mathcal{A}_{-\gamma,\mu}) = \sigma(\mathcal{A}_{\gamma,\mu}).$$

Spectral analysis of the Bloch operators

We analyze the spectra of the operators $\mathcal{A}_{\gamma,\mu}$ in two steps, first for values of γ outside a fixed, but arbitrary, neighborhood of 0 (Lemma 4.1), and then for small γ (Lemma 4.2).

Lemma 4.1 *For any $\gamma_1 \in (0, 1/2)$, there exist positive constants μ_1 and δ_1, such that the spectrum of $\mathcal{A}_{\gamma,\mu}$ satisfies*

$$\sigma(\mathcal{A}_{\gamma,\mu}) \subset \{\lambda \in \mathbb{C}\, ;\, \operatorname{Re}\lambda < -\delta_1\},$$

for any $\gamma \in (-1/2, 1/2]$, $|\gamma| > \gamma_1$ and $|\mu| < \mu_1$.

Proof. We use a perturbation argument in which we regard the operator $\mathcal{A}_{\gamma,\mu}$ as a small perturbation of the operator $\mathcal{A}_{\gamma,0}$, which has constant coefficients. The spectrum of $\mathcal{A}_{\gamma,0}$ is easily obtained from the linear stability analysis in Section 2, in which we restrict to the Fourier modes $k = \sqrt{2-\alpha}\,(n+\gamma)$, so that

$$\sigma(\mathcal{A}_{\gamma,0}) = \left\{\lambda_\pm(n,\gamma) = -1 \pm \sqrt{1-(2-\alpha)^2((n+\gamma)^2-1)^2}\right\}.$$

The results in Section 2 also show that the two eigenvalues $\lambda_\pm(n,\gamma)$ are either negative or have negative real parts except for the eigenvalues $\lambda_+(\pm 1, 0)$ which vanish. Consequently, for any $\gamma_1 \in (0, 1/2)$, the exists $\delta_0 > 0$, such that the spectrum of $\mathcal{A}_{\gamma,0}$ satisfies

$$\sigma(\mathcal{A}_{\gamma,0}) \subset \{\lambda \in \mathbb{C}\, ;\, \operatorname{Re}\lambda < -\delta_0\}, \tag{4.3}$$

for any $\gamma \in (-1/2, 1/2]$, $|\gamma| > \gamma_1$.

For small μ, the operator $\mathcal{A}_{\gamma,\mu}$ is a bounded perturbation of the operator $\mathcal{A}_{\gamma,0}$ with uniform bound in γ of order $\mathcal{O}(|\mu|^{1/2})$. Consequently, for any $\delta > 0$ and any $\gamma \in (-1/2, 1/2]$, we have that

$$\sigma(\mathcal{A}_{\gamma,\mu}) \subset \{\lambda \in \mathbb{C}\, ;\, \operatorname{dist}(\lambda, \sigma(\mathcal{A}_{\gamma,0})) < \delta\},$$

provided μ is sufficiently small. Together with the property (4.3) this proves the lemma. ■

Lemma 4.2 *There exist positive constants γ_2, μ_2, and δ_2 such that, for any $|\gamma| < \gamma_2$ and $|\mu| < \mu_2$, the spectrum of $\mathcal{A}_{\gamma,\mu}$ decomposes into two disjoint subsets*

$$\sigma(\mathcal{A}_{\gamma,\mu}) = \sigma_-(\mathcal{A}_{\gamma,\mu}) \cup \sigma_0(\mathcal{A}_{\gamma,\mu}), \tag{4.4}$$

with the following properties:

(i) $\sigma_-(\mathcal{A}_{\gamma,\mu}) \subset \{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\,\lambda < -\delta_2\}$ *and* $\sigma_0(\mathcal{A}_{\gamma,\mu}) \subset \{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\,\lambda > -\delta_2\}$*;*

