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Enumeration of hypermaps of a given genus*

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Abstract

This paper addresses the enumeration of rooted and unrooted hypermaps of a given genus. For rooted hypermaps the enumeration method consists of considering the more general family of multirooted hypermaps, in which darts other than the root dart are distinguished. We give functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the vertices containing the distinguished darts. We solve these equations to get parametric expressions of the generating functions of rooted hypermaps of low genus. We also count unrooted hypermaps of given genus by number of darts, vertices, hyperedges and faces.

Keywords: Enumeration, surface, genus, rooted hypermap, unrooted hypermap.

Math. Subj. Class.: 05C30, 05A15

1 Introduction

A *(combinatorial) hypermap* is a triple (D, R, L) where D is a finite set of *darts* and R and L are permutations on D such that the group $\langle R, L\rangle$ generated by R and L acts transitively on D. A *(combinatorial ordinary) map* is a hypermap (D, R, L) whose permutation L is a fixed-point-free involution on D. For a hypermap (resp. map) the orbits of R, L and

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RL (L followed by R) are respectively called *vertices*, *hyperedges* (resp. *edges*) and *faces*. The *degree* of a vertex, edge, hyperedge or face is the number of darts it contains. The equivalence of combinatorial maps and topological maps having been established in [14], we use the word "map" to mean "combinatorial map" throughout this paper. The *genus* g of a map is given by the Euler-Poincaré formula [7]

$$v - e + f = 2(1 - g), \tag{1.1}$$

where v is the number of vertices, e is the number of edges and f is the number of faces. The genus of a hypermap with t darts, v vertices, e hyperedges and f faces was defined in [13] by the formula

$$v + e + f = t + 2(1 - g). \tag{1.2}$$

An *isomorphism* between two maps or hypermaps (D, R, L) and (D', R', L') is a bijection from D onto D' that takes R into R' and L into L'; it corresponds to an orientation-preserving homeomorphism between two topological maps. A *sensed* hypermap (resp. map) is an isomorphism class of hypermaps (resp. maps). We admit the existence of a unique hypermap (resp. map) with an empty set of darts D, called the *empty* hypermap (resp. map). For both of these objects $v = f = 1$ and $g = e = 0$. A *rooted* hypermap (resp. map) is either the empty hypermap (resp. map) or a tuple (D, x, R, L) where (D, R, L) is a non-empty combinatorial hypermap (resp. map) and $x \in D$ is a distinguished dart, called the *root*.

The enumeration of maps and hypermaps has several non-trivial applications. One such application is based on the correspondence between hypermaps and algebraic curves established by the Belyi theorem [16]. For instance, the formula for the number of plane trees was used by A. Zvonkin in the computer generation of Shabat polynomials of bounded degree [16]. Another area where the map enumeration plays an important role is theoretical physics, in particular in 2-dimensional gravitation models. Roughly speaking, map enumeration is used to compute matrix integrals determining the properties of gravitational fields (see for instance the works of B. Eynard [9]). Some hypermaps have been shown to be related to contextuality in quantum physics [21]. Also, A. Mednykh and R. Nedela have applied the enumeration of rooted (resp. unrooted) hypermaps to the enumeration of subgroups (resp. conjugacy classes of subgroups) of the triangle group with three generators x, y, z and the relation $xyz = 1$ [20].

We enumerate rooted hypermaps of a given genus by number of darts, vertices, hyperedges and faces. To do so we consider more general families of rooted hypermaps and bipartite maps, in which other vertices or darts than the root dart are distinguished. We also use the genus-preserving bijection between hypermaps and 2-vertex-coloured bipartite maps presented in [23]. But since bipartite maps have all their **faces** of even degree and we're using the degrees of the **vertices** as parameters, we must instead study the face-vertex dual of a 2-coloured bipartite map, that is, a map whose faces are coloured in two colours (white and black) so that no two faces that share an edge have the same colour. All these maps are *Eulerian* – that is, all their vertices are of even degree – but not all Eulerian maps are 2-face-colourable. For example, the map on the torus with one vertex, one face and two edges is Eulerian because its only vertex is of degree 4, but its face cannot be coloured because it shares both edges with itself. Therefore we call the maps we are studying *face-bipartite*.

A *sequenced (rooted) map* is a rooted map with some vertices other than the *root vertex* (the vertex that contains the root) distinguished from each other and from all the other vertices. The labels that distinguish these vertices can be taken to be $1, 2, \dots, k$, where k is the number of distinguished vertices. A *sequenced (rooted) hypermap* is defined similarly. We state (in Section 4) a bijective decomposition for the set $\mathcal{H}(g, t, f, e, n, D)$ of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the sequence of degrees of the distinguished vertices equal to $D = (d_1, d_2, \dots, d_{|D|})$, where d_i is the degree of the distinguished vertex with label i. We obtain a bijective decomposition of the set $\mathcal{F}(g, e, w, b, n, D)$ of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the sequence of half-degrees of the distinguished vertices equal to D. Then we apply face-vertex duality to obtain a bijective decomposition of the corresponding set of 2-coloured bipartite maps with distinguished faces. Next we use the bijection in [23] to obtain a bijective decomposition for hypermaps with distinguished faces, and finally we again apply face-vertex duality to obtain a bijective decomposition of $\mathcal{H}(g, t, f, e, n, D)$.

A *mutirooted hypermap* is a hypermap in which a non-empty sequence of darts with pairwise distinct initial vertices is distinguished. We relate multirooted hypermaps to sequenced hypermaps and thus obtain a recurrence for the number of multirooted hypermaps and functional equations for the generating series counting multirooted hypermaps of a given genus by number of darts, vertices, edges, faces and the degrees of the initial vertices of the distinguished darts.

The paper is organized as follows. Section 2 fixes some notations, recalls a known decomposition for sequenced rooted maps and describes the bijection between hypermaps and bipartite maps presented in [23]. Sections 3 and 4 respectively enumerate sequenced face-bipartite maps and sequenced rooted hypermaps of a given genus. In Section 5 we consider multirooted hypermaps and we give equations for the generating functions that count these objects. In Section 6 we give functional equations relating the generating functions for rooted hypermaps with that for multirooted hypermaps. Then we show how to solve these equations. In Section 7 we obtain parametric expressions for the generating functions that count rooted hypermaps with a given small positive genus. Section 8 presents enumeration algorithms for sensed unrooted hypermaps counted by number of darts, vertices and hyperedges. Appendix A (resp. B) contains a table for numbers of rooted (resp. unrooted) hypermaps of genus g with d darts, v vertices and e hyperedges for $d \leq 14$.

2 Background

2.1 Notations

We first introduce the notations and conventions we use throughout the paper. Let D and D' be two lists of integers. The inclusion $D' \subseteq D$ means that D' is a sublist of D. In this case $D - D'$ is the complementary sublist of D' in D. For instance, the sublists of $D = [1, 1, 2]$ are the empty list $[\,]$, $[1]$ (twice), $[2]$, $[1, 1]$, $[1, 2]$ (twice) and D itself. Their complementary sublists in the same order are D, $[1, 2]$ (twice), $[1, 1]$, $[2]$, $[1]$ (twice) and $[\,]$. We denote by $D.D'$ the concatenation of the lists D and D'. If i is an integer and D is a list of integers, then $i.D$ is a shortcut for $[i].D$. For $1 \leq j \leq |D|$ we denote by d_j the j-th element of the list D of length $|D|$ and by $D - \{d_j\}$ the list obtained from D by removing its j-th element d_j. Let ρ be a positive integer. The abbreviation $D_{1..\rho}$ denotes the list $[d_1, \dots, d_\rho]$. The abbreviation $v_{1..\rho}^{D_{1..\rho}}$ denotes $v_1^{d_1} \dots v_\rho^{d_\rho}$.

The sign $+$ (resp. $\sum$) denotes (resp. generalized) disjoint set union in the following decompositions and (resp. generalized) arithmetic sum in the following equations. By convention, a disjoint set union (resp. sum) over an empty domain is equal to the empty set (resp. zero). For any logical formula φ the notation Δ_φ means the singleton set containing only the empty hypermap or map (depending on the context) and the empty set if φ is false. The notation δ_φ means 1 if φ is true and 0 if φ is false.

2.2 Bijective decomposition of the set of sequenced maps

In 1962 W. T. Tutte [22] presented a bijective decomposition of a planar map with all the vertices distinguished and a root in every vertex. In 1972 T. R. Walsh and A. B. Lehman [27] generalized this decomposition to maps of higher genus and used it to count rooted maps of a given genus by number of vertices and faces. In 1987 D. Arquès [3] used this latter decomposition to find a closed-form formula for the number of rooted maps of genus 1 by number of vertices and faces. In 1991 E. A. Bender and E. A. Canfield [4] presented a more efficient decomposition that roots only a single vertex and distinguishes only as many other vertices as necessary and used it to obtain explicit formulas for counting rooted maps of genus 2 and 3. In 1998 the first author [11] modified this decomposition and used it to obtain a bijective decomposition satisfied by the set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with D the list of degrees of the distinguished vertices was obtained in [11]. Since this bijective decomposition contains an error, we present the correct bijective decomposition here, and we derive it to make the derivation more accessible than the contents of a Ph. D. thesis.

Theorem 2.1. *The set $\mathcal{M}(g, e, f, n, D)$ of sequenced orientable maps of genus g with e edges and f faces, with the root vertex of degree n and with the list D of degrees of the distinguished vertices is defined by the bijective decomposition*

$$\begin{aligned}
&\mathcal{M}(g,e,f,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ f_1+f_2=f\\ n_1+n_2=n-2\\ D_1\subseteq D}} \mathcal{M}(g_1,e_1,f_1,n_1,D_1)\times\mathcal{M}(g_2,e_2,f_2,n_2,D-D_1)\\
&\quad + \sum_{p=1}^{n-3} \mathcal{M}(g-1,e-1,f,n-2-p,p.D)\times\{1,\dots,p\} \\
&\quad + \sum_{p=n-1}^{p=2e-2} \mathcal{M}(g,e-1,f,p,D)\\
&\quad + \sum_{j=1}^{|D|} \mathcal{M}(g,e-1,f,d_j+n-2,D-\{d_j\}) + \Delta_{(g,e,f,n,D)=(0,0,1,0,[\,])}.
\end{aligned} \tag{2.1}$$

Proof. If a map m has at least one edge, we reduce by 1 the number of edges by the face-vertex dual of deleting the root edge. There are two cases of this operation, depending upon whether the root edge is a loop or a link, and each of these cases breaks down into two sub-cases.

Case 1: The root edge is a loop. We delete the root edge and split the root vertex into two parts, s_1 and s_2. If r is the root, then s_1 consists of the darts $R(r)$, $R^2(r)$, ..., $R^{-1}(L(r))$

and s_2 consists of the darts $R(L(r))$, $R^2(L(r))$, …, $R^{-1}(r)$. This case breaks down into two cases, depending upon whether or not this operation disconnects the map.

Case 1a: This operation disconnects the map into two maps, m_1 containing s_1 and m_2 containing s_2. If m_1 has at least 1 edge, its root is $r_1 = R(r)$, and if m_2 has at least 1 edge, its root is $r_2 = R(L(r))$. Let g_1, e_1, f_1, n_1, D_1 and g_2, e_2, f_2, n_2, D_2 be the parameters of the maps m_1 and m_2, respectively, corresponding to g, e, f, n, D. This operation reduces by 1 the total number of edges; so $e_1 + e_2 = e - 1$. It leaves unchanged the total number of faces because r and $L(r)$ simply get deleted from the cycle(s) of RL (L followed by R) containing them; so $f_1 + f_2 = f$. It increases by 1 the total number of vertices; so from Formula (1.1), which relates the genus of a map to the number of its vertices, faces and edges, it can easily be deduced that $g_1 + g_2 = g$. It decreases by 2 the total number of darts in s_1 and s_2 since r and $L(r)$, which belonged to the root vertex, get eliminated; so $n_1 + n_2 = n - 2$. Finally, D_1 can be any sublist of D and D_2 is just the complementary sublist, denoted by $D - D_1$. This operation is uniquely reversible; so the set of ordered pairs of sequenced maps obtained in this case is

$$\sum_{\substack{g_1+g_2=g\\ e_1+e_2=e-1\\ f_1+f_2=f\\ n_1+n_2=n-2\\ D_1\subseteq D}} \mathcal{M}(g_1, e_1, f_1, n_1, D_1) \times \mathcal{M}(g_2, e_2, f_2, n_2, D - D_1), \tag{2.2}$$

where Σ means the union of disjoint sets.

Case 1b: This operation does not disconnect the map, but instead turns it into a new map m' with $e - 1$ edges and f faces and, since the number of vertices increases by 1, the genus of m' is $g - 1$, so that this case only occurs when $g \geq 1$. Neither s_1 nor s_2 can be of degree 0 (otherwise the map would be disconnected); so we can choose for m' the root $r_1 = R(r)$ belonging to s_1. Let p be the degree of s_2. Since the sum of the degrees of s_1 and s_2 is $n - 2$, the degree of s_1, the root vertex, is $n - 2 - p$. We distinguish the vertex s_2 so that this operation can be reversed, and we put its degree p at the beginning of the list D, turning it into $p.D$. Now this operation is reversible in p distinct ways, since any of the p darts of s_2 can be chosen to be $R(L(r))$ when we merge the vertices s_1 and s_2 and replace the deleted root edge. Now p can be any integer from 1 up to $n-3$ (so that $n-2-p \geq 1$). For both p and $n - 2 - p$ to be at least 1, n must be at least 4. The set of sequenced maps obtained in this case is

$$\sum_{p=1}^{n-3} \mathcal{M}(g - 1, e - 1, f, n - 2 - p, p.D) \times \{1, \ldots, p\}. \tag{2.3}$$

Case 2: The root edge is a link. We contract the root edge, merging its two incident vertices s_1 containing the root r and s_2 containing $L(r)$ into a single vertex s with root $R(r)$. This operation decreases by 1 the number of edges and doesn't change the number of faces, since r and $L(r)$ simply get deleted from the cycle(s) containing them. Since the number of vertices is decreased by 1, the genus remains the same. This case breaks down into two sub-cases, depending upon whether or not s_2 is one of the distinguished vertices.

Case 2a: The vertex s_2 is not one of the distinguished vertices. Let p be the degree of the new vertex s. Then $p = n - 2 +$ the degree of s_2, and since the degree of s_2 must be at least 1, we have $p \geq n-1$. Also, the new map has $2e-2$ darts; so $p \leq 2e-2$. This operation is uniquely reversible for each value of p; so the set of maps so obtained is

$$\sum_{p=n-1}^{p=2e-2} \mathcal{M}(g, e-1, f, p, D). \tag{2.4}$$

Case 2b: The vertex s_2 is one of the distinguished vertices. It can be any one of the $|D|$ distinguished vertices. If it is the jth distinguished vertex, then its degree is d_j. Then since it gets merged with s_1 into the new root vertex, d_j gets dropped from D. Finally, the degree of s is $d_j + n - 2$. This operation too is uniquely reversible; so the set of maps so obtained is

$$\sum_{j=1}^{|D|} \mathcal{M}(g, e-1, f, d_j + n - 2, D - \{d_j\}). \tag{2.5}$$

Finally, suppose that m has no edges. It is of genus 0, has 1 face, its one vertex is of degree 0 and its list D is empty because it has no distinguished vertices; so it constitutes the singleton

$$\Delta_{(g,e,f,n,D)=(0,0,1,0,[\,])}. \tag{2.6}$$

Then $\mathcal{M}(g, e, f, n, D)$ is the disjoint union of the sets given by (2.2)–(2.6). □

2.3 Bipartite maps and hypermaps

To motivate the transformation of (2.2)–(2.6) into the corresponding equations for sequenced hypermaps we briefly describe the bijection in [23] that takes a hypermap h into a 2-coloured bipartite map $m = I(h)$, its *incidence map*. The bijection I takes the darts, vertices and hyperedges of h into the edges, white vertices and black vertices of m. A root (distinguished dart) of h corresponds to a distinguished **edge** of m; to make it correspond to a root of m we impose the condition that a root of m belongs to a white vertex. The permutation R in h corresponds to R in m acting on a dart in a white vertex and the permutation L in h corresponds to R in m acting on a dart in a black vertex. The permutation L in m doesn't correspond to any permutation in h; rather, since it takes a dart belonging to a vertex of one colour into a dart belonging to a vertex of the opposite colour, it toggles R in m between R and L in h. A face (cycle of RL) in h corresponds to a face in m with twice the degree. To see this, we follow one application of RL in h starting with a dart d, which corresponds to an edge in m but we make it correspond to the dart d' in that edge that also belongs to a white vertex. Then the L in h takes d' first into $L(d')$, which belongs to a black vertex, and then into $RL(d')$ and the following R in h takes $RL(d')$ first into $LRL(d')$, which belongs to a white vertex, and then into $RLRL(d')$. Since the genus of a hypermap with t darts, v vertices, e hyperedges and f faces is defined by (1.2), m has the same genus as h.