(ii) the set $\sigma_0(\mathcal{A}_{\gamma,\mu})$ consists of two simple and real eigenvalues $\lambda_{1,2}(\gamma,\mu)$ with Taylor expansions

$$\lambda_1(\gamma,\mu) = -\frac{2}{(2-\alpha)^2}\,\mu - 2(2-\alpha)^2\gamma^2 + o(|\mu| + |\gamma|^2),$$
$$\lambda_2(\gamma,\mu) = -2(2-\alpha)^2\gamma^2 + o(|\gamma|^2).$$

Proof. For small γ and μ, the operator $\mathcal{A}_{\gamma,\mu}$ is a small relatively bounded perturbation of the operator $\mathcal{A}_{0,0}$. The latter operator is precisely the operator $\mathcal{A}_0^*$ considered in Section 3, for which we have the spectral decomposition (3.3). A standard perturbation argument then implies the spectral decomposition (4.4) for $\mathcal{A}_{\gamma,\mu}$, with the property *(i)*, and such that $\sigma_0(\mathcal{A}_{\gamma,\mu})$ consists of precisely two eigenvalues $\lambda_{1,2}(\gamma,\mu)$, which are the continuation of the double eigenvalue 0 of $\mathcal{A}_{0,0}$, for sufficiently small γ and μ. It remains to compute the Taylor expansions of these eigenvalues in *(ii)*.

Consider a basis $\{\zeta_1(\gamma,\mu),\ \zeta_2(\gamma,\mu)\}$ of the two-dimensional spectral subspace of $\mathcal{A}_{\gamma,\mu}$ associated to $\sigma_0(\mathcal{A}_{\gamma,\mu})$, which is a smooth continuation for small γ and μ of the two-dimensional kernel of $\mathcal{A}_{0,0}$, and denote by $\mathcal{M}(\gamma,\mu)$ the 2×2 matrix representing the action of $\mathcal{A}_{\gamma,\mu}$ on this basis. Since $\mathcal{A}_{\gamma,\mu}$ depends smoothly upon γ and $|\mu|^{1/2}$, we have the same smoothness properties for the vectors in the basis and for the matrix $\mathcal{M}(\gamma,\mu)$. The eigenvalues $\lambda_{1,2}(\gamma,\mu)$ in $\sigma_0(\mathcal{A}_{\gamma,\mu})$ are the two eigenvalues of the matrix $\mathcal{M}(\gamma,\mu)$, and our purpose is to compute a Taylor expansion in γ and $|\mu|^{1/2}$ of these two eigenvalues, from a Taylor expansion of the matrix $\mathcal{M}(\gamma,\mu)$.

At $\gamma=\mu=0$, we have $\mathcal{M}(0,0)=0$, and we take as basis the even and odd real eigenvectors

$$\zeta_1(0,0) = \begin{pmatrix}\alpha \\ 2-\alpha\end{pmatrix}\cos(y), \qquad \zeta_2(0,0) = \begin{pmatrix}\alpha \\ 2-\alpha\end{pmatrix}\sin(y),$$

which are the real and imaginary parts, respectively, of the vector ζ used for the bifurcation analysis in Section 3. With this choice for the eigenvectors at $\gamma=\mu=0$, due to the symmetries (4.2), which are inherited by the matrix $\mathcal{M}(\gamma,\mu)$, the elements on the main diagonal and on the antidiagonal of the matrix $\mathcal{M}(\gamma,\mu)$ will be even real and odd purely imaginary in γ, respectively. Furthermore, the real and imaginary parts of the vectors $\zeta_1(\gamma,\mu)$ and $\zeta_2(\gamma,\mu)$ will be even and odd in γ, respectively, and have opposite parities in y: the real part of $\zeta_1(\gamma,\mu)$ and the imaginary part of $\zeta_2(\gamma,\mu)$ are even functions in y, whereas the imaginary part of $\zeta_1(\gamma,\mu)$ and the real part of $\zeta_2(\gamma,\mu)$ are odd functions in y.

We start by computing an expansion of $\mathcal{M}(\gamma,\mu)$ for the particular values $\mu=0$ and $\gamma=0$. For $\mu=0$, the operator $\mathcal{A}(\gamma,0)$ has constant coefficients, so that we can explicitly compute the basis and the matrix $\mathcal{M}(\gamma, 0)$. We find

$$\zeta_1(\gamma,0) = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} \cos(y) - \frac{2(2-\alpha)F_1^2}{\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(y)\, i\gamma + \mathcal{O}(\gamma^2),$$

$$\zeta_2(\gamma,0) = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} \sin(y) + \frac{2(2-\alpha)F_1^2}{\alpha} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(y)\, i\gamma + \mathcal{O}(\gamma^2),$$

and

$$\mathcal{M}(\gamma,0) = \begin{pmatrix} -2(2-\alpha)^2\gamma^2 + \mathcal{O}(\gamma^4) & \mathcal{O}(|\gamma|^3) \\ \mathcal{O}(|\gamma|^3) & -2(2-\alpha)^2\gamma^2 + \mathcal{O}(\gamma^4) \end{pmatrix}.$$

For $\gamma = 0$, the symmetries above imply that the matrix $\mathcal{M}(0,\mu)$ is diagonal,

$$\mathcal{M}(0,\mu) = \begin{pmatrix} \lambda_1(0,\mu) & 0 \\ 0 & \lambda_2(0,\mu) \end{pmatrix},$$

so that $\zeta_1(0,\mu)$ and $\zeta_2(0,\mu)$ are eigenvectors associated to the eigenvalues $\lambda_1(0,\mu)$ and $\lambda_2(0,\mu)$, respectively. Due to the translation invariance of the Lugiato-Lefever equation, the derivative $(\psi'_{r\mu}(y), \psi'_{i\mu}(y))$ of the periodic wave, which is an odd function, belongs to the kernel of the operator $\mathcal{A}(0,\mu)$. Consequently, $(\psi'_{r\mu}(y), \psi'_{i\mu}(y))$ is proportional to the odd eigenvector,

$$\zeta_2(0,\mu) = -\frac{1}{\psi_1|\mu|^{1/2}} \begin{pmatrix} \psi'_{r\mu}(y) \\ \psi'_{i\mu}(y) \end{pmatrix} = \begin{pmatrix} \alpha \\ 2-\alpha \end{pmatrix} \sin(y) + \mathcal{O}(|\mu|^{1/2}),$$

and the corresponding eigenvalue vanishes, $\lambda_2(0,\mu) = 0$. For the computation of the eigenvalue $\lambda_1(0,\mu)$, we use the center manifold constructed in Section 3. The key observation is that the linear operator $\mathcal{A}(0,\mu)$ coincides with the linear operator obtained by linearizing the dynamical system (3.4) at the periodic wave $(\psi_{r\mu}(y), \psi_{i\mu}(y))$. Since the center manifold is locally invariant for the flow of the dynamical system (3.4), the two eigenvalues $\lambda_1(0,\mu)$ and $\lambda_2(0,\mu)$ are the two eigenvalues of the 2×2 matrix obtained by linearizing the reduced system (3.5) at the real equilibrium r_μ which corresponds to the even periodic wave. A direct computation gives

$$\lambda_1(0,\mu) = -\frac{2}{(2-\alpha)^2}\,\mu + \mathcal{O}(|\mu|^{3/2}), \qquad \lambda_2(0,\mu) = 0,$$

so that

$$\mathcal{M}(0,\mu) = \begin{pmatrix} -\frac{2}{(2-\alpha)^2}\,\mu + \mathcal{O}(|\mu|^{3/2}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Summarizing the above results, we conclude that

$$\begin{aligned} \mathcal{M}(\gamma,\mu) \;=\; & \begin{pmatrix} -2(2-\alpha)^2\gamma^2 - \frac{2}{(2-\alpha)^2}\,\mu & ic_1\gamma|\mu|^{1/2} \\ ic_2\gamma|\mu|^{1/2} & -2(2-\alpha)^2\gamma^2 \end{pmatrix} \\ & + \begin{pmatrix} \mathcal{O}(\gamma^4 + \gamma^2|\mu|^{1/2} + |\mu|^{3/2}) & \mathcal{O}(|\gamma|^3 + |\gamma\mu|) \\ \mathcal{O}(|\gamma|^3 + |\gamma\mu|) & \mathcal{O}(\gamma^4 + \gamma^2|\mu|) \end{pmatrix}, \end{aligned}$$