Since the root of an incidence map of a rooted hypermap must belong to a white vertex, we impose the condition on a rooted 2-face-coloured face-bipartite map that the root belong to a white face and we transform (2.2)–(2.6) into the corresponding bijective decomposition for these maps.

3 Sequenced face-bipartite maps

Let $\mathcal{F}(g, e, w, b, n, D)$ be the set of sequenced orientable face-bipartite maps of genus g with e edges, w white faces, b black faces, with the root face of degree $2n$ and with the list of half-degrees of the distinguished vertices equal to D. For any dart d we denote by $f(d)$ the face containing d and we note that the face $f(R(d)) = f(L(d))$ must have the opposite colour from $f(d)$ because those two faces share the edge $\{d, L(d)\}$.

Theorem 3.1. *The set $\mathcal{F}(g, e, w, b, n, D)$ satisfies the bijective decomposition*

$$\begin{aligned}
&\mathcal{F}(g, e, w, b, n, D) = \\
&\quad \sum_{\substack{g_1+g_2=g \\ e_1+e_2=e-1 \\ w_1+b_2=b \\ w_2+b_1=w \\ n_1+n_2=n-1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1) \\
&\quad + \sum_{p=1}^{n-2} \mathcal{F}(g-1, e-1, b, w, n-1-p, p.D) \times \{1, \dots, p\} \qquad (3.1) \\
&\quad + \sum_{p=n}^{p=e-1} \mathcal{F}(g, e-1, b, w, p, D) \\
&\quad + \sum_{j=1}^{|D|} \mathcal{F}(g, e-1, b, w, d_j + n - 1, D - \{d_j\}) + \Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[\,])}.
\end{aligned}$$

Proof. **Case 1: The root edge is a loop.** By definition, $f(r)$, where r is the root of the map m, is white, so that since $r_1 = R(r)$, $f(r_1)$ must be black. But when the loop is removed and the vertex s containing r is split, r_1 becomes a root; so $f(r_1)$ must change colour and so must all the faces of the new map m' (in case 1b) or the map m_1 containing r_1 (in case 1a). In case 1a, the other map m_2 has $r_2 = RL(r)$ as a root and $f(r_2)$ is white; so its faces stay the same colour. This implies that in case 1a $w_1 + b_2 = b$ and $w_2 + b_1 = w$, whereas in case 1b w and b switch in going from m to m'.

In case 1a, we have, as for general maps, $g_1 + g_2 = g$, $e_1 + e_2 = e - 1$ and D_1 is any subset of D, but instead of $n_1 + n_2 = n - 2$ we have $n_1 + n_2 = n - 1$ because the degrees satisfy the equation $2n_1 + 2n_2 = 2n - 2$. The analogue of formula (2.2) is thus

$$\sum_{\substack{g_1+g_2=g \\ e_1+e_2=e-1 \\ w_1+b_2=b \\ w_2+b_1=w \\ n_1+n_2=n-1 \\ D_1 \subseteq D}} \mathcal{F}(g_1, e_1, w_1, b_1, n_1, D_1) \times \mathcal{F}(g_2, e_2, w_2, b_2, n_2, D - D_1). \qquad (3.2)$$

In case 1b, the reduced map m' is still of genus $g-1$ and has $e-1$ edges, but the degree of s_2 is now $2p$ instead of p and the degree of the new root vertex s_1 is $2(n-1-p)$; so the parameter $n-2-p$ in (2.3) changes to $n-1-p$. Also, $1 \leq 2p \leq 2n-3$, but since $2p$ is even, we have $1 \leq p \leq n-2$ instead of $1 \leq p \leq n-3$, and the condition that $n \geq 4$

changes to $n \geq 3$. The analogue of formula (2.3) is thus

$$\sum_{p=1}^{n-2} \mathcal{F}(g-1, e-1, b, w, n-1-p, p.D) \times \{1, \ldots, p\}. \tag{3.3}$$

Case 2: The root edge is a link. Since the new root $R(r)$ belongs to a black face, all the faces change colour; so b and w switch.

In case 2a, we have $2n - 1 \leq 2p \leq 2e - 2$, but since $2p$ is even, we now have $n \leq p \leq e - 1$; so the analogue of (2.4) is

$$\sum_{p=n}^{p=e-1} \mathcal{F}(g, e-1, b, w, p, D). \tag{3.4}$$

In case 2b, the degree of the new root vertex is $2d_j + 2n - 2$; so the analogue of (2.5) is

$$\sum_{j=1}^{|D|} \mathcal{F}(g, e-1, b, w, d_j + n - 1, D - \{d_j\}). \tag{3.5}$$

Finally, the map with no edges has one white face and no black ones; so the analogue of (2.6) is

$$\Delta_{(g,e,w,b,n,D)=(0,0,1,0,0,[\,])}. \tag{3.6}$$

□

After deriving this bijective decomposition, we became aware of the article [8], which presents a similar bijective decomposition but for multi-rooted face-bipartite maps, which are like sequenced face-bipartite maps except that every distinguished vertex has a root. However, we present our derivation here for several reasons: it makes our article self-contained, we obtained it independently of [8] and our main purpose is to count hypermaps rather than face-bipartite maps. Now [8] does present a construction that converts a hypermap into a face-bipartite map. However, that construction is not proved and it is far more complicated than the one in [23], which is not cited in [8]. We also recently became aware of the article [6], which generalizes the results of [15] by computing the generating functions for edge-labelled bipartite maps on an orientable surface of genus g with an unbounded number of faces and including the degrees of these faces as parameters.

4 Sequenced rooted hypermaps

Theorem 3.1 holds for rooted 2-coloured bipartite maps with distinguished faces, where e is the number of edges, w is the number of white vertices, b is the number of black vertices, n is half the degree of the root face and D is the list of half-degrees of the distinguished faces. By the bijection described in Section 2.3, it also holds for rooted hypermaps with distinguished faces, where e is the number of darts, w is the number of vertices, b is the number of hyperedges, n is the degree of the root face and D is the list of degrees of the distinguished faces. By duality, the theorem also holds for sequenced hypermaps, where e is the number of darts, w is the number of faces, b is the number of hyperedges, n is the degree of the root vertex and D is the list of degrees of the distinguished vertices. To make the letters correspond to the objects they represent, we change $\mathcal{F}$ to $\mathcal{H}$, e to t, w to f and b to e. We thus obtain the following results.

Theorem 4.1 (Bijective decomposition for sequenced hypermaps)**.** *Let $\mathcal{H}(g,t,f,e,n,D)$ be the set of sequenced orientable hypermaps of genus g with t darts, f faces and e hyperedges, with the root vertex of degree n and with the list of degrees of the distinguished vertices equal to $D=(d_1,d_2,\ldots,d_{|D|})$, where d_i is the degree of the distinguished vertex with label i. The set $\mathcal{H}(g,t,f,e,n,D)$ satisfies the bijective decomposition*

$$\begin{aligned}
&\mathcal{H}(g,t,f,e,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ t_1+t_2=t-1\\ f_1+e_2=e\\ f_2+e_1=f\\ n_1+n_2=n-1\\ D_1\subseteq D}} \mathcal{H}(g_1,t_1,f_1,e_1,n_1,D_1)\times\mathcal{H}(g_2,t_2,f_2,e_2,n_2,D-D_1) \\
&\quad + \sum_{p=1}^{n-2} \mathcal{H}(g-1,t-1,e,f,n-1-p,p.D)\times\{1,\ldots,p\} \\
&\quad + \sum_{p=n}^{p=t-1} \mathcal{H}(g,t-1,e,f,p,D) \\
&\quad + \sum_{j=1}^{|D|} \mathcal{H}(g,t-1,e,f,d_j+n-1,D-\{d_j\}) + \Delta_{(g,t,f,e,n,D)=(0,0,1,0,0,[\,])}.
\end{aligned} \tag{4.1}$$

Corollary 4.2 (Recurrence between numbers of sequenced hypermaps)**.** *Let $H(g,t,f,e,n,D)$ be the number of rooted sequenced hypermaps of genus g with t darts, f faces and e hyperedges such that the root vertex is of degree n and D is the list of degrees of the distinguished vertices. Then $H(0,0,1,0,0,[\,])=1$ and if $t\geq 1$, then*

$$\begin{aligned}
&H(g,t,f,e,n,D) = \\
&\quad \sum_{\substack{g_1+g_2=g\\ t_1+t_2=t-1\\ f_1+e_2=e\\ f_2+e_1=f\\ n_1+n_2=n-1\\ D_1\subseteq D}} H(g_1,t_1,f_1,e_1,n_1,D_1)\,H(g_2,t_2,f_2,e_2,n_2,D-D_1) \\
&\quad + \delta_{n\geq 3}\delta_{g\geq 1}\sum_{p=1}^{n-2} p\,H(g-1,t-1,e,f,n-1-p,p.D) \\
&\quad + \sum_{p=n}^{p=t-1} H(g,t-1,e,f,p,D) \\
&\quad + \sum_{j=1}^{|D|} H(g,t-1,e,f,d_j+n-1,D-\{d_j\}).
\end{aligned} \tag{4.2}$$

5 Multirooted hypermaps

For $\rho\geq 1$ a ρ-rooted hypermap is a hypermap in which a sequence of ρ darts with pairwise distinct initial vertices is distinguished. A multirooted hypermap is a ρ-rooted hypermap for some $\rho \geq 1$. This section addresses the enumeration of multirooted hypermaps.

Theorem 5.1 (Recurrence between numbers of multirooted hypermaps)**.** *Let $H_m(g,t,f,e,D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that D is the list of degrees of the distinguished vertices. Then $H_m(0,0,1,0,[]) = 1$ and if $t \geq 1$, then*

$$\begin{aligned} &H_m(g,t,f,e,n.D) = \\ &\sum_{\substack{g_1+g_2=g\\ t_1+t_2=t-1\\ f_1+e_2=e\\ f_2+e_1=f\\ n_1+n_2=n-1\\ D_1\subseteq D}} H_m(g_1,t_1,f_1,e_1,n_1.D_1)\,H_m(g_2,t_2,f_2,e_2,n_2.(D-D_1)) \\ &\quad + \delta_{n\geq 3}\delta_{g\geq 1}\sum_{p=1}^{n-2} H_m(g-1,t-1,e,f,(n-1-p).p.D) \\ &\quad + \sum_{p=n}^{p=t-1} H_m(g,t-1,e,f,p.D) \\ &\quad + \sum_{j=1}^{|D|} d_j\, H_m(g,t-1,e,f,(d_j+n-1).(D-\{d_j\})). \end{aligned} \tag{5.1}$$

Proof. A multirooted hypermap is similar to a sequenced rooted hypermap except that for each distinguished non-root vertex a dart starting from it is distinguished. If the degree of the jth distinguished vertex is d_j, then there are d_j ways of distinguishing a dart of this vertex. It follows that for each sequenced rooted hypermap, there are $\Pi_{j=1}^{|D|} d_j$ multirooted hypermaps. Let $H_m(g,t,f,e,D)$ be the number of multirooted hypermaps of genus g with t darts, f faces and e hyperedges such that such that D is the list of degrees of the initial vertex of the distinguished darts. Then

$$H_m(g,t,f,e,n.D) = H(g,t,f,e,n,D)\,\Pi_{j=1}^{|D|} d_j. \tag{5.2}$$

Solving (5.2) for $H(g,t,f,e,n,D)$ and substituting into (4.2) proves the theorem. □

For $\rho \geq 1$ let

$$H_g(v_1,\ldots,v_\rho,x,y,u,z) = \sum_{\substack{t\geq 0,\, f\geq 1,\, e\geq 0\\ d_1\geq 1,\ldots,d_\rho\geq 1\\ v=t+2(1-g)-e-f}} H_m(g,t,f,e,D_{1..\rho})v_{1..\rho}^{D_{1..\rho}} x^f y^e u^v z^t \tag{5.3}$$

be the generating function that counts multirooted hypermaps of genus g with ρ distinguished darts if $g \geq 0$, and 0 otherwise. For $1 \leq i \leq \rho$, the exponent d_i of the variable v_i in this series is the degree of the initial vertex of the i-th distinguished dart. The exponent f of the variable x is the number of faces, the exponent e of the variable y is the number of hyperedges, the exponent t of the variable z is the number of darts and the exponent v of the variable u is the number of vertices (v is computable from the other parameters by Formula (1.2)).

Corollary 5.2 (Functional equations for multirooted hypermaps)**.** *For $g \geq 0$ and $\rho \geq 1$ the generating functions H_g of multirooted hypermaps of genus g are defined by the following functional equations:*

$$\begin{aligned}
H_g(v_1, W, x, y, u, z) = \\
&\frac{yv_1z}{xu}\sum_{j=0}^{g}\sum_{X\subseteq W} H_j(v_1, X, y, x, u, z)H_{g-j}(v_1, W-X, x, y, u, z) \\
&+ \frac{v_1z}{u}H_{g-1}(v_1, v_1, W, y, x, u, z) \\
&+ \frac{v_1uz}{v_1-1}\left(H_g(v_1, W, y, x, u, z) - H_g(1, W, y, x, u, z)\right) \\
&+ v_1uz\sum_{j=2}^{j=\rho} v_j\frac{\partial}{\partial v_j}\left(v_j\frac{H_g(v_j, W-\{v_j\}, y, x, u, z) - H_g(v_1, W-\{v_j\}, y, x, u, z)}{v_j - v_1}\right) \\
&+ xu\delta_{g=0}\delta_{\rho=1},
\end{aligned} \tag{5.4}$$

where $W = v_2, \ldots, v_\rho$.