in which it remains to compute the real constants c_1 and c_2. Consider the expansion of the linear operator

$$\mathcal{A}(\gamma,\mu) = \mathcal{A}_{00} + \mathcal{A}_{10} i\gamma + \mathcal{A}_{01}|\mu|^{1/2} + \mathcal{A}_{20}\gamma^2 + \mathcal{A}_{11}\, i\gamma|\mu|^{1/2} + \mathcal{A}_{02}|\mu| + \mathcal{O}((|\gamma| + |\mu|^{1/2})^3),$$

and the expansions of the two vectors in the basis

$$\zeta_j(\gamma,\mu) = \zeta_{j00} + \zeta_{j10} i\gamma + \zeta_{j01}|\mu|^{1/2} + \zeta_{j20}\gamma^2 + \zeta_{j11}\, i\gamma|\mu|^{1/2} + \zeta_{j02}|\mu| + \mathcal{O}((|\gamma| + |\mu|^{1/2})^3), \quad j = 1,2,$$

in which the vectors $\zeta_{j00} = \zeta_j(0,0)$, $j = 1,2$ are known. Inserting these expansions into the equality

$$\mathcal{A}_{\gamma,\mu}\begin{pmatrix}\zeta_1(\gamma,\mu) & \zeta_2(\gamma,\mu)\end{pmatrix} = \mathcal{M}(\gamma,\mu)\begin{pmatrix}\zeta_1(\gamma,\mu)\\ \zeta_2(\gamma,\mu)\end{pmatrix},$$

we find at orders $\mathcal{O}(|\gamma|)$ and $\mathcal{O}(|\mu|^{1/2})$ the equalities

$$\mathcal{A}_{00}\zeta_{j10} + \mathcal{A}_{10}\zeta_{j00} = 0, \qquad \mathcal{A}_{00}\zeta_{j01} + \mathcal{A}_{01}\zeta_{j00} = 0, \quad j = 1,2,$$

which allow to compute the first order terms ζ_{j10} and ζ_{j01}, $j = 1,2$, in the expansions above, and at order $\mathcal{O}(|\gamma||\mu|^{1/2})$ the equalities

$$\mathcal{A}_{00}\zeta_{111} + \mathcal{A}_{10}\zeta_{101} + \mathcal{A}_{01}\zeta_{110} = c_1\zeta_{200},$$
$$\mathcal{A}_{00}\zeta_{211} + \mathcal{A}_{10}\zeta_{201} + \mathcal{A}_{01}\zeta_{210} = c_2\zeta_{100}.$$

which determine ζ_{111}, ζ_{211}, and the coefficients c_1 and c_2. Taking into account the action of the operators $\mathcal{A}_{ij}$ on the different Fourier modes, and the known expressions of $\zeta_{j00} = \zeta_j(0,0)$, $j = 1,2$, which only involve the Fourier modes ± 1, we conclude that $c_1 = c_2 = 0$. Consequently,

$$\begin{aligned}\mathcal{M}(\gamma,\mu) \;=\; &\begin{pmatrix} -2(2-\alpha)^2\gamma^2 - \dfrac{2}{(2-\alpha)^2}\,\mu & 0 \\ 0 & -2(2-\alpha)^2\gamma^2 \end{pmatrix}\\ &+ \begin{pmatrix} \mathcal{O}(\gamma^4 + \gamma^2|\mu|^{1/2} + |\mu|^{3/2}) & \mathcal{O}(|\gamma|^3 + |\gamma\mu|) \\ \mathcal{O}(|\gamma|^3 + |\gamma\mu|) & \mathcal{O}(\gamma^4 + \gamma^2|\mu|) \end{pmatrix},\end{aligned}$$

which implies the expansions of the eigenvalues in *(ii)*, and completes the proof of the lemma. ■