Proof. By summation according to (5.3) of the recurrence between numbers of multirooted hypermaps from Theorem 5.1. □

By vertex-hyperedge duality, we have

$$H_g(v_1, W, y, x, u, z) = H_g(v_1, W, x, y, u, z) + \delta_{g=0}\delta_{\rho=1}(yu - xu) \tag{5.5}$$

and thus another functional equation without x, y swaps is:

$$\begin{aligned}
H_g(v_1, W, x, y, u, z) = \\
&\frac{yv_1z}{xu}\sum_{j=0}^{g}\sum_{X\subseteq W}\Big(\big(H_j(v_1, X, x, y, u, z) + \delta_{j=0}\delta_{|X|=0}(yu - xu)\big) \\
&\qquad\qquad H_{g-j}(v_1, W-X, x, y, u, z)\Big) \\
&+ \frac{v_1z}{u}H_{g-1}(v_1, v_1, W, x, y, u, z) \\
&+ \frac{v_1uz}{v_1-1}\left(H_g(v_1, W, x, y, u, z) - H_g(1, W, x, y, u, z)\right) \\
&+ v_1uz\sum_{j=2}^{j=\rho} v_j\frac{\partial}{\partial v_j}\left(v_j\frac{H_g(v_j, W-\{v_j\}, x, y, u, z) - H_g(v_1, W-\{v_j\}, x, y, u, z)}{v_j - v_1}\right) \\
&+ xu\delta_{g=0}\delta_{\rho=1}.
\end{aligned} \tag{5.6}$$

The former equation is given here for maximal generality. However, a consequence of the genus formula (1.2) is that three variables among the four variables x, y, u and z are sufficient. In the remainder of the paper we consider the generating functions

$$H_g(v_1, W, x, y, u) = H_g(v_1, W, x, y, u, 1)$$

with one fewer variable. They are defined by the following functional equations:

$$\begin{aligned}
&H_g(v_1, W, x, y, u) = \\
&\quad \frac{yv_1}{xu}\sum_{j=0}^{g}\sum_{X\subseteq W}\left(H_j(v_1, X, x, y, u) + \delta_{j,0}\delta_{|X|,0}(yu - xu)\right) H_{g-j}(v_1, W - X, x, y, u) \\
&\quad + \frac{v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) \qquad (5.7)\\
&\quad + \frac{v_1 u}{v_1 - 1}\left(H_g(v_1, W, x, y, u) - H_g(1, W, x, y, u)\right) \\
&\quad + v_1 u \sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j}\left(v_j \frac{H_g(v_j, W - \{v_j\}, x, y, u) - H_g(v_1, W - \{v_j\}, x, y, u)}{v_j - v_1}\right) \\
&\quad + xu\delta_{g=0}\delta_{\rho=1}.
\end{aligned}$$

For $g, \rho \neq 0, 1$, after grouping in the left-hand side the terms containing $H_g(v_1, W, x, y, u)$ in (5.7), one gets

$$\begin{aligned}
&\frac{A(v_1, x, y, u)}{v_1} H_g(v_1, W, x, y, u) = \\
&\quad x(1 - v_1)\sum_{j=0}^{g}\sum_{\substack{X\subseteq W\\(j,X)\neq(0,[])\\(j,X)\neq(g,W)}} H_j(v_1, X, x, y, u)H_{g-j}(v_1, W - X, x, y, u) \\
&\quad + \frac{1 - v_1}{u} H_{g-1}(v_1, v_1, W, x, y, u) + uH_g(1, W, x, y, u) \\
&\quad + uT_g(v_1, W, x, y, u) \qquad (5.8)
\end{aligned}$$

with

$$A(v, x, y, u) = vu + (1 - v)(1 - yv + xv - 2vH_0(v, x, y, u)/u) \qquad (5.9)$$

and

$$\begin{aligned}
&T_g(v_1, W, x, y, u) = \\
&\quad (1 - v_1)\sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j}\left(\frac{v_j}{v_j - v_1}\Big(H_g(v_j, W - \{v_j\}, x, y, u)\right. \\
&\qquad\qquad \left. - H_g(v_1, W - \{v_j\}, x, y, u)\Big)\right). \qquad (5.10)
\end{aligned}$$

6 Rooted hypermap generating functions

Let $h_g(v, e, f)$ be the number of rooted genus-g hypermaps with v vertices, e hyperedges and f faces. Let

$$H_g(x, y, u) = \sum_{v,e,f\geq 1} h_g(v, e, f)x^v y^e u^f \qquad (6.1)$$

be the ordinary generating function for counting rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable x is the number of vertices, the exponent of variable y is the number of hyperedges, and the exponent of variable u is the number of faces.

Rooted hypermaps being 1-rooted hypermaps,

$$H_g(x, y, u) = H_g(1, x, y, u), \tag{6.2}$$

where $H_g(v_1, \ldots, v_\rho, x, y, u)$ is the generating function counting ρ-rooted genus-g hypermaps defined in Section 5 for $\rho \geq 1$.

We first recall in Section 6.1 a known parametric expression of the generating function that counts rooted planar hypermaps. Then we explain in Section 6.2 how to solve the functional equation of the generating functions $H_g(x, y, u)$ that count rooted hypermaps with a given positive genus g.

6.1 Rooted planar hypermaps

The following proposition is a reformulation of [1, Theorem 3], with the correspondence $s = x$, $f = u$ and $a = y$ for variables, $\lambda = p$, $\mu = q$ and $\nu = r$ for parameters, and $H_0 = sf(1 + J)$ for generating functions.

Proposition 6.1 ([1]). *The ordinary generating function $H_0(x, y, u)$ that counts rooted planar hypermaps by number of vertices (exponent of x), hyperedges (exponent of y) and faces (exponent of u) is the unique solution of the following parametric system:*

$$H_0(x, y, u) = 1 + pqr(1 - p - q - r) \tag{6.3}$$

with

$$\begin{cases} x = p(1 - q - r) \\ u = q(1 - p - r) \\ y = r(1 - p - q). \end{cases} \tag{6.4}$$

Proof. The generating function $H_0(v, x, y, u)$ that counts rooted planar hypermaps (genus 0) by number of vertices (exponent of x), hyperedges (exponent of y), faces (exponent of u) and degree of the root vertex (exponent of v) satisfies the functional equation

$$\begin{aligned} H_0(v, x, y, u) \quad &= \quad \frac{yv}{xu}\left(H_0(v, x, y, u) + yu - xu\right) H_0(v, x, y, u) \\ &\quad + \frac{vu}{v-1}\left(H_0(v, x, y, u) - H_0(1, x, y, u)\right) + xu \end{aligned} \tag{6.5}$$

obtained by instantiation of (5.7) with $g = 0$, $\rho = 1$ and $v_1 = v$.

This equation can be solved by the *quadratic method* [10, page 515]. The idea is to define auxiliary functions $A(v, x, y, u)$ and $B(v, x, y, u)$ by (5.9) and

$$B(v, x, y, u) \quad = \quad A(v, x, y, u)^2 \tag{6.6}$$

and look for a function $V(x, y, u)$ such that

$$A(V(x, y, u), x, y, u) = 0, \tag{6.7}$$

implying that $B(V(x, y, u), x, y, u) = 0$ and $\partial_v B(v, x, y, u)_{|v=V(x,y,u)} = 0$.

We get from (6.5), (5.9) and (6.6) that

$$\begin{aligned} &B(v, x, y, u) = \\ &\quad 1 - 2yv - 2xv - 2v^3y - 2v^3x - 2v^2u + v^4y^2 - 2v^3y^2 + y^2v^2 + v^4x^2 \\ &\quad - 2v^3x^2 + x^2v^2 + v^2u^2 + 4v^3yx - 2yv^2x - 2yv^2u + 2v^3yu - 2v^4yx \\ &\quad - 2v^3xu + 2xv^2u + 4v^2x + 4v^2y + 2vu + 4v^3H_0(1, x, y, u) \\ &\quad - 4v^2H_0(1, x, y, u) - 2v + v^2. \end{aligned} \tag{6.8}$$

The constraints $B(V(x,y,u),x,y,u) = 0$ and $\partial_v B(v,x,y,u)_{|v=V(x,y,u)} = 0$ respectively are

$$\begin{aligned}1 - 2yV - 2xV - 2V^3y - 2V^3x - 2V^2u + V^4y^2 - 2V^3y^2 + y^2V^2\\ + V^4x^2 - 2V^3x^2 + x^2V^2 + V^2u^2 + 4V^3yx - 2yV^2x - 2yV^2u\\ + 2V^3yux - 2V^4y - 2V^3xu + 2xV^2u + 4V^2x + 4V^2y + 2Vu\\ + 4V^3H_0(1,x,y,u) - 4V^2H_0(1,x,y,u) - 2V + V^2 &= 0 \qquad (6.9)\end{aligned}$$

and

$$\begin{aligned}-2 + 8yV + 8xV + 4V^3y^2 - 6y^2V^2 + 4V^3x^2 - 6x^2V^2 - 6V^2x\\ - 6V^2y - 4Vu + 2y^2V + 2x^2V + 2Vu^2 - 4yVu + 4xVu - 4yVx\\ + 12yV^2x + 6yV^2u - 8V^3yx - 6xV^2u + 2V - 2x - 2y + 2u &= 0. \qquad (6.10)\end{aligned}$$

It can be checked that both equations are satisfied by

$$V = 1/(1-q) \qquad (6.11)$$

with x, u, y and $H_0(1,x,y,u)$ related to p, q and r by (6.4) and (6.3). □

6.2 Rooted hypermaps with positive genus

The following additional notations are used in this section. Let ρ be a positive integer. Let $H_j[n_1,\dots,n_\rho]$ denote the partial derivative of the function $H_j(v_1,\dots,v_\rho,x,y,u)$ with respect to the variables $v_1,\dots,v_\rho$ to the respective orders $n_1,\dots,n_\rho$, computed at $v_1 = \dots = v_\rho = V$. The abbreviation $[\rho]$ denotes the list $[2,\dots,\rho]$ if $\rho \geq 2$ and the empty list $[\,]$ if $\rho = 1$. The abbreviation $N_{[\rho]}$ denotes the list $[n_2,\dots,n_\rho]$. For any sublist $X \subseteq [\rho]$ of $[\rho]$, $[\rho] - X$ denotes the sublist of the elements of $[\rho]$ that are not in X, N_X denotes the list of those n_i in $N_{[\rho]}$ such that i is in X and N_j denotes the list $[n_2,\dots,n_{j-1},n_{j+1},\dots,n_\rho]$.

6.2.1 Equation for rooted hypermaps and recurrence relations

The special case of Formula (5.8) for $g \geq 1$, $\rho = 1$ and $v_1 = V$ is the following formula:

$$uH_g(1,x,y,u) = (V-1)\left(x\sum_{j=1}^{g-1} H_j(V,x,y,u)H_{g-j}(V,x,y,u) + H_{g-1}(V,V,x,y,u)/u\right)$$

i.e.

$$uH_g(1,x,y,u) = (V-1)\left(x\sum_{j=1}^{g-1} H_j[0]H_{g-j}[0] + H_{g-1}[0,0]/u\right). \qquad (6.12)$$

In order to derive from (6.12) a value for $H_g(1,x,y,u)$, we are looking for a value for $H_j[0]$, $H_{g-j}[0]$ and $H_{g-1}[0,0]$. More generally, we will derive from the following proposition a closed form for the expressions $H_g[n_1,\dots,n_\rho]$.

Proposition 6.2. *For $g \geq 0$, $\rho \geq 1$ and $n_1, \dots, n_\rho \geq 0$ the function $H_g[n_1, \dots, n_\rho]$ is defined by*

$$
\begin{aligned}
&\frac{(n_1+1)A[1]}{V} H_g[n_1, N_{[\rho]}] = \\
&\quad \sum_{\substack{i+j+k=n_1+1\\ i>0,\, k<n_1}} \binom{n_1+1}{i,j} \frac{(-1)^{j+1} j!}{V^{j+1}} A[i] H_g[k, N_{[\rho]}] \\
&\quad + x \sum_{\substack{k+l+m=n_1+1\\ 0\leq j\leq g\\ X\subseteq[\rho]\\ (j,X)\neq(0,[\,])\\ (j,X)\neq(g,[\rho])}} \binom{n_1+1}{k,l} M[m] H_j[k, N_X] H_{g-j}[l, N_{[\rho]-X}] \\
&\quad + \frac{1}{u} \sum_{i+j+k=n_1+1} \binom{n_1+1}{i,j} M[k] H_{g-1}[i, j, N_{[\rho]}] \qquad (6.13)\\
&\quad + u \sum_{j=2}^{\rho} \frac{(n_1+1)! n_j!}{(n_1+n_j+2)!} \Big(n_j F_g[n_1+n_j+2, N_j] \\
&\qquad\qquad + \tfrac{V(n_j+1)}{n_1+n_j+3} F_g[n_1+n_j+3, N_j] \Big),
\end{aligned}
$$

where

$$F_g(v_1, \dots, v_h, x, y, u) = L(v_1) H_g(v_1, \dots, v_h, x, y, u) \qquad (6.14)$$

for $h \geq 1$, $M(v) = 1 - v$ and $L(v) = v(1-v)$.

Proof. Equation (6.13) is obtained from Equation (5.8) as follows:

1. Partial derivation of (5.8) with respect to the variables $v_1, v_2, \dots, v_\rho$ to the respective orders $n_1+1, n_2, \dots, n_\rho$.
2. Evaluation of this differential equation at $v_1 = \dots = v_\rho = V$. The function $H_g[n_1+1, \dots, n_\rho]$ is multiplied by $A[0]$ in the resulting equation, and $A[0]$ is known to be zero (6.7). The functions $T_g[\dots]$ are replaced by expressions with the functions $F_g[\dots]$ thanks to Lemma 6.3 below.
3. In the left-hand side of the resulting equation, isolation of the single term involving the function $H_g[n_1, \dots, n_\rho]$.

By inspection one can check that the right-hand side of (6.13) depends only on some functions $H_g[k, n_2, \dots, n_\rho]$ with $k < n_1$, some functions $H_g[n'_1, \dots, n'_{\rho'}]$ with $\rho' < \rho$ and some functions $H_j[\dots]$ for $j < g$. Thus, (6.13) in a recursive definition of the family of functions $H_g[n_1, \dots, n_\rho]$ for $g \geq 0$, $\rho \geq 1$ and $n_1, \dots, n_\rho \geq 0$. □

The following lemma relates the partial derivatives of T_g at $v = V$ with the ones of F_g.

Lemma 6.3. *For $\rho \geq 2$ and $g, n_1, \dots, n_\rho \geq 0$,*

$$
\begin{aligned}
&T_g[n_1+1, N_{[\rho]}] = \\
&\quad \sum_{j=2}^{j=\rho} \frac{(n_1+1)! n_j!}{(n_1+n_j+2)!} \Big(n_j F_g[n_1+n_j+2, N_j] \\
&\qquad\qquad + \tfrac{V(n_j+1)}{n_1+n_j+3} F_g[n_1+n_j+3, N_j] \Big). \qquad (6.15)
\end{aligned}
$$

Proof. We can easily prove that

$$\frac{\partial}{\partial v_j}\left[\frac{(v_j - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1}\right] = 0. \tag{6.16}$$

Then, $T_g(v_1, \dots, v_\rho, x, y, u)$ equals

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial}{\partial v_j}\Bigg((v_j - v_1)^{-1}\Big(v_j(1 - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u) \\ -\ v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u)\Big)\Bigg). \tag{6.17}$$

It also holds that

$$\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}}\left[\frac{v_j(v_j - v_1)H_g(v_j, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1}\right] \quad = \quad 0, \tag{6.18}$$

so that $\frac{\partial^{n_1+1}}{\partial v_1^{n_1+1}} T_g(v_1, \dots, v_\rho, x, y, u)$ equals

$$\sum_{j=2}^{j=r} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1}\partial v_j}\Bigg((v_j - v_1)^{-1}\Big(v_j(1 - v_j)H_g(v_j, [\rho] - \{v_j\}, x, y, u) \\ - v_1(1 - v_1)H_g(v_1, [\rho] - \{v_j\}, x, y, u)\Big)\Bigg) \tag{6.19}$$

i.e.

$$\sum_{j=2}^{j=\rho} v_j \frac{\partial^{n_1+2}}{\partial v_1^{n_1+1}\partial v_j}\left(\frac{F_g(v_j, [\rho] - \{v_j\}, x, y, u) - F_g(v_1, [\rho] - \{v_j\}, x, y, u)}{v_j - v_1}\right). \tag{6.20}$$

Formula (6.15) is a consequence of

$$\frac{\partial^{n_1+n_2}}{\partial x_1^{n_1}\partial x_2^{n_2}}\left(\frac{\psi(x_1) - \psi(x_2)}{x_1 - x_2}\right)_{x_1=x_2=a} = \frac{n_1!n_2!}{(n_1 + n_2 + 1)!}\psi^{(n_1+n_2+1)}(a). \tag{6.21}$$

□

The formula

$$F_g[n, N] = \sum_{k+l=n}\binom{n}{k} L[k]H_g[l, N] \tag{6.22}$$

is an easy consequence of (6.14). Thus the right-hand side of (6.13) only depends on some functions $H_g[k, \dots, n_\rho]$ with $k < n_1$, some functions $H_g[n'_1, \dots, n'_{\rho'}]$ with $\rho' < \rho$, some functions $H_j[\dots]$ for $j < g$ and some functions $A[i]$. A relation between $A[i]$ and some functions $H_0[j]$ is established in Section 6.2.2.

6.2.2 Case $g = 0$ and $\rho = 1$

The function $A[i]$ can be related to some functions $H_0[j]$ as follows: With $M(v) = 1 - v$ and $L(v) = v(1 - v)$, Equation (5.9) is

$$A(v, x, y, u) = vu + M(v) + L(v)(-y + x - 2xH_0(v, x, y, u)). \tag{6.23}$$

Its instantiation at $v = V$ gives

$$H_0[0] = \frac{1-q}{1-q-r}. \tag{6.24}$$

For $k \geq 1$, the k-th partial derivative of (6.23) in v is

$$\begin{aligned}\frac{\partial^k}{\partial v^k} A(v,x,y,u) &= \frac{\partial^k}{\partial v^k}(vu) + \frac{\partial^k}{\partial v^k} M(v) \\ &\quad + \frac{\partial^k}{\partial v^k}\left[L(v)(-y+x-2xH_0(v,x,y,u))\right]\end{aligned} \tag{6.25}$$

and its instantiation in $v = V$ is

$$\begin{aligned}A[k] &= \frac{\partial^k}{\partial v^k}(vu)_{|v=V} + M[k] \\ &\quad + \sum_{i+j=k} \binom{k}{i} L[i] \left(\frac{\partial^j}{\partial v^j}(-y+x-2xH_0(v,x,y,u))_{|v=V}\right).\end{aligned} \tag{6.26}$$

Solving (6.26) for $k = 1$ gives

$$H_0[1] = \frac{(1-q)^2(A[1]+1-p-q-r)}{2pq(1-q-r)}. \tag{6.27}$$

For $k \geq 2$, one gets

$$A[k] = -2x \sum_{i+j=k} \binom{k}{i} L[i] H_0[j]$$

since $M[k] = 0$, i.e.

$$A[k] = -2x\left(L[0]H_0[k] + kL[1]H_0[k-1] + \frac{k(k-1)}{2}L[2]H_0[k-2]\right) \tag{6.28}$$

since $L[k] = 0$ if $k \geq 3$.

7 Explicit formulas for small genera

This section proposes explicit parametric expressions for the generating functions that count rooted hypermaps of small positive genus. In Section 7.1 we count by number of vertices, hyperedges and faces; the number of darts can be obtained from these parameters by Formula (1.2). In Section 7.2 we count by number of darts alone.

7.1 Rooted hypermap series enumerated with three parameters

For $g = 1, \ldots, 5$ we have computed an explicit expression of $H_g(x,y,u)$ parameterized by p, q and r, with $x = p(1-q-r)$, $u = q(1-p-r)$ and $y = r(1-p-q)$, by application of formulas in Section 6. For $g \geq 3$, the expressions are too large to be included in the present text, but a Maple file with all the generating functions up to genus 5 is available from the first author on request.

A parametric expression of $H_1(x, y, u)$ is

$$H_1(x, y, u) = \frac{p\, q\, r\, (1-p)\, (1-q)\, (1-r)}{\left[(1-p-q-r)^2 - 4pqr\right]^2}. \tag{7.1}$$

This expression can be derived from [2, Theorem 3], with the correspondence $s = x$, $f = u$, and $a = y$ between variables and the correspondence $H_1(x, y, u) = xuK_1(1, x, y, u)$ between generating functions.

A parametric expression of $H_2(x, y, u)$ is

$$H_2(x, y, u) = \frac{p\, q\, r\, (1-p)\, (1-q)\, (1-r)\, P_2(p, q, r)}{\left[(1-p-q-r)^2 - 4pqr\right]^7} \tag{7.2}$$

where

$$\begin{aligned}
P_2(p, q, r) = {} & 76p^6q^2r^2 - 8p^4q^4r^2 - 8p^4q^2r^4 + 76p^2q^6r^2 - 8p^2q^4r^4 + 76p^2q^2r^6 \\
& + 40p^7qr - 76p^6q^2r - 76p^6qr^2 - 112p^5q^3r - 228p^5q^2r^2 - 112p^5qr^3 \\
& + 8p^4q^4r + 16p^4q^3r^2 + 16p^4q^2r^3 + 8p^4qr^4 - 112p^3q^5r + 16p^3q^4r^2 \\
& + 40p^3q^3r^3 + 16p^3q^2r^4 - 112p^3qr^5 - 76p^2q^6r - 228p^2q^5r^2 \\
& + 16p^2q^4r^3 + 16p^2q^3r^4 - 228p^2q^2r^5 - 76p^2qr^6 + 40pq^7r - 76pq^6r^2 \\
& - 112pq^5r^3 + 8pq^4r^4 - 112pq^3r^5 - 76pq^2r^6 + 40pqr^7 + p^8 - 20p^7q \\
& - 20p^7r - 35p^6q^2 - 64p^6qr - 35p^6r^2 + 56p^5q^3 + 396p^5q^2r + 396p^5qr^2 \\
& + 56p^5r^3 + 140p^4q^4 + 264p^4q^3r + 393p^4q^2r^2 + 264p^4qr^3 + 140p^4r^4 \\
& + 56p^3q^5 + 264p^3q^4r - 92p^3q^3r^2 - 92p^3q^2r^3 + 264p^3qr^4 + 56p^3r^5 \\
& - 35p^2q^6 + 396p^2q^5r + 393p^2q^4r^2 - 92p^2q^3r^3 + 393p^2q^2r^4 + 396p^2qr^5 \\
& - 35p^2r^6 - 20pq^7 - 64pq^6r + 396pq^5r^2 + 264pq^4r^3 + 264pq^3r^4 \\
& + 396pq^2r^5 - 64pqr^6 - 20pr^7 + q^8 - 20q^7r - 35q^6r^2 + 56q^5r^3 \\
& + 140q^4r^4 + 56q^3r^5 - 35q^2r^6 - 20qr^7 + r^8 + 6p^7 + 105p^6q + 105p^6r \\
& + 21p^5q^2 - 116p^5qr + 21p^5r^2 - 420p^4q^3 - 821p^4q^2r - 821p^4qr^2 \\
& - 420p^4r^3 - 420p^3q^4 - 648p^3q^3r - 316p^3q^2r^2 - 648p^3qr^3 - 420p^3r^4 \\
& + 21p^2q^5 - 821p^2q^4r - 316p^2q^3r^2 - 316p^2q^2r^3 - 821p^2qr^4 + 21p^2r^5 \\
& + 105pq^6 - 116pq^5r - 821pq^4r^2 - 648pq^3r^3 - 821pq^2r^4 - 116pqr^5 \\
& + 105pr^6 + 6q^7 + 105q^6r + 21q^5r^2 - 420q^4r^3 - 420q^3r^4 + 21q^2r^5 \\
& + 105qr^6 + 6r^7 - 49p^6 - 189p^5q - 189p^5r + 315p^4q^2 + 479p^4qr \\
& + 315p^4r^2 + 910p^3q^3 + 1162p^3q^2r + 1162p^3qr^2 + 910p^3r^3 + 315p^2q^4 \\
& + 1162p^2q^3r + 720p^2q^2r^2 + 1162p^2qr^3 + 315p^2r^4 - 189pq^5 + 479pq^4r \\
& + 1162pq^3r^2 + 1162pq^2r^3 + 479pqr^4 - 189pr^5 - 49q^6 - 189q^5r \\
& + 315q^4r^2 + 910q^3r^3 + 315q^2r^4 - 189qr^5 - 49r^6 + 112p^5 + 70p^4q \\
& + 70p^4r - 770p^3q^2 - 876p^3qr - 770p^3r^2 - 770p^2q^3 - 1380p^2q^2r \\
& - 1380p^2qr^2 - 770p^2r^3 + 70pq^4 - 876pq^3r - 1380pq^2r^2 - 876pqr^3 \\
& + 70pr^4 + 112q^5 + 70q^4r - 770q^3r^2 - 770q^2r^3 + 70qr^4 + 112r^5
\end{aligned}$$

$$- 105p^4 + 210p^3q + 210p^3r + 735p^2q^2 + 1034p^2qr + 735p^2r^2 + 210pq^3 + 1034pq^2r + 1034pqr^2 + 210pr^3 - 105q^4 + 210q^3r + 735q^2r^2 + 210qr^3 - 105r^4 + 14p^3 - 315p^2q - 315p^2r - 315pq^2 - 672pqr - 315pr^2 + 14q^3 - 315q^2r - 315qr^2 + 14r^3 + 49p^2 + 175pq + 175pr + 49q^2 + 175qr + 49r^2 - 36p - 36q - 36r + 8.$$

Remark: For $g = 0$, the formula

$$H_0(x, y, u) = pqr(1 - p - q - r) \tag{7.3}$$

can be derived from [1], with the correspondence $s = x$, $f = u$, and $a = y$ between variables and the correspondence $H_0(x, y, u) = xuK_0(1, x, y, u)$ between generating functions.

7.2 Rooted hypermap series enumerated by number of darts

Let $H_g(z)$ be the ordinary generating function of rooted hypermaps on the orientable surface of genus $g \geq 0$, where the exponent of variable z is the number d of darts.

7.2.1 Generating functions

For g from 0 to 6, a parametric expression of $H_g(z)$, where $z = \tau(1-2\tau)$ and $\tau = 0$ when $z = 0$, is

$$H_0(z) = \frac{\tau^3\,(1 - 3\,\tau)}{z^2}, \tag{7.4}$$

$$H_1(z) = \frac{\tau^3}{(1-\tau)\,(1-4\,\tau)^2}, \tag{7.5}$$

$$H_2(z) = \frac{4\,z^2\,\tau^3\,(51\,\tau^4 - 77\,\tau^3 + 48\,\tau^2 - 15\,\tau + 2)}{(1-\tau)^5\,(1-4\,\tau)^7}, \tag{7.6}$$

$$H_3(z) = \frac{4\,z^4\,\tau^3\,P_3(z)}{(1-\tau)^9\,(1-4\,\tau)^{12}}, \tag{7.7}$$

$$H_4(z) = \frac{4\,z^6\,\tau^3\,P_4(z)}{(1-\tau)^{13}\,(1-4\,\tau)^{17}}, \tag{7.8}$$

$$H_5(z) = \frac{4\,z^8\,\tau^3\,P_5(z)}{(1-\tau)^{17}\,(1-4\,\tau)^{22}}, \tag{7.9}$$

$$H_6(z) = \frac{4\,z^{10}\,\tau^3 P_6(z)}{(1-\tau)^{21}\,(1-4\,\tau)^{27}}, \tag{7.10}$$

with

$$P_3(z) = 28496\,\tau^9 - 36888\,\tau^8 - 13164\,\tau^7 + 61676\,\tau^6 - 61872\,\tau^5 + 35172\,\tau^4 - 13168\,\tau^3 + 3360\,\tau^2 - 552\,\tau + 45,$$

$$\begin{aligned}P_4(z) &= 32375616\,\tau^{14} + 15509760\,\tau^{13} - 243313744\,\tau^{12} + 442844592\,\tau^{11}\\&\quad - 389268768\,\tau^{10} + 170357328\,\tau^{9} + 1281984\,\tau^{8} - 53553072\,\tau^{7}\\&\quad + 39814032\,\tau^{6} - 17597520\,\tau^{5} + 5541192\,\tau^{4} - 1320920\,\tau^{3} + 239697\,\tau^{2}\\&\quad - 30456\,\tau + 2016,\\P_5(z) &= 61742404608\,\tau^{19} + 239043447552\,\tau^{18} - 1163002515456\,\tau^{17}\\&\quad + 1403096348736\,\tau^{16} + 338393916800\,\tau^{15} - 2962590413376\,\tau^{14}\\&\quad + 4243997599488\,\tau^{13} - 3552865706240\,\tau^{12} + 2000782619136\,\tau^{11}\\&\quad - 761565230016\,\tau^{10} + 165542511744\,\tau^{9} + 7568059872\,\tau^{8}\\&\quad - 23295865824\,\tau^{7} + 11016156244\,\tau^{6} - 3336459144\,\tau^{5} + 761835465\,\tau^{4}\\&\quad - 141393220\,\tau^{3} + 21738240\,\tau^{2} - 2490480\,\tau + 151200\end{aligned}$$

and

$$\begin{aligned}P_6(z) &= 178054771302400\,\tau^{24} + 1584534210564096\,\tau^{23} - 4933663711730688\,\tau^{22}\\&\quad - 2073822560019456\,\tau^{21} + 28025505345377280\,\tau^{20}\\&\quad - 55010184951564288\,\tau^{19} + 54283457920223232\,\tau^{18}\\&\quad - 22997164994372352\,\tau^{17} - 13439214645718272\,\tau^{16}\\&\quad + 31734000656779264\,\tau^{15} - 29719458122609664\,\tau^{14}\\&\quad + 18704646148809216\,\tau^{13} - 8736443315384448\,\tau^{12}\\&\quad + 3098312828500416\,\tau^{11} - 813298324826016\,\tau^{10} + 138473163256176\,\tau^{9}\\&\quad - 4043551301232\,\tau^{8} - 6580517850696\,\tau^{7} + 2630924485729\,\tau^{6}\\&\quad - 626336383104\,\tau^{5} + 112079088144\,\tau^{4} - 17314508592\,\tau^{3} + 2485496880\,\tau^{2}\\&\quad - 284717376\,\tau + 17107200.\end{aligned}$$

We have also computed the generating functions for $7 \le g \le 11$. Their expressions are too large to be included in the present text, but a Maple file is available from the first author on request.

A. Mednykh and R. Nedela used our formulas (7.4) to (7.7) to find explicit formulas for the number of rooted hypermaps for genus $g = 0, 1, 2$ and 3 [19].

7.3 Other parameterization

In a private communication to the second author, P. Zograf suggests the parameterization

$$z = \frac{t}{(1+2t)^2}. \tag{7.11}$$

After adding the condition that $t = 0$ when $z = 0$, it corresponds to

$$t = \frac{1 - 4z - \sqrt{1-8z}}{8z}. \tag{7.12}$$

These two parameterizations are equivalent. The one can be transformed into the other by means of the following substitutions:

$$\tau = \frac{t}{1+2t} \tag{7.13}$$

and

$$t = \frac{\tau}{1-2\tau}. \tag{7.14}$$

By means of these substitutions, the following parametric expressions in t can be obtained from the parametric expressions (7.4)–(7.10) for $H_g(t)$ in τ:

$$\begin{aligned}
H_0(z) &= t\,(1-t),\\
H_1(z) &= \frac{t^3}{(1+t)(1-2t)^2},\\
H_2(z) &= \frac{4\,t^5\,(1+2t)\,(t^4-t^3+6\,t^2+t+2)}{(1+t)^5(1-2t)^7},\\
H_3(z) &= 4\,t^7\,(1+2t)\,(1+t)^{-9}(1-2t)^{-12}\,(80\,t^9-120\,t^8+1500\,t^7+1036\,t^6\\
&\quad +3768\,t^5+2820\,t^4+2288\,t^3+1008\,t^2+258\,t+45),\\
H_4(z) &= 4\,t^9\,(1+2t)\,(1+t)^{-13}(1-2t)^{-17}\,(16768\,t^{14}-33536\,t^{13}\\
&\quad +653776\,t^{12}+786480\,t^{11}+4358016\,t^{10}+6151056\,t^9+10059552\,t^8\\
&\quad +10217040\,t^7+8418240\,t^6+5227024\,t^5+2365888\,t^4+800128\,t^3\\
&\quad +181665\,t^2+25992\,t+2016),\\
H_5(z) &= 4\,t^{11}\,(1+2t)\,(1+t)^{-17}(1-2t)^{-22}\,(6732800\,t^{19}-16832000\,t^{18}\\
&\quad +450011520\,t^{17}+773106240\,t^{16}+5764983552\,t^{15}+11910647232\,t^{14}\\
&\quad +29130502912\,t^{13}+46090300928\,t^{12}+63452543616\,t^{11}\\
&\quad +68713116608\,t^{10}+60654218080\,t^9+43591208976\,t^8\\
&\quad +25142796864\,t^7+11637842232\,t^6+4232899206\,t^5+1181820745\,t^4\\
&\quad +245635580\,t^3+35501760\,t^2+3255120\,t+151200),\\
H_6(z) &= 4\,t^{13}\,(1+2t)\,(1+t)^{-21}(1-2t)^{-27}\,(4424052736\,t^{24}-13272158208\,t^{23}\\
&\quad +452750478336\,t^{22}+1012254206976\,t^{21}+9488911137792\,t^{20}\\
&\quad +25803592571904\,t^{19}+83891900050944\,t^{18}+180120643165440\,t^{17}\\
&\quad +346626234587904\,t^{16}+535272874975232\,t^{15}+701152993531392\,t^{14}\\
&\quad +771688966862592\,t^{13}+716686355273472\,t^{12}+563018634260736\,t^{11}\\
&\quad +372549313187520\,t^{10}+207088794784752\,t^9+96021082581732\,t^8\\
&\quad +36765061031004\,t^7+11475757049569\,t^6+2863185376896\,t^5\\
&\quad +556090776432\,t^4+80913152016\,t^3+8274846384\,t^2+536428224\,t\\
&\quad +17107200).
\end{aligned}$$

For $0 \le g \le 3$, these expressions correspond to $F_g(t)$ in Zograf's communication. Moreover, they reveal an extra factorization by $4(1+2t)$ for $g \ge 2$.