Spectral stability and instability

The results in Lemma 4.1, Lemma 4.2, and the connection between the spectra of the operator $\mathcal{A}_\mu$ and of the Bloch operators $\mathcal{A}_{\gamma,\mu}$, imply that the periodic waves are stable with respect to general bounded perturbations in the case $\alpha < 41/30$ (supercritical bifurcation) and unstable in the case $\alpha > 41/30$ (subcritical bifurcation). The instability is due to the eigenvalue $\lambda_1(0,\mu)$ of the operator $\mathcal{A}_{0,\mu}$, which is positive in this case, since $\mu < 0$. This recovers the instability result for co-periodic perturbations already found in Theorem 1, on the one hand, and implies that $\lambda_1(\gamma,\mu)$ is positive for sufficiently small γ, thus the periodic wave is also unstable for periodic perturbations with nearby periods, on the other hand. For localized and bounded perturbations this leads to an instability interval $(0, \lambda_1(0, \mu)]$ in the spectrum of the operator $\mathcal{A}_\mu$. More precisely, we have the following result.

Theorem 2 *Consider the Lugiato-Lefever equation (1.1) in the case $\beta = -1$ of anomalous dispersion. Assume that $\alpha < 2$, $\alpha \neq 41/30$, and $F^2 = F_1^2 + \mu$. Consider the periodic solutions $\psi_\mu(x)$ of the Lugiato-Lefever equation (1.1) given by (3.10), for $\mu > 0$ if $\alpha < 41/30$ and for $\mu < 0$ if $\alpha > 41/30$, and the associated linear operator $\mathcal{A}_\mu$ in (4.1), acting in either $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ or $C_b(\mathbb{R}) \times C_b(\mathbb{R})$. For any μ sufficiently small the following properties hold.*

(i) If $\alpha < 41/30$ and $\mu > 0$, then the spectrum of the linear operator $\mathcal{A}_\mu$ lies in the closed left half complex plane, and the periodic solution $\psi_\mu(x)$ is spectrally stable with respect to bounded perturbations.

(ii) If $\alpha > 41/30$ and $\mu < 0$, then the spectrum of the linear operator $\mathcal{A}_\mu$ lies in the closed left half complex plane, except for a finite interval of positive real numbers $(0, c_\mu]$, with $c_\mu = \mathcal{O}(|\mu|)$, and the periodic solution $\psi_\mu(x)$ is spectrally unstable with respect to co-periodic, localized, and bounded perturbations.

5 The case of normal dispersion

In this section we consider the Lugiato-Lefever equation (1.1) in the case $\beta = 1$ of normal dispersion. The key difference with the case $\beta = -1$ of anomalous dispersion occurs in the linear stability analysis of constant solutions, which have now different stability properties. As a consequence, periodic waves will bifurcate at different parameter values, but the bifurcation analysis in Section 3 and the stability analysis in Section 4 stay the same, including computations.

Constant solutions

In both cases of normal and of anomalous dispersion, the constant solutions of the Lugiato-Lefever equation (1.1) are the same (see Figure 2.1). The first difference between the two cases arises in the linear stability analysis of these constant solutions. For a constant solution $\psi^* = \psi_r^* + i\psi_i^*$, with modulus square ρ^*, the linear operator $\mathcal{A}^*$ has the same form (2.2), but the linear operator $\mathcal{L}^*$ changes, the terms ∂_x^2 having now a coefficient -1,

$$\mathcal{L}^* = \begin{pmatrix} -\partial_x^2 - \alpha + 3\psi_r^{*2} + \psi_i^{*2} & 2\psi_r^*\psi_i^* \\ 2\psi_r^*\psi_i^* & -\partial_x^2 - \alpha + \psi_r^{*2} + 3\psi_i^{*2} \end{pmatrix}.$$