8 Efficient enumeration of rooted and sensed unrooted hypermaps by number of darts, vertices and hyperedges

We recall that a sensed map or hypermap is an equivalence class of (unrooted) maps or hypermaps under orientation-preserving isomorphism.

Before enumerating sensed hypermaps we first need to enumerate rooted hypermaps. We use an efficient method of counting rooted hypermaps by number of darts, faces, vertices and hyperedges or, equivalently [23], 2-coloured bipartite maps rooted at a white vertex by number of edges, faces, white vertices and black vertices, presented by Kazarian and Zograf [15], and then count sensed 2-coloured bipartite maps and hypermaps with the same parameters using the same method we used [26, 12] to count sensed maps by number of edges, faces and vertices. The recurrence (formula (11) in [15]), with f changed to H, is as follows. Define $H_{g,d}$ to be a homogeneous polynomial in the three variables t, u, and v. The coefficient of $t^f u^b v^w$ in $H_{g,d}$ is the number of 2-coloured bipartite maps of genus g with d edges, f faces, b black vertices and w white vertices rooted at a white vertex or, equivalently, the number of rooted hypermaps of genus g with d darts, f faces, b hyperedges and w vertices. Then $H_{0,1} = tuv$ and

$$\begin{aligned}(d+1)H_{g,d} &= \\ &(2d-1)(t+u+v)H_{g,d-1} \\ &+ (d-2)\left(2(tu+tv+uv) - (t^2+u^2+v^2)\right) H_{g,d-2} \\ &+ (d-1)^2(d-2)H_{g-1,d-2} + \sum_{i=0}^{g}\sum_{j=1}^{d-3}(4+6j)(d-2-j)H_{i,j}H_{g-i,d-2-j}.\end{aligned} \tag{8.1}$$

In [26] we collaborated with Mednykh to enumerate rooted and sensed maps. Mednykh enumerated maps of genus up to 11 by number of edges alone, while we enumerated maps of genus up to 10 by number of edges and vertices. The method we used to enumerate rooted maps is presented in [25]. The method we used to enumerate sensed maps is based on Liskovets' refinement [17] of the method Mednykh and Nedela used to enumerate sensed map of genus up to 3 by number of edges [18]. Later we used a more efficient method of enumerating rooted maps, presented in [5], to enumerate rooted and sensed maps of genus up to 50 [12].

To describe here the modifications we made to pass from maps to 2-coloured bipartite maps we need to briefly discuss a few of the concepts described in more detail in [26]. All the automorphisms of a map on an orientable surface are periodic. If the period is $L > 1$, then the automorphism divides the map into L isomorphic copies of a smaller map, called the *quotient map*. Most of the *cells* (vertices, edges and faces) are in orbits of length L under the automorphism; those that aren't are called *branch points*. For example, if a map is drawn on the surface of a sphere which undergoes a rotation through $360/L$ degrees, the two cells through which the axis of rotation pass are fixed; so they are each in an orbit of length 1 for any L. For maps of higher genus, not all the branch points are on orbits of length 1. For example, if a torus is represented as a square with opposite edges identified in pairs, and is rotated by 90 degrees (period 4), then the centre of the square is a branch point of orbit length 1 and so is the point represented by all four corners of the square, but the middle of the sides of the square are two branch points of orbit length 2: the point represented by the middle of both vertical sides of the square is taken by the rotation onto the point represented by the middle of both horizontal sides, and vice versa; so it takes two rotations to take either of these points back onto itself. Also, if the middle of an edge is a branch point, then the quotient map contains half of that edge – a *dangling semi-edge*.

An automorphism of a map M of genus G is characterized by the following parameters: the period L, the genus g of its quotient map and the number of branch points of each orbit length. If each orbit length is replaced by its *branch index* (L divided by the orbit length), we obtain what is called an *orbifold signature* in [18]. In [18] a method is presented for determining which orbifold signatures could characterize an automorphism of a map of genus G (a *G-admissible orbifold*) and how many such automorphisms could be characterized by that orbifold signature; a variant of that method is presented in [17], and this is the one we use except that we deal with orbit lengths instead of branch indices. The method used in [18] to enumerate sensed maps of genus G with E edges by number of edges can be roughly described as follows. For each G-admissible orbifold O, let g be the genus of the quotient map, L be the period and q_i be the number of branch points with branch index i. Then the number $\nu_O(d)$ of rooted maps with d darts that could serve as a quotient map for an automorphism with that signature once the branch points are pasted onto the map in all possible ways is given by

$$\nu_O(d) = \sum_{s=0}^{q_2} \binom{d}{s} \binom{(d-s)/2+2-2g}{q_2-s, q_3, \ldots, q_L} N_g((d-s)/2), \tag{8.2}$$

where $N_g(n)$ is the number of rooted maps of genus g with n edges (0 if n is not an integer). Here s is the number of dangling semi-edges in the quotient map m, all of which must be in orbits of length $L/2$ so that they represent normal edges in the original map M. The binomial coefficient is the number of ways of inserting dangling semi-edges into the rooted map multiplied by $d/(d-s)$ because there are d ways to root the map once the dangling edges have been inserted and only $d-s$ ways to root it without the dangling edges. The multinomial coefficient is the number of ways to distribute the branch points with the various branch indices among the non-edges of the quotient map; the number at the top of the multinomial coefficient is the number of non-edges and is given by the Euler-Poincaré formula (1.1). Then the number of sensed maps of genus G with E edges is

$$\frac{1}{2E} \sum_{L|E} \sum_{O} \mathrm{Epi}_0(\pi_1(O), Z_L)\, \nu_O(2E/L), \tag{8.3}$$

where O runs over all the G-admissible orbifolds with period L and $\mathrm{Epi}_0(\pi_1(O), Z_L)$ is the number of automorphisms that have the orbifold signature of O.

In [26] we distributed the branch points that aren't on dangling semi-edges among the vertices and faces separately. The quotient map of a bipartite map can't contain any dangling semi-edges; otherwise the lifted map would have an edge joining two vertices of the same colour. Here we distribute the branch points among the white vertices, black vertices and faces, and, like in [26], we don't use a formula like (8.3); instead we compute the contribution of each orbifold signature to the number of sensed 2-coloured bipartite maps whose number of white vertices, black vertices, faces and edges are allowed to vary within a user-defined upper bound on the number of edges.

Suppose that the quotient map is of genus g and has w white vertices, b black vertices and f faces. Then the number e of edges can be calculated from the formula

$$f - e + w + b = 2(1-g) \tag{8.4}$$

and the number d of darts is $2e$. Suppose also that among the branch points of orbit length i, w_i are on a white vertex, b_i are on a black vertex and f_i are in a face. We denote by w_L, b_L and f_L the number of white vertices, black vertices and faces, respectively, that do not contain a branch point. The original map will have W white vertices, B black vertices and F faces, where

$$W = \sum_{i=1}^{L} i w_i, \quad B = \sum_{i=1}^{L} i b_i \quad \text{and} \quad F = \sum_{i=1}^{L} i f_i, \tag{8.5}$$

and the total number E of edges is equal to $L\,e = F + W + B - 2(1-g)$.

The binomial coefficient in (8.2) disappears because the quotient map can't contain any dangling semi-edges. The multinomial coefficient must be replaced by the number of ways to distribute the branch points among the white vertices, black vertices and faces. Then (8.2) becomes

$$\nu_O(d,w,b,f) = \binom{w}{w_1,w_2,\ldots,w_L}\binom{b}{b_1,b_2,\ldots,b_L}\binom{f}{f_1,f_2,\ldots,f_L}N_g(d,w,b,f), \quad (8.6)$$

where d is the number of edges in the quotient maps on both sides of the formula (or the number of darts in the corresponding hypermaps) and $N_g(d,w,b,f)$ is the number of 2-coloured bipartite maps with d edges with w white vertices, b black vertices and f faces, rooted at a white vertex. For this number to be positive, the sum of all the w_i cannot exceed w with a similar bound on the sum of all the b_i and the sum of all the f_i; so w, b and f each starts at its respective sum and increases by 1 until the number E of edges in the original map exceeds a user-defined maximum. With each increase of w, b or f, one of the multinomial coefficients in (8.6) gets updated using a single multiplication and division. The product of these three multinomial coefficients must be computed for all sets of non-negative integers such that for each i, $w_i + b_i + f_i$ is equal to the total number of branch points of orbit length i.

Once (8.6) is multiplied by the number of automorphisms with the current orbifold signature, we get the contribution of that signature and the numbers w_i, b_i and f_i to E times the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices. This contribution is added to the appropriate element of an array, initially 0, and when all the contributions have been tallied, for each E, F, W and B the corresponding array element is divided by E (not $2E$ because the root must be incident to a white vertex) to give the number of sensed 2-coloured bipartite maps of genus G with E edges, F faces, B black vertices and W white vertices or, equivalently, the number of sensed hypermaps of genus G with E darts, F faces, B hyperedges and W vertices.

This enumeration was done with a program written in C++ using CLN to treat big integers. It enumerated rooted and sensed hypermaps of genus up to 24 with up to 50 darts as fast as it could display the numbers on the screen. The numbers coincide with those obtained by generating the hypermaps [24]. The source code is available from the second author on request.

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A First numbers of rooted hypermaps

The following sections show the numbers h of rooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

A.1 Genus 0

d	v	e	f	h
1	1	1	1	1
1			sum	1
2	1	1	2	1
2	1	2	1	1
2	2	1	1	1
2			sum	3
3	1	1	3	1
3	1	2	2	3
3	2	1	2	3
3	1	3	1	1
3	2	2	1	3
3	3	1	1	1
3			sum	12
4	1	1	4	1
4	1	2	3	6
4	2	1	3	6
4	1	3	2	6
4	2	2	2	17
4	3	1	2	6
4	1	4	1	1
4	2	3	1	6
4	3	2	1	6
4	4	1	1	1
4			sum	56
5	1	1	5	1
5	1	2	4	10
5	2	1	4	10
5	1	3	3	20
5	2	2	3	55
5	3	1	3	20
5	1	4	2	10
5	2	3	2	55
5	3	2	2	55
5	4	1	2	10
5	1	5	1	1
5	2	4	1	10
5	3	3	1	20
5	4	2	1	10
5	5	1	1	1
5			sum	288
6	1	1	6	1
6	1	2	5	15
6	2	1	5	15
6	1	3	4	50
6	2	2	4	135
6	3	1	4	50
6	1	4	3	50
6	2	3	3	262
6	3	2	3	262
6	4	1	3	50
6	1	5	2	15
6	2	4	2	135
6	3	3	2	262
6	4	2	2	135
6	5	1	2	15
6	1	6	1	1
6	2	5	1	15
6	3	4	1	50
6	4	3	1	50
6	5	2	1	15
6	6	1	1	1
6			sum	1584
7	1	1	7	1
7	1	2	6	21
7	2	1	6	21
7	1	3	5	105
7	2	2	5	280
7	3	1	5	105
7	1	4	4	175
7	2	3	4	889
7	3	2	4	889
7	4	1	4	175
7	1	5	3	105
7	2	4	3	889
7	3	3	3	1694
7	4	2	3	889
7	5	1	3	105
7	1	6	2	21
7	2	5	2	280
7	3	4	2	889
7	4	3	2	889
7	5	2	2	280
7	6	1	2	21
7	1	7	1	1
7	2	6	1	21
7	3	5	1	105
7	4	4	1	175
7	5	3	1	105
7	6	2	1	21
7	7	1	1	1
7			sum	9152
8	1	1	8	1
8	1	2	7	28
8	2	1	7	28
8	1	3	6	196
8	2	2	6	518
8	3	1	6	196
8	1	4	5	490
8	2	3	5	2436
8	3	2	5	2436
8	4	1	5	490
8	1	5	4	490
8	2	4	4	3985
8	3	3	4	7500
8	4	2	4	3985
8	5	1	4	490
8	1	6	3	196
8	2	5	3	2436
8	3	4	3	7500
8	4	3	3	7500
8	5	2	3	2436
8	6	1	3	196
8	1	7	2	28
8	2	6	2	518
8	3	5	2	2436
8	4	4	2	3985
8	5	3	2	2436
8	6	2	2	518
8	7	1	2	28
8	1	8	1	1
8	2	7	1	28
8	3	6	1	196
8	4	5	1	490
8	5	4	1	490
8	6	3	1	196
8	7	2	1	28
8	8	1	1	1
8			sum	54912
9	1	1	9	1
9	1	2	8	36
9	2	1	8	36
9	1	3	7	336
9	2	2	7	882
9	3	1	7	336
9	1	4	6	1176
9	2	3	6	5754
9	3	2	6	5754
9	4	1	6	1176
9	1	5	5	1764
9	2	4	5	13941
9	3	3	5	26004
9	4	2	5	13941
9	5	1	5	1764
9	1	6	4	1176
9	2	5	4	13941
9	3	4	4	42015
9	4	3	4	42015
9	5	2	4	13941
9	6	1	4	1176
9	1	7	3	336
9	2	6	3	5754
9	3	5	3	26004
9	4	4	3	42015
9	5	3	3	26004
9	6	2	3	5754
9	7	1	3	336
9	1	8	2	36
9	2	7	2	882
9	3	6	2	5754
9	4	5	2	13941
9	5	4	2	13941
9	6	3	2	5754
9	7	2	2	882
9	8	1	2	36
9	1	9	1	1
9	2	8	1	36