Then the sign of the coefficient of k^2 changes in $\mathcal{L}^*(k)$ and in the formula of $a(k)$ in the spectrum of linear operator $\mathcal{A}^*$,

$$\sigma(\mathcal{A}^*) = \left\{\lambda \in \mathbb{C}\ ;\ \lambda^2 + 2\lambda + a(k) = 0,\ k \in \mathbb{R}\right\},$$

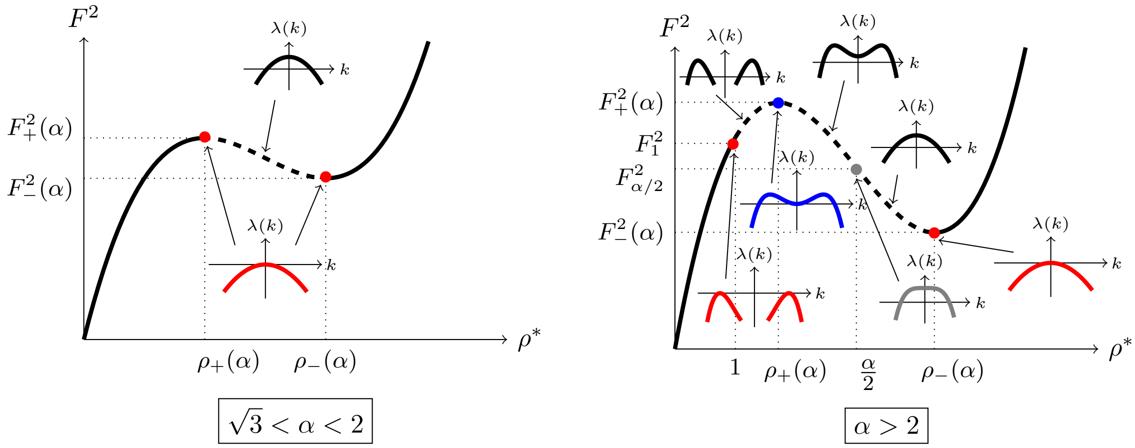

Figure 5.1: Stability of constant solutions of the Lugiato-Lefever equation in the case of normal dispersion ($\beta = 1$), for $\alpha > \sqrt{3}$ (for $\alpha \leqslant \sqrt{3}$ the unique constant solution is stable). Stable branches are represented by solid lines, unstable branches by dashed lines, and we set $F_1^2 = (1-\alpha)^2 + 1$ and $F_{\alpha/2}^2 = \alpha(4+\alpha^2)/8$. The shape of the largest eigenvalue $\lambda(k)$ is given at the points where the solutions loose their stability and for the unstable solutions.

in which

$$a(k) = k^4 - 2(\alpha - 2\rho^*)k^2 + \alpha^2 - 4\alpha\rho^* + 3\rho^{*2} + 1.$$

This implies a change in the stability properties of the constant solutions, which are summarized in Figure 5.1 . A Turing instability occurs now for $\alpha > 2$, $\rho^* = 1$, and $F^2 = (1-\alpha)^2+1$, for wavenumbers $k = \pm\sqrt{\alpha - 2}$, and a zero-mode instability occurs for $\alpha > \sqrt{3}$, $\rho^* = \rho_-(\alpha)$, $F^2 = F_-(\alpha)$, and for $\sqrt{3} < \alpha < 2$, $\rho^* = \rho_+(\alpha)$, $F^2 = F_+(\alpha)$.

Periodic waves

As in Section 3, we consider the onset of Turing instability which occurs in this case for $\alpha > 2$, but the same values $\rho^* = 1$ and $F = F_1$, $F_1^2 = (1-\alpha)^2 + 1$. We fix $\alpha > 2$, take as bifurcation parameter $F^2 = F_1^2 + \mu$, and denote by $\psi_\mu^* = \psi_{r\mu}^* + i\psi_{i\mu}^*$ and $\rho_\mu^* = |\psi_\mu^*|^2$ the corresponding constant solution of the Lugiato-Lefever equation (1.1) and its square modulus, respectively, given by (2.1).

In this case, the Turing instability occurs for modes $k = \pm\sqrt{\alpha - 2}$. We therefore look for solutions of the Lugiato-Lefever equation (1.1) close to the branch of constant solutions ψ_μ^* of the form

$$\psi(x,t) = \psi_\mu^* + (u + iv)(y,t), \qquad y = \sqrt{\alpha - 2}\, x.$$

Inserting this Ansatz into the Lugiato-Lefever equation (1.1) we obtain exactly the same equation (3.1) as in the case of anomalous dispersion. Since the constant solutions are the same in the two cases, the only difference could occur for the coefficient of the term ∂_y^2 in the formula of the linear operator $\mathcal{L}_\mu^*$. But since $y = \sqrt{\alpha - 2}\, x$ in this case, instead of $y = \sqrt{2 - \alpha}\, x$ in the case of anomalous dispersion, this coefficient is also the same. Consequently, all arguments, including computations, in Section 3 remain valid, with the only difference that now $\alpha > 2$. This implies that in this case the reduced system (3.5) undergoes a subcritical steady bifurcation with $O(2)$ symmetry with the properties *(iii)* and *(iv)* in Theorem 1.

The stability analysis in Section 4 is also the same. For the same reason as above, the linear operator $\mathcal{A}_\mu$ in (4.1) does not change, and we conclude that the periodic waves are unstable with the properties in Theorem 2 *(ii)*. We summarize these results in the following theorem.

Theorem 3 *Consider the Lugiato-Lefever equation (1.1) in the case $\beta = 1$ of normal dispersion. Assume that $\alpha > 2$ and $F^2 = F_1^2 + \mu$, $F_1^2 = (1-\alpha)^2 + 1$. For any $\mu < 0$ sufficiently small the following properties hold.*

(i) The equation possesses an even periodic solution $\psi_\mu(x)$ with wavenumber $\sqrt{\alpha-2}$ and Taylor expansion (3.10), which is unstable with respect to co-periodic perturbations.

(ii) The spectrum of the linear operator $\mathcal{A}_\mu$ obtained by linearizing the equation at this periodic solution and acting in either $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ or $C_b(\mathbb{R}) \times C_b(\mathbb{R})$ lies in the closed left half complex plane, except for a finite interval of positive real numbers $(0, c_\mu]$, with $c_\mu = \mathcal{O}(|\mu|)$. Consequently, the periodic solution $\psi_\mu(x)$ is spectrally unstable with respect to localized and bounded perturbations.

6 Discussion

We conclude with a brief discussion of the nonlinear stability problem and of the local bifurcations which occur at the onset of zero-mode instability.

Nonlinear stability: co-periodic and subharmonic perturbations

Besides showing the existence of periodic waves, the local bifurcation result in Theorem 1 also implies their nonlinear stability with respect to perturbations which belong to the domain $\mathcal{Y}$ of the linear operator $\mathcal{A}_\mu^*$ in the dynamical system (3.1), i.e., for co-periodic perturbations which are H^2. Since at the bifurcation point $\mu = 0$ the operator $\mathcal{A}_0^*$ does not have unstable spectrum, the leading order dynamics is given by the behavior of the solutions on the center manifold. Therefore, the bifurcating periodic waves, which correspond to the circle of equilibria on the center manifold, are unstable in the subcritical bifurcation, whereas they are stable in the supercritical bifurcation. In this latter case, for initial data $\psi(x,0) = \psi_\mu(x) + \phi_0(x)$, sufficiently close to a periodic wave $\psi_\mu(x)$, the solution $\psi(x,t)$ of the Lugiato-Lefever equation converges to a translated periodic wave $\psi_\mu(x+a)$, for some $a \in \mathbb{R}$,

$$\|\psi(\cdot,t) - \psi_\mu(\cdot + a)\|_{H^2_{\rm per}} \to 0, \quad \text{as } t \to \infty.$$

The decay rate is given by the convergence rate towards equilibria on the center manifold, hence it is slowly exponential, $\mathcal{O}(e^{-d\mu})$, for some $d > 0$.