9	3	7	1	336
9	4	6	1	1176
9	5	5	1	1764
9	6	4	1	1176
9	7	3	1	336
9	8	2	1	36
9	9	1	1	1
9			sum	339456
10	1	1	10	1
10	1	2	9	45
10	2	1	9	45
10	1	3	8	540
10	2	2	8	1410
10	3	1	8	540
10	1	4	7	2520
10	2	3	7	12180
10	3	2	7	12180
10	4	1	7	2520
10	1	5	6	5292
10	2	4	6	40935
10	3	3	6	75840
10	4	2	6	40935
10	5	1	6	5292
10	1	6	5	5292
10	2	5	5	60626
10	3	4	5	179860
10	4	3	5	179860
10	5	2	5	60626
10	6	1	5	5292
10	1	7	4	2520
10	2	6	4	40935
10	3	5	4	179860
10	4	4	4	288025
10	5	3	4	179860
10	6	2	4	40935
10	7	1	4	2520
10	1	8	3	540
10	2	7	3	12180
10	3	6	3	75840
10	4	5	3	179860
10	5	4	3	179860
10	6	3	3	75840
10	7	2	3	12180
10	8	1	3	540
10	1	9	2	45
10	2	8	2	1410
10	3	7	2	12180
10	4	6	2	40935
10	5	5	2	60626
10	6	4	2	40935
10	7	3	2	12180
10	8	2	2	1410
10	9	1	2	45
10	1	10	1	1
10	2	9	1	45
10	3	8	1	540
10	4	7	1	2520
10	5	6	1	5292
10	6	5	1	5292
10	7	4	1	2520
10	8	3	1	540
10	9	2	1	45
10	10	1	1	1
10			sum	2149888
11	1	1	11	1
11	1	2	10	55
11	2	1	10	55
11	1	3	9	825
11	2	2	9	2145
11	3	1	9	825
11	1	4	8	4950
11	2	3	8	23694
11	3	2	8	23694
11	4	1	8	4950
11	1	5	7	13860
11	2	4	7	105435
11	3	3	7	194304
11	4	2	7	105435
11	5	1	7	13860
11	1	6	6	19404
11	2	5	6	216601
11	3	4	6	634865
11	4	3	6	634865
11	5	2	6	216601
11	6	1	6	19404
11	1	7	5	13860
11	2	6	5	216601
11	3	5	5	931854
11	4	4	5	1482250
11	5	3	5	931854
11	6	2	5	216601
11	7	1	5	13860
11	1	8	4	4950
11	2	7	4	105435
11	3	6	4	634865
11	4	5	4	1482250
11	5	4	4	1482250
11	6	3	4	634865
11	7	2	4	105435
11	8	1	4	4950
11	1	9	3	825
11	2	8	3	23694
11	3	7	3	194304
11	4	6	3	634865
11	5	5	3	931854
11	6	4	3	634865
11	7	3	3	194304
11	8	2	3	23694
11	9	1	3	825
11	1	10	2	55
11	2	9	2	2145
11	3	8	2	23694
11	4	7	2	105435
11	5	6	2	216601
11	6	5	2	216601
11	7	4	2	105435
11	8	3	2	23694
11	9	2	2	2145
11	10	1	2	55
11	1	11	1	1
11	2	10	1	55
11	3	9	1	825
11	4	8	1	4950
11	5	7	1	13860
11	6	6	1	19404
11	7	5	1	13860
11	8	4	1	4950
11	9	3	1	825
11	10	2	1	55
11	11	1	1	1
11			sum	13891584
12	1	1	12	1
12	1	2	11	66
12	2	1	11	66
12	1	3	10	1210
12	2	2	10	3135
12	3	1	10	1210
12	1	4	9	9075
12	2	3	9	43098
12	3	2	9	43098
12	4	1	9	9075
12	1	5	8	32670
12	2	4	8	245223
12	3	3	8	449988
12	4	2	8	245223
12	5	1	8	32670
12	1	6	7	60984
12	2	5	7	666996
12	3	4	7	1936308
12	4	3	7	1936308
12	5	2	7	666996
12	6	1	7	60984
12	1	7	6	60984
12	2	6	6	925190
12	3	5	6	3915576
12	4	4	6	6195560
12	5	3	6	3915576
12	6	2	6	925190
12	7	1	6	60984
12	1	8	5	32670
12	2	7	5	666996
12	3	6	5	3915576
12	4	5	5	9032898
12	5	4	5	9032898
12	6	3	5	3915576
12	7	2	5	666996
12	8	1	5	32670
12	1	9	4	9075
12	2	8	4	245223
12	3	7	4	1936308
12	4	6	4	6195560
12	5	5	4	9032898
12	6	4	4	6195560
12	7	3	4	1936308
12	8	2	4	245223
12	9	1	4	9075
12	1	10	3	1210
12	2	9	3	43098
12	3	8	3	449988
12	4	7	3	1936308
12	5	6	3	3915576
12	6	5	3	3915576
12	7	4	3	1936308
12	8	3	3	449988
12	9	2	3	43098
12	10	1	3	1210
12	1	11	2	66
12	2	10	2	3135
12	3	9	2	43098
12	4	8	2	245223
12	5	7	2	666996
12	6	6	2	925190
12	7	5	2	666996
12	8	4	2	245223
12	9	3	2	43098
12	10	2	2	3135
12	11	1	2	66
12	1	12	1	1
12	2	11	1	66
12	3	10	1	1210
12	4	9	1	9075
12	5	8	1	32670
12	6	7	1	60984
12	7	6	1	60984

12	8	5	1	32670
12	9	4	1	9075
12	10	3	1	1210
12	11	2	1	66
12	12	1	1	1
12			sum	91287552
13	1	1	13	1
13	1	2	12	78
13	2	1	12	78
13	1	3	11	1716
13	2	2	11	4433
13	3	1	11	1716
13	1	4	10	15730
13	2	3	10	74217
13	3	2	10	74217
13	4	1	10	15730
13	1	5	9	70785
13	2	4	9	525525
13	3	3	9	960960
13	4	2	9	525525
13	5	1	9	70785
13	1	6	8	169884
13	2	5	8	1827683
13	3	4	8	5264545
13	4	3	8	5264545
13	5	2	8	1827683
13	6	1	8	169884
13	1	7	7	226512
13	2	6	7	3356522
13	3	5	7	14019928
13	4	4	7	22089600
13	5	3	7	14019928
13	6	2	7	3356522
13	7	1	7	226512
13	1	8	6	169884
13	2	7	6	3356522
13	3	6	6	19315114
13	4	5	6	44136820
13	5	4	6	44136820
13	6	3	6	19315114
13	7	2	6	3356522
13	8	1	6	169884
13	1	9	5	70785
13	2	8	5	1827683
13	3	7	5	14019928
13	4	6	5	44136820
13	5	5	5	64013222
13	6	4	5	44136820
13	7	3	5	14019928
13	8	2	5	1827683
13	9	1	5	70785
13	1	10	4	15730
13	2	9	4	525525
13	3	8	4	5264545
13	4	7	4	22089600
13	5	6	4	44136820
13	6	5	4	44136820
13	7	4	4	22089600
13	8	3	4	5264545
13	9	2	4	525525
13	10	1	4	15730
13	1	11	3	1716
13	2	10	3	74217
13	3	9	3	960960
13	4	8	3	5264545
13	5	7	3	14019928
13	6	6	3	19315114
13	7	5	3	14019928
13	8	4	3	5264545
13	9	3	3	960960
13	10	2	3	74217
13	11	1	3	1716
13	1	12	2	78
13	2	11	2	4433
13	3	10	2	74217
13	4	9	2	525525
13	5	8	2	1827683
13	6	7	2	3356522
13	7	6	2	3356522
13	8	5	2	1827683
13	9	4	2	525525
13	10	3	2	74217
13	11	2	2	4433
13	12	1	2	78
13	1	13	1	1
13	2	12	1	78
13	3	11	1	1716
13	4	10	1	15730
13	5	9	1	70785
13	6	8	1	169884
13	7	7	1	226512
13	8	6	1	169884
13	9	5	1	70785
13	10	4	1	15730
13	11	3	1	1716
13	12	2	1	78
13	13	1	1	1
13			sum	608583680
14	1	1	14	1
14	1	2	13	91
14	2	1	13	91
14	1	3	12	2366
14	2	2	12	6097
14	3	1	12	2366
14	1	4	11	26026
14	2	3	11	122122
14	3	2	11	122122
14	4	1	11	26026
14	1	5	10	143143
14	2	4	10	1053052
14	3	3	10	1919918
14	4	2	10	1053052
14	5	1	10	143143
14	1	6	9	429429
14	2	5	9	4557553
14	3	4	9	13043030
14	4	3	9	13043030
14	5	2	9	4557553
14	6	1	9	429429
14	1	7	8	736164
14	2	6	8	10701873
14	3	5	8	44221632
14	4	4	8	69432090
14	5	3	8	44221632
14	6	2	8	10701873
14	7	1	8	736164
14	1	8	7	736164
14	2	7	7	14168988
14	3	6	7	80231508
14	4	5	7	181925268
14	5	4	7	181925268
14	6	3	7	80231508
14	7	2	7	14168988
14	8	1	7	736164
14	1	9	6	429429
14	2	8	6	10701873
14	3	7	6	80231508
14	4	6	6	249321114
14	5	5	6	360078558
14	6	4	6	249321114
14	7	3	6	80231508
14	8	2	6	10701873
14	9	1	6	429429
14	1	10	5	143143
14	2	9	5	4557553
14	3	8	5	44221632
14	4	7	5	181925268
14	5	6	5	360078558
14	6	5	5	360078558
14	7	4	5	181925268
14	8	3	5	44221632
14	9	2	5	4557553
14	10	1	5	143143
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14	2	10	4	1053052
14	3	9	4	13043030
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14	5	7	4	181925268
14	6	6	4	249321114
14	7	5	4	181925268
14	8	4	4	69432090
14	9	3	4	13043030
14	10	2	4	1053052
14	11	1	4	26026
14	1	12	3	2366
14	2	11	3	122122
14	3	10	3	1919918
14	4	9	3	13043030
14	5	8	3	44221632
14	6	7	3	80231508
14	7	6	3	80231508
14	8	5	3	44221632
14	9	4	3	13043030
14	10	3	3	1919918
14	11	2	3	122122
14	12	1	3	2366
14	1	13	2	91
14	2	12	2	6097
14	3	11	2	122122
14	4	10	2	1053052
14	5	9	2	4557553
14	6	8	2	10701873
14	7	7	2	14168988
14	8	6	2	10701873
14	9	5	2	4557553
14	10	4	2	1053052
14	11	3	2	122122
14	12	2	2	6097
14	13	1	2	91
14	1	14	1	1
14	2	13	1	91
14	3	12	1	2366
14	4	11	1	26026
14	5	10	1	143143
14	6	9	1	429429
14	7	8	1	736164
14	8	7	1	736164
14	9	6	1	429429
14	10	5	1	143143
14	11	4	1	26026
14	12	3	1	2366
14	13	2	1	91
14	14	1	1	1
14			sum	4107939840

A.2 Genus 1

d	v	e	f	h
3	1	1	1	1
3			sum	1
4	1	1	2	5
4	1	2	1	5
4	2	1	1	5
4			sum	15
5	1	1	3	15
5	1	2	2	40
5	2	1	2	40
5	1	3	1	15
5	2	2	1	40
5	3	1	1	15
5			sum	165
6	1	1	4	35
6	1	2	3	175
6	2	1	3	175
6	1	3	2	175
6	2	2	2	456
6	3	1	2	175
6	1	4	1	35
6	2	3	1	175
6	3	2	1	175
6	4	1	1	35
6			sum	1611
7	1	1	5	70
7	1	2	4	560
7	2	1	4	560
7	1	3	3	1050
7	2	2	3	2695
7	3	1	3	1050
7	1	4	2	560
7	2	3	2	2695
7	3	2	2	2695
7	4	1	2	560
7	1	5	1	70
7	2	4	1	560
7	3	3	1	1050
7	4	2	1	560
7	5	1	1	70
7			sum	14805
8	1	1	6	126
8	1	2	5	1470
8	2	1	5	1470
8	1	3	4	4410
8	2	2	4	11199
8	3	1	4	4410
8	1	4	3	4410
8	2	3	3	20684
8	3	2	3	20684
8	4	1	3	4410
8	1	5	2	1470
8	2	4	2	11199
8	3	3	2	20684
8	4	2	2	11199
8	5	1	2	1470
8	1	6	1	126
8	2	5	1	1470
8	3	4	1	4410
8	4	3	1	4410
8	5	2	1	1470
8	6	1	1	126
8			sum	131307
9	1	1	7	210
9	1	2	6	3360
9	2	1	6	3360
9	1	3	5	14700
9	2	2	5	37035
9	3	1	5	14700
9	1	4	4	23520
9	2	3	4	108285
9	3	2	4	108285
9	4	1	4	23520
9	1	5	3	14700
9	2	4	3	108285
9	3	3	3	197896
9	4	2	3	108285
9	5	1	3	14700
9	1	6	2	3360
9	2	5	2	37035
9	3	4	2	108285
9	4	3	2	108285
9	5	2	2	37035
9	6	1	2	3360
9	1	7	1	210
9	2	6	1	3360
9	3	5	1	14700
9	4	4	1	23520
9	5	3	1	14700
9	6	2	1	3360
9	7	1	1	210
9			sum	1138261
10	1	1	8	330
10	1	2	7	6930
10	2	1	7	6930
10	1	3	6	41580
10	2	2	6	104115
10	3	1	6	41580
10	1	4	5	97020
10	2	3	5	440440
10	3	2	5	440440
10	4	1	5	97020
10	1	5	4	97020
10	2	4	4	697250
10	3	3	4	1264310
10	4	2	4	697250
10	5	1	4	97020
10	1	6	3	41580
10	2	5	3	440440
10	3	4	3	1264310
10	4	3	3	1264310
10	5	2	3	440440
10	6	1	3	41580
10	1	7	2	6930
10	2	6	2	104115
10	3	5	2	440440
10	4	4	2	697250
10	5	3	2	440440
10	6	2	2	104115
10	7	1	2	6930
10	1	8	1	330
10	2	7	1	6930
10	3	6	1	41580
10	4	5	1	97020
10	5	4	1	97020
10	6	3	1	41580
10	7	2	1	6930
10	8	1	1	330
10			sum	9713835
11	1	1	9	495
11	1	2	8	13200
11	2	1	8	13200
11	1	3	7	103950
11	2	2	7	259017
11	3	1	7	103950
11	1	4	6	332640
11	2	3	6	1493525
11	3	2	6	1493525
11	4	1	6	332640
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11	2	4	5	3420835
11	3	3	5	6165478
11	4	2	5	3420835
11	5	1	5	485100
11	1	6	4	332640
11	2	5	4	3420835
11	3	4	4	9684433
11	4	3	4	9684433
11	5	2	4	3420835
11	6	1	4	332640
11	1	7	3	103950
11	2	6	3	1493525
11	3	5	3	6165478
11	4	4	3	9684433
11	5	3	3	6165478
11	6	2	3	1493525
11	7	1	3	103950
11	1	8	2	13200
11	2	7	2	259017
11	3	6	2	1493525
11	4	5	2	3420835
11	5	4	2	3420835
11	6	3	2	1493525
11	7	2	2	259017
11	8	1	2	13200
11	1	9	1	495
11	2	8	1	13200
11	3	7	1	103950
11	4	6	1	332640
11	5	5	1	485100
11	6	4	1	332640
11	7	3	1	103950
11	8	2	1	13200
11	9	1	1	495
11			sum	81968469
12	1	1	10	715
12	1	2	9	23595
12	2	1	9	23595
12	1	3	8	235950
12	2	2	8	585585
12	3	1	8	235950
12	1	4	7	990990
12	2	3	7	4410120

12	3	2	7	4410120
12	4	1	7	990990
12	1	5	6	1981980
12	2	4	6	13768300
12	3	3	6	24695580
12	4	2	6	13768300
12	5	1	6	1981980
12	1	6	5	1981980
12	2	5	5	19920390
12	3	4	5	55785870
12	4	3	5	55785870
12	5	2	5	19920390
12	6	1	5	1981980
12	1	7	4	990990
12	2	6	4	13768300
12	3	5	4	55785870
12	4	4	4	87100531
12	5	3	4	55785870
12	6	2	4	13768300
12	7	1	4	990990
12	1	8	3	235950
12	2	7	3	4410120
12	3	6	3	24695580
12	4	5	3	55785870
12	5	4	3	55785870
12	6	3	3	24695580
12	7	2	3	4410120
12	8	1	3	235950
12	1	9	2	23595
12	2	8	2	585585
12	3	7	2	4410120
12	4	6	2	13768300
12	5	5	2	19920390
12	6	4	2	13768300
12	7	3	2	4410120
12	8	2	2	585585
12	9	1	2	23595
12	1	10	1	715
12	2	9	1	23595
12	3	8	1	235950
12	4	7	1	990990
12	5	6	1	1981980
12	6	5	1	1981980
12	7	4	1	990990
12	8	3	1	235950
12	9	2	1	23595
12	10	1	1	715
12			sum	685888171
13	1	1	11	1001
13	1	2	10	40040
13	2	1	10	40040
13	1	3	9	495495
13	2	2	9	1225653
13	3	1	9	495495
13	1	4	8	2642640
13	2	3	8	11674663
13	3	2	8	11674663
13	4	1	8	2642640
13	1	5	7	6936930
13	2	4	7	47604648
13	3	3	7	85050784
13	4	2	7	47604648
13	5	1	7	6936930
13	1	6	6	9513504
13	2	5	6	93880696
13	3	4	6	260619268
13	4	3	6	260619268
13	5	2	6	93880696
13	6	1	6	9513504
13	1	7	5	6936930
13	2	6	5	93880696
13	3	5	5	374805834
13	4	4	5	582408775
13	5	3	5	374805834
13	6	2	5	93880696
13	7	1	5	6936930
13	1	8	4	2642640
13	2	7	4	47604648
13	3	6	4	260619268
13	4	5	4	582408775
13	5	4	4	582408775
13	6	3	4	260619268
13	7	2	4	47604648
13	8	1	4	2642640
13	1	9	3	495495
13	2	8	3	11674663
13	3	7	3	85050784
13	4	6	3	260619268
13	5	5	3	374805834
13	6	4	3	260619268
13	7	3	3	85050784
13	8	2	3	11674663
13	9	1	3	495495
13	1	10	2	40040
13	2	9	2	1225653
13	3	8	2	11674663
13	4	7	2	47604648
13	5	6	2	93880696
13	6	5	2	93880696
13	7	4	2	47604648
13	8	3	2	11674663
13	9	2	2	1225653
13	10	1	2	40040
13	1	11	1	1001
13	2	10	1	40040
13	3	9	1	495495
13	4	8	1	2642640
13	5	7	1	6936930
13	6	6	1	9513504
13	7	5	1	6936930
13	8	4	1	2642640
13	9	3	1	495495
13	10	2	1	40040
13	11	1	1	1001
13			sum	5702382933
14	1	1	12	1365
14	1	2	11	65065
14	2	1	11	65065
14	1	3	10	975975
14	2	2	10	2407405
14	3	1	10	975975
14	1	4	9	6441435
14	2	3	9	28283255
14	3	2	9	28283255
14	4	1	9	6441435
14	1	5	8	21471450
14	2	4	8	145864355
14	3	3	8	259750218
14	4	2	8	145864355
14	5	1	8	21471450
14	1	6	7	38648610
14	2	5	7	375707570
14	3	4	7	1035514340
14	4	3	7	1035514340
14	5	2	7	375707570
14	6	1	7	38648610
14	1	7	6	38648610
14	2	6	6	512104880
14	3	5	6	2020140430
14	4	4	6	3126887407
14	5	3	6	2020140430
14	6	2	6	512104880
14	7	1	6	38648610
14	1	8	5	21471450
14	2	7	5	375707570
14	3	6	5	2020140430
14	4	5	5	4475516612
14	5	4	5	4475516612
14	6	3	5	2020140430
14	7	2	5	375707570
14	8	1	5	21471450
14	1	9	4	6441435
14	2	8	4	145864355
14	3	7	4	1035514340
14	4	6	4	3126887407
14	5	5	4	4475516612
14	6	4	4	3126887407
14	7	3	4	1035514340
14	8	2	4	145864355
14	9	1	4	6441435
14	1	10	3	975975
14	2	9	3	28283255
14	3	8	3	259750218
14	4	7	3	1035514340
14	5	6	3	2020140430
14	6	5	3	2020140430
14	7	4	3	1035514340
14	8	3	3	259750218
14	9	2	3	28283255
14	10	1	3	975975
14	1	11	2	65065
14	2	10	2	2407405
14	3	9	2	28283255
14	4	8	2	145864355
14	5	7	2	375707570
14	6	6	2	512104880
14	7	5	2	375707570
14	8	4	2	145864355
14	9	3	2	28283255
14	10	2	2	2407405
14	11	1	2	65065
14	1	12	1	1365
14	2	11	1	65065
14	3	10	1	975975
14	4	9	1	6441435
14	5	8	1	21471450
14	6	7	1	38648610
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14	8	5	1	21471450
14	9	4	1	6441435
14	10	3	1	975975
14	11	2	1	65065
14	12	1	1	1365
14			sum	47168678571