This result which holds for co-periodic perturbations can be extended to subharmonic perturbations by enlarging the phase space $\mathcal{X} = L^2(0, 2\pi) \times L^2(0, 2\pi)$ of 2π-periodic functions to the phase space $\mathcal{X}_N = L^2(0, 2\pi N) \times L^2(0, 2\pi N)$ of $2\pi N$-periodic functions, for some arbitrary, but fixed N. The key difference is that now the spectrum of the operator $\mathcal{A}_0^*$ possesses additional eigenvalues,

$$\sigma(\mathcal{A}_0^*) = \left\{ \lambda_\pm(n, N) = -1 \pm \sqrt{1 - (2-\alpha)^2(n^2/N^2 - 1)^2},\ n \in \mathbb{Z} \right\}.$$

These additional eigenvalues have negative real parts, so that the spectral decomposition (3.3) still holds, but with a constant $\delta_N \to 0$, as $N \to \infty$, and 0 remains a double semi-simple eigenvalue with the same associated eigenvectors ζ and $\overline{\zeta}$ which belong to $\mathcal{X} \subset \mathcal{X}_N$. Since $\mathcal{X}$ is an invariant subspace for the dynamics of (3.1), the resulting center manifold lies in $\mathcal{X}$ and the reduced dynamics is described by the same reduced system (3.5). Consequently, the result in Theorem 1 holds in $\mathcal{X}_N$, implying in particular the stability of the periodic waves with respect to subharmonic perturbations. However, this result is not uniform in N, since the spectral gap in the decomposition (3.3) tends to 0, as $N \to \infty$. For a given periodic wave ψ_μ^*, we can therefore conclude on stability for a finite number of integers N, only. A stability result which holds for arbitrary values N, would allow to conclude on stability, at least spectral, with respect to general bounded perturbations, but such a result cannot be obtained using the center manifold approach in Section 3. The spectral analysis in Section 4 shows the spectral stability for localized and bounded perturbations, but leaves open the question of nonlinear stability.

Zero-mode instability

The local bifurcations induced by the zero-mode instability of the constant solutions found in Section 2 can be analyzed using the same approach as in Section 3. Focusing on parameter values close to zero-mode instability, the Lugiato-Lefever equation is written as dynamical system of the form (3.1) in the same way, and we may take as phase space $L^2(0, L) \times L^2(0, L)$, for any arbitrary, but fixed L, hence considering functions which are L-periodic in the spatial variable x. With this choice, the corresponding linear operator has purely point spectrum, again, which lies in the open left half complex plane, except for 0 which is now a simple eigenvalue. Applying the center manifold theorem, we obtain a one-dimensional manifold on which the dynamics is governed by a scalar ordinary differential equation. However, since the eigenvector associated with the simple eigenvalue 0 is a constant function, and since the subspace of constant functions is invariant for the dynamics of (3.1), the center manifold lies in this subspace and the only bifurcating solutions found in this way are the constant solutions of (1.1). We point out that in this approach we cannot replace the phase space $L^2(0, L) \times L^2(0, L)$ by a space of functions defined on the real line, e.g., $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, because with such a choice the linear operator $\mathcal{A}_0^*$ has continuous spectrum and no spectral gap allowing to use the center manifold reduction.

As mentioned in the Introduction, an alternative approach for the existence problem is the spatial dynamics approach used in [3]. In the setting from [3], this zero-mode instability corresponds to a reversible 0^2-bifurcation, in which bounded steady solutions bifurcate which are localized or periodic. The Turing instability studied in Section 3, corresponds to a reversible 1:1-resonance, or $(i\omega)^2$-bifurcation, and besides periodic solutions, localized and quasi-periodic solutions bifurcate in this case, as well. While this spatial dynamics approach provides a very detailed description of the set of bounded solutions, with no restriction to periodic waves, it does not give any information about the stability of these solutions, which is an open problem.

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