A.3 Genus 2

d	v	e	f	h
5	1	1	1	8
5			sum	8
6	1	1	2	84
6	1	2	1	84
6	2	1	1	84
6			sum	252
7	1	1	3	469
7	1	2	2	1183
7	2	1	2	1183
7	1	3	1	469
7	2	2	1	1183
7	3	1	1	469
7			sum	4956
8	1	1	4	1869
8	1	2	3	8526
8	2	1	3	8526
8	1	3	2	8526
8	2	2	2	21229
8	3	1	2	8526
8	1	4	1	1869
8	2	3	1	8526
8	3	2	1	8526
8	4	1	1	1869
8			sum	77992
9	1	1	5	5985
9	1	2	4	42588
9	2	1	4	42588
9	1	3	3	77028
9	2	2	3	189999
9	3	1	3	77028
9	1	4	2	42588
9	2	3	2	189999
9	3	2	2	189999
9	4	1	2	42588
9	1	5	1	5985
9	2	4	1	42588
9	3	3	1	77028
9	4	2	1	42588
9	5	1	1	5985
9			sum	1074564
10	1	1	6	16401
10	1	2	5	167013
10	2	1	5	167013
10	1	3	4	471240
10	2	2	4	1154095
10	3	1	4	471240
10	1	4	3	471240
10	2	3	3	2068070
10	3	2	3	2068070
10	4	1	3	471240
10	1	5	2	167013
10	2	4	2	1154095
10	3	3	2	2068070
10	4	2	2	1154095
10	5	1	2	167013
10	1	6	1	16401
10	2	5	1	167013
10	3	4	1	471240
10	4	3	1	471240
10	5	2	1	167013
10	6	1	1	16401
10			sum	13545216
11	1	1	7	39963
11	1	2	6	550011
11	2	1	6	550011
11	1	3	5	2221065
11	2	2	5	5409019
11	3	1	5	2221065
11	1	4	4	3465000
11	2	3	4	15014846
11	3	2	4	15014846
11	4	1	4	3465000
11	1	5	3	2221065
11	2	4	3	15014846
11	3	3	3	26717482
11	4	2	3	15014846
11	5	1	3	2221065
11	1	6	2	550011
11	2	5	2	5409019
11	3	4	2	15014846
11	4	3	2	15014846
11	5	2	2	5409019
11	6	1	2	550011
11	1	7	1	39963
11	2	6	1	550011
11	3	5	1	2221065
11	4	4	1	3465000
11	5	3	1	2221065
11	6	2	1	550011
11	7	1	1	39963
11			sum	160174960
12	1	1	8	88803
12	1	2	7	1585584
12	2	1	7	1585584
12	1	3	6	8654646
12	2	2	6	20981337
12	3	1	6	8654646
12	1	4	5	19324305
12	2	3	5	82897296
12	3	2	5	82897296
12	4	1	5	19324305
12	1	5	4	19324305
12	2	4	4	128420004
12	3	3	4	227256510
12	4	2	4	128420004
12	5	1	4	19324305
12	1	6	3	8654646
12	2	5	3	82897296
12	3	4	3	227256510
12	4	3	3	227256510
12	5	2	3	82897296
12	6	1	3	8654646
12	1	7	2	1585584
12	2	6	2	20981337
12	3	5	2	82897296
12	4	4	2	128420004
12	5	3	2	82897296
12	6	2	2	20981337
12	7	1	2	1585584
12	1	8	1	88803
12	2	7	1	1585584
12	3	6	1	8654646
12	4	5	1	19324305
12	5	4	1	19324305
12	6	3	1	8654646
12	7	2	1	1585584
12	8	1	1	88803
12			sum	1805010948
13	1	1	9	183183
13	1	2	8	4114110
13	2	1	8	4114110
13	1	3	7	29135106
13	2	2	7	70367479
13	3	1	7	29135106
13	1	4	6	87933846
13	2	3	6	374127663
13	3	2	6	374127663
13	4	1	6	87933846
13	1	5	5	125855730
13	2	4	5	824962502
13	3	3	5	1453414846
13	4	2	5	824962502
13	5	1	5	125855730
13	1	6	4	87933846
13	2	5	4	824962502
13	3	4	4	2239280420
13	4	3	4	2239280420
13	5	2	4	824962502
13	6	1	4	87933846
13	1	7	3	29135106
13	2	6	3	374127663
13	3	5	3	1453414846
13	4	4	3	2239280420
13	5	3	3	1453414846
13	6	2	3	374127663
13	7	1	3	29135106
13	1	8	2	4114110
13	2	7	2	70367479
13	3	6	2	374127663
13	4	5	2	824962502
13	5	4	2	824962502
13	6	3	2	374127663
13	7	2	2	70367479
13	8	1	2	4114110
13	1	9	1	183183
13	2	8	1	4114110
13	3	7	1	29135106
13	4	6	1	87933846
13	5	5	1	125855730
13	6	4	1	87933846
13	7	3	1	29135106
13	8	2	1	4114110
13	9	1	1	183183
13			sum	19588944336
14	1	1	10	355355
14	1	2	9	9798789
14	2	1	9	9798789
14	1	3	8	87291204
14	2	2	8	210164227
14	3	1	8	87291204
14	1	4	7	341825484
14	2	3	7	1444432612

14	3	2	7	1444432612
14	4	1	7	341825484
14	1	5	6	661320660
14	2	4	6	4286172247
14	3	3	6	7523770016
14	4	2	6	4286172247
14	5	1	6	661320660
14	1	6	5	661320660
14	2	5	5	6100939726
14	3	4	5	16427471172
14	4	3	5	16427471172
14	5	2	5	6100939726
14	6	1	5	661320660
14	1	7	4	341825484
14	2	6	4	4286172247
14	3	5	4	16427471172
14	4	4	4	25199010256
14	5	3	4	16427471172
14	6	2	4	4286172247
14	7	1	4	341825484
14	1	8	3	87291204
14	2	7	3	1444432612
14	3	6	3	7523770016
14	4	5	3	16427471172
14	5	4	3	16427471172
14	6	3	3	7523770016
14	7	2	3	1444432612
14	8	1	3	87291204
14	1	9	2	9798789
14	2	8	2	210164227
14	3	7	2	1444432612
14	4	6	2	4286172247
14	5	5	2	6100939726
14	6	4	2	4286172247
14	7	3	2	1444432612
14	8	2	2	210164227
14	9	1	2	9798789
14	1	10	1	355355
14	2	9	1	9798789
14	3	8	1	87291204
14	4	7	1	341825484
14	5	6	1	661320660
14	6	5	1	661320660
14	7	4	1	341825484
14	8	3	1	87291204
14	9	2	1	9798789
14	10	1	1	355355
14			sum	206254571236

A.4 Genus 3

d	v	e	f	h
7	1	1	1	180
7			sum	180
8	1	1	2	3044
8	1	2	1	3044
8	2	1	1	3044
8			sum	9132
9	1	1	3	26060
9	1	2	2	63600
9	2	1	2	63600
9	1	3	1	26060
9	2	2	1	63600
9	3	1	1	26060
9			sum	268980
10	1	1	4	152900
10	1	2	3	659340
10	2	1	3	659340
10	1	3	2	659340
10	2	2	2	1595480
10	3	1	2	659340
10	1	4	1	152900
10	2	3	1	659340
10	3	2	1	659340
10	4	1	1	152900
10			sum	6010220
11	1	1	5	696905
11	1	2	4	4606910
11	2	1	4	4606910
11	1	3	3	8141100
11	2	2	3	19571123
11	3	1	3	8141100
11	1	4	2	4606910
11	2	3	2	19571123
11	3	2	2	19571123
11	4	1	2	4606910
11	1	5	1	696905
11	2	4	1	4606910
11	3	3	1	8141100
11	4	2	1	4606910
11	5	1	1	696905
11			sum	112868844
12	1	1	6	2641925
12	1	2	5	24656775
12	2	1	5	24656775
12	1	3	4	66805310
12	2	2	4	159762815
12	3	1	4	66805310
12	1	4	3	66805310
12	2	3	3	280514670
12	3	2	3	280514670
12	4	1	3	66805310
12	1	5	2	24656775
12	2	4	2	159762815
12	3	3	2	280514670
12	4	2	2	159762815
12	5	1	2	24656775
12	1	6	1	2641925
12	2	5	1	24656775
12	3	4	1	66805310
12	4	3	1	66805310
12	5	2	1	24656775
12	6	1	1	2641925
12			sum	1877530740
13	1	1	7	8691683
13	1	2	6	108452916
13	2	1	6	108452916
13	1	3	5	414918075
13	2	2	5	988043771
13	3	1	5	414918075
13	1	4	4	636184120
13	2	3	4	2646424729
13	3	2	4	2646424729
13	4	1	4	636184120
13	1	5	3	414918075
13	2	4	3	2646424729
13	3	3	3	4623070842
13	4	2	3	2646424729
13	5	1	3	414918075
13	1	6	2	108452916
13	2	5	2	988043771
13	3	4	2	2646424729
13	4	3	2	2646424729
13	5	2	2	988043771
13	6	1	2	108452916
13	1	7	1	8691683
13	2	6	1	108452916
13	3	5	1	414918075
13	4	4	1	636184120
13	5	3	1	414918075
13	6	2	1	108452916
13	7	1	1	8691683
13			sum	28540603884
14	1	1	8	25537655
14	1	2	7	409732895
14	2	1	7	409732895
14	1	3	6	2096068975
14	2	2	6	4973691275
14	3	1	6	2096068975
14	1	4	5	4538348815
14	2	3	5	18733893115
14	3	2	5	18733893115
14	4	1	5	4538348815
14	1	5	4	4538348815
14	2	4	4	28579309570
14	3	3	4	49719495672
14	4	2	4	28579309570
14	5	1	4	4538348815
14	1	6	3	2096068975
14	2	5	3	18733893115
14	3	4	3	49719495672
14	4	3	3	49719495672
14	5	2	3	18733893115
14	6	1	3	2096068975
14	1	7	2	409732895
14	2	6	2	4973691275
14	3	5	2	18733893115
14	4	4	2	28579309570
14	5	3	2	18733893115
14	6	2	2	4973691275
14	7	1	2	409732895
14	1	8	1	25537655
14	2	7	1	409732895
14	3	6	1	2096068975
14	4	5	1	4538348815
14	5	4	1	4538348815
14	6	3	1	2096068975
14	7	2	1	409732895
14	8	1	1	25537655
14			sum	404562365316

A.5 Genus 4

d	v	e	f	h
9	1	1	1	8064
9			sum	8064
10	1	1	2	193248
10	1	2	1	193248
10	2	1	1	193248
10			sum	579744
11	1	1	3	2286636
11	1	2	2	5458464
11	2	1	2	5458464
11	1	3	1	2286636
11	2	2	1	5458464
11	3	1	1	2286636
11			sum	23235300
12	1	1	4	18128396
12	1	2	3	75220860
12	2	1	3	75220860
12	1	3	2	75220860
12	2	2	2	178462816
12	3	1	2	75220860
12	1	4	1	18128396
12	2	3	1	75220860
12	3	2	1	75220860
12	4	1	1	18128396
12			sum	684173164
13	1	1	5	109425316
13	1	2	4	687238552
13	2	1	4	687238552
13	1	3	3	1194737544
13	2	2	3	2820651496
13	3	1	3	1194737544
13	1	4	2	687238552
13	2	3	2	2820651496
13	3	2	2	2820651496
13	4	1	2	687238552
13	1	5	1	109425316
13	2	4	1	687238552
13	3	3	1	1194737544
13	4	2	1	687238552
13	5	1	1	109425316
13			sum	16497874380
14	1	1	6	539651112
14	1	2	5	4736419688
14	2	1	5	4736419688
14	1	3	4	12465308856
14	2	2	4	29310854804
14	3	1	4	12465308856
14	1	4	3	12465308856
14	2	3	3	50713072144
14	3	2	3	50713072144
14	4	1	3	12465308856
14	1	5	2	4736419688
14	2	4	2	29310854804
14	3	3	2	50713072144
14	4	2	2	29310854804
14	5	1	2	4736419688
14	1	6	1	539651112
14	2	5	1	4736419688
14	3	4	1	12465308856
14	4	3	1	12465308856
14	5	2	1	4736419688
14	6	1	1	539651112
14			sum	344901105444

A.6 Genus 5

d	v	e	f	h
11	1	1	1	604800
11			sum	604800
12	1	1	2	19056960
12	1	2	1	19056960
12	2	1	1	19056960
12			sum	57170880
13	1	1	3	292271616
13	1	2	2	686597184
13	2	1	2	686597184
13	1	3	1	292271616
13	2	2	1	686597184
13	3	1	1	292271616
13			sum	2936606400
14	1	1	4	2961802480
14	1	2	3	11947069680
14	2	1	3	11947069680
14	1	3	2	11947069680
14	2	2	2	27934773440
14	3	1	2	11947069680
14	1	4	1	2961802480
14	2	3	1	11947069680
14	3	2	1	11947069680
14	4	1	1	2961802480
14			sum	108502598960

A.7 Genus 6

d	v	e	f	h
13	1	1	1	68428800
13			sum	68428800
14	1	1	2	2699672832
14	1	2	1	2699672832
14	2	1	1	2699672832
14			sum	8099018496

These tables extend to 14 darts the part of Appendix B of [24] about rooted hypermaps.

B First numbers of unrooted hypermaps

The following sections show the numbers H of unrooted hypermaps of genus g with d darts, v vertices, e edges and $d - v - e + 2(1 - g)$ faces, for $g \leq 6$ and $d \leq 14$.

B.1 Genus 0

d	v	e	f	H
1	1	1	1	1
1			sum	1
2	1	1	2	1
2	1	2	1	1
2	2	1	1	1
2			sum	3
3	1	1	3	1
3	1	2	2	1
3	2	1	2	1
3	1	3	1	1
3	2	2	1	1
3	3	1	1	1
3			sum	6
4	1	1	4	1
4	1	2	3	2
4	2	1	3	2
4	1	3	2	2
4	2	2	2	5
4	3	1	2	2
4	1	4	1	1
4	2	3	1	2
4	3	2	1	2
4	4	1	1	1
4			sum	20
5	1	1	5	1
5	1	2	4	2
5	2	1	4	2
5	1	3	3	4
5	2	2	3	11
5	3	1	3	4
5	1	4	2	2
5	2	3	2	11
5	3	2	2	11
5	4	1	2	2
5	1	5	1	1
5	2	4	1	2
5	3	3	1	4
5	4	2	1	2
5	5	1	1	1
5			sum	60
6	1	1	6	1
6	1	2	5	3
6	2	1	5	3
6	1	3	4	10
6	2	2	4	24
6	3	1	4	10
6	1	4	3	10
6	2	3	3	46
6	3	2	3	46
6	4	1	3	10
6	1	5	2	3
6	2	4	2	24
6	3	3	2	46
6	4	2	2	24
6	5	1	2	3
6	1	6	1	1
6	2	5	1	3
6	3	4	1	10
6	4	3	1	10
6	5	2	1	3
6	6	1	1	1
6			sum	291
7	1	1	7	1
7	1	2	6	3
7	2	1	6	3
7	1	3	5	15
7	2	2	5	40
7	3	1	5	15
7	1	4	4	25
7	2	3	4	127
7	3	2	4	127
7	4	1	4	25
7	1	5	3	15
7	2	4	3	127
7	3	3	3	242
7	4	2	3	127
7	5	1	3	15
7	1	6	2	3
7	2	5	2	40
7	3	4	2	127
7	4	3	2	127
7	5	2	2	40
7	6	1	2	3
7	1	7	1	1
7	2	6	1	3
7	3	5	1	15
7	4	4	1	25
7	5	3	1	15
7	6	2	1	3
7	7	1	1	1
7			sum	1310
8	1	1	8	1
8	1	2	7	4
8	2	1	7	4
8	1	3	6	26
8	2	2	6	67
8	3	1	6	26
8	1	4	5	64
8	2	3	5	309
8	3	2	5	309
8	4	1	5	64
8	1	5	4	64
8	2	4	4	505
8	3	3	4	946
8	4	2	4	505
8	5	1	4	64
8	1	6	3	26
8	2	5	3	309
8	3	4	3	946
8	4	3	3	946
8	5	2	3	309
8	6	1	3	26
8	1	7	2	4
8	2	6	2	67
8	3	5	2	309
8	4	4	2	505
8	5	3	2	309
8	6	2	2	67
8	7	1	2	4
8	1	8	1	1
8	2	7	1	4
8	3	6	1	26
8	4	5	1	64
8	5	4	1	64
8	6	3	1	26
8	7	2	1	4
8	8	1	1	1
8			sum	6975
9	1	1	9	1
9	1	2	8	4
9	2	1	8	4
9	1	3	7	38
9	2	2	7	98
9	3	1	7	38
9	1	4	6	132
9	2	3	6	640
9	3	2	6	640
9	4	1	6	132
9	1	5	5	196
9	2	4	5	1549
9	3	3	5	2890
9	4	2	5	1549
9	5	1	5	196
9	1	6	4	132
9	2	5	4	1549
9	3	4	4	4671
9	4	3	4	4671
9	5	2	4	1549
9	6	1	4	132
9	1	7	3	38
9	2	6	3	640
9	3	5	3	2890
9	4	4	3	4671
9	5	3	3	2890
9	6	2	3	640
9	7	1	3	38
9	1	8	2	4
9	2	7	2	98
9	3	6	2	640
9	4	5	2	1549
9	5	4	2	1549
9	6	3	2	640
9	7	2	2	98
9	8	1	2	4
9	1	9	1	1
9	2	8	1	4

9	3	7	1	38
9	4	6	1	132
9	5	5	1	196
9	6	4	1	132
9	7	3	1	38
9	8	2	1	4
9	9	1	1	1
9			sum	37746
10	1	1	10	1
10	1	2	9	5
10	2	1	9	5
10	1	3	8	56
10	2	2	8	144
10	3	1	8	56
10	1	4	7	256
10	2	3	7	1226
10	3	2	7	1226
10	4	1	7	256
10	1	5	6	536
10	2	4	6	4111
10	3	3	6	7606
10	4	2	6	4111
10	5	1	6	536
10	1	6	5	536
10	2	5	5	6081
10	3	4	5	18019
10	4	3	5	18019
10	5	2	5	6081
10	6	1	5	536
10	1	7	4	256
10	2	6	4	4111
10	3	5	4	18019
10	4	4	4	28852
10	5	3	4	18019
10	6	2	4	4111
10	7	1	4	256
10	1	8	3	56
10	2	7	3	1226
10	3	6	3	7606
10	4	5	3	18019
10	5	4	3	18019
10	6	3	3	7606
10	7	2	3	1226
10	8	1	3	56
10	1	9	2	5
10	2	8	2	144
10	3	7	2	1226
10	4	6	2	4111
10	5	5	2	6081
10	6	4	2	4111
10	7	3	2	1226
10	8	2	2	144
10	9	1	2	5
10	1	10	1	1
10	2	9	1	5
10	3	8	1	56
10	4	7	1	256
10	5	6	1	536
10	6	5	1	536
10	7	4	1	256
10	8	3	1	56
10	9	2	1	5
10	10	1	1	1
10			sum	215602
11	1	1	11	1
11	1	2	10	5
11	2	1	10	5
11	1	3	9	75
11	2	2	9	195
11	3	1	9	75
11	1	4	8	450
11	2	3	8	2154
11	3	2	8	2154
11	4	1	8	450
11	1	5	7	1260
11	2	4	7	9585
11	3	3	7	17664
11	4	2	7	9585
11	5	1	7	1260
11	1	6	6	1764
11	2	5	6	19691
11	3	4	6	57715
11	4	3	6	57715
11	5	2	6	19691
11	6	1	6	1764
11	1	7	5	1260
11	2	6	5	19691
11	3	5	5	84714
11	4	4	5	134750
11	5	3	5	84714
11	6	2	5	19691
11	7	1	5	1260
11	1	8	4	450
11	2	7	4	9585
11	3	6	4	57715
11	4	5	4	134750
11	5	4	4	134750
11	6	3	4	57715
11	7	2	4	9585
11	8	1	4	450
11	1	9	3	75
11	2	8	3	2154
11	3	7	3	17664
11	4	6	3	57715
11	5	5	3	84714
11	6	4	3	57715
11	7	3	3	17664
11	8	2	3	2154
11	9	1	3	75
11	1	10	2	5
11	2	9	2	195
11	3	8	2	2154
11	4	7	2	9585
11	5	6	2	19691
11	6	5	2	19691
11	7	4	2	9585
11	8	3	2	2154
11	9	2	2	195
11	10	1	2	5
11	1	11	1	1
11	2	10	1	5
11	3	9	1	75
11	4	8	1	450
11	5	7	1	1260
11	6	6	1	1764
11	7	5	1	1260
11	8	4	1	450
11	9	3	1	75
11	10	2	1	5
11	11	1	1	1
11			sum	1262874
12	1	1	12	1
12	1	2	11	6
12	2	1	11	6
12	1	3	10	104
12	2	2	10	265
12	3	1	10	104
12	1	4	9	765
12	2	3	9	3605
12	3	2	9	3605
12	4	1	9	765
12	1	5	8	2736
12	2	4	8	20472
12	3	3	8	37545
12	4	2	8	20472
12	5	1	8	2736
12	1	6	7	5102
12	2	5	7	55633
12	3	4	7	161455
12	4	3	7	161455
12	5	2	7	55633
12	6	1	7	5102
12	1	7	6	5102
12	2	6	6	77174
12	3	5	6	326432
12	4	4	6	516507
12	5	3	6	326432
12	6	2	6	77174
12	7	1	6	5102
12	1	8	5	2736
12	2	7	5	55633
12	3	6	5	326432
12	4	5	5	752940
12	5	4	5	752940
12	6	3	5	326432
12	7	2	5	55633
12	8	1	5	2736
12	1	9	4	765
12	2	8	4	20472
12	3	7	4	161455
12	4	6	4	516507
12	5	5	4	752940
12	6	4	4	516507
12	7	3	4	161455
12	8	2	4	20472
12	9	1	4	765
12	1	10	3	104
12	2	9	3	3605
12	3	8	3	37545
12	4	7	3	161455
12	5	6	3	326432
12	6	5	3	326432
12	7	4	3	161455
12	8	3	3	37545
12	9	2	3	3605
12	10	1	3	104
12	1	11	2	6
12	2	10	2	265
12	3	9	2	3605
12	4	8	2	20472
12	5	7	2	55633
12	6	6	2	77174
12	7	5	2	55633
12	8	4	2	20472
12	9	3	2	3605
12	10	2	2	265
12	11	1	2	6
12	1	12	1	1
12	2	11	1	6
12	3	10	1	104
12	4	9	1	765
12	5	8	1	2736
12	6	7	1	5102
12	7	6	1	5102

12	8	5	1	2736
12	9	4	1	765
12	10	3	1	104
12	11	2	1	6
12	12	1	1	1
12			sum	7611156
13	1	1	13	1
13	1	2	12	6
13	2	1	12	6
13	1	3	11	132
13	2	2	11	341
13	3	1	11	132
13	1	4	10	1210
13	2	3	10	5709
13	3	2	10	5709
13	4	1	10	1210
13	1	5	9	5445
13	2	4	9	40425
13	3	3	9	73920
13	4	2	9	40425
13	5	1	9	5445
13	1	6	8	13068
13	2	5	8	140591
13	3	4	8	404965
13	4	3	8	404965
13	5	2	8	140591
13	6	1	8	13068
13	1	7	7	17424
13	2	6	7	258194
13	3	5	7	1078456
13	4	4	7	1699200
13	5	3	7	1078456
13	6	2	7	258194
13	7	1	7	17424
13	1	8	6	13068
13	2	7	6	258194
13	3	6	6	1485778
13	4	5	6	3395140
13	5	4	6	3395140
13	6	3	6	1485778
13	7	2	6	258194
13	8	1	6	13068
13	1	9	5	5445
13	2	8	5	140591
13	3	7	5	1078456
13	4	6	5	3395140
13	5	5	5	4924094
13	6	4	5	3395140
13	7	3	5	1078456
13	8	2	5	140591
13	9	1	5	5445
13	1	10	4	1210
13	2	9	4	40425
13	3	8	4	404965
13	4	7	4	1699200
13	5	6	4	3395140
13	6	5	4	3395140
13	7	4	4	1699200
13	8	3	4	404965
13	9	2	4	40425
13	10	1	4	1210
13	1	11	3	132
13	2	10	3	5709
13	3	9	3	73920
13	4	8	3	404965
13	5	7	3	1078456
13	6	6	3	1485778
13	7	5	3	1078456
13	8	4	3	404965
13	9	3	3	73920
13	10	2	3	5709
13	11	1	3	132
13	1	12	2	6
13	2	11	2	341
13	3	10	2	5709
13	4	9	2	40425
13	5	8	2	140591
13	6	7	2	258194
13	7	6	2	258194
13	8	5	2	140591
13	9	4	2	40425
13	10	3	2	5709
13	11	2	2	341
13	12	1	2	6
13	1	13	1	1
13	2	12	1	6
13	3	11	1	132
13	4	10	1	1210
13	5	9	1	5445
13	6	8	1	13068
13	7	7	1	17424
13	8	6	1	13068
13	9	5	1	5445
13	10	4	1	1210
13	11	3	1	132
13	12	2	1	6
13	13	1	1	1
13			sum	46814132
14	1	1	14	1
14	1	2	13	7
14	2	1	13	7
14	1	3	12	172
14	2	2	12	440
14	3	1	12	172
14	1	4	11	1868
14	2	3	11	8741
14	3	2	11	8741
14	4	1	11	1868
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14	2	4	10	75283
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14	4	5	7	12995424
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14	6	3	7	5731330
14	7	2	7	1012271
14	8	1	7	52634
14	1	9	6	30711
14	2	8	6	764633
14	3	7	6	5731330
14	4	6	6	17809776
14	5	5	6	25720986
14	6	4	6	17809776
14	7	3	6	5731330
14	8	2	6	764633
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14	1	10	5	10247
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14	8	3	5	3159069
14	9	2	5	325652
14	10	1	5	10247
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14	4	8	4	4960016
14	5	7	4	12995424
14	6	6	4	17809776
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14	12	2	2	440
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14	3	12	1	172
14	4	11	1	1868
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14	6	9	1	30711
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14			sum	293447817

B.2 Genus 1

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4	2	1	1	2
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5	1	2	2	8
5	2	1	2	8
5	1	3	1	3
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5	3	1	1	3
5			sum	33
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6	2	1	3	31
6	1	3	2	31
6	2	2	2	78
6	3	1	2	31
6	1	4	1	7
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6			sum	285
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7	4	1	2	80
7	1	5	1	10
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7	3	3	1	150
7	4	2	1	80
7	5	1	1	10
7			sum	2115
8	1	1	6	17
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8	4	1	3	557
8	1	5	2	187
8	2	4	2	1409
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8	4	2	2	1409
8	5	1	2	187
8	1	6	1	17
8	2	5	1	187
8	3	4	1	557
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8	5	2	1	187
8	6	1	1	17
8			sum	16533
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9	1	2	6	374
9	2	1	6	374
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9	2	2	5	4115
9	3	1	5	1634
9	1	4	4	2616
9	2	3	4	12033
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9	3	5	1	1634
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9			sum	126501
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11	1	2	8	1200
11	2	1	8	1200
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11	3	1	7	9450
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11	3	2	6	135775
11	4	1	6	30240
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11	3	3	5	560498
11	4	2	5	310985
11	5	1	5	44100
11	1	6	4	30240
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11	3	4	4	880403
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11	6	1	4	30240
11	1	7	3	9450
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11	3	5	3	560498
11	4	4	3	880403
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11	6	2	3	135775
11	7	1	3	9450
11	1	8	2	1200
11	2	7	2	23547
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11	5	4	2	310985
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11	1	9	1	45
11	2	8	1	1200
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11	7	3	1	9450
11	8	2	1	1200
11	9	1	1	45
11			sum	7451679
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12	1	2	9	1976
12	2	1	9	1976
12	1	3	8	19694
12	2	2	8	48846
12	3	1	8	19694
12	1	4	7	82652
12	2	3	7	367645

12	3	2	7	367645
12	4	1	7	82652
12	1	5	6	165262
12	2	4	6	1147628
12	3	3	6	2058329
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12	5	2	5	1660331
12	6	1	5	165262
12	1	7	4	82652
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12	7	1	4	82652
12	1	8	3	19694
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12	6	3	3	2058329
12	7	2	3	367645
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12	1	9	2	1976
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12	4	6	2	1147628
12	5	5	2	1660331
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12	7	3	2	367645
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12	6	5	1	165262
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12	8	3	1	19694
12	9	2	1	1976
12	10	1	1	62
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13	1	3	9	38115
13	2	2	9	94281
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13	3	3	7	6542368
13	4	2	7	3661896
13	5	1	7	533610
13	1	6	6	731808
13	2	5	6	7221592
13	3	4	6	20047636
13	4	3	6	20047636
13	5	2	6	7221592
13	6	1	6	731808
13	1	7	5	533610
13	2	6	5	7221592
13	3	5	5	28831218
13	4	4	5	44800675
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13	6	3	4	20047636
13	7	2	4	3661896
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13	6	4	3	20047636
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13	2	9	2	94281
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14	3	2	9	2020530
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14	2	4	8	10419653
14	3	3	8	18554641
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14	5	2	7	26837442
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14	7	2	5	26837442
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14	2	8	4	10419653
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14	7	3	4	73967488
14	8	2	4	10419653
14	9	1	4	460245
14	1	10	3	69765
14	2	9	3	2020530
14	3	8	3	18554641
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14	5	6	3	144298902
14	6	5	3	144298902
14	7	4	3	73967488
14	8	3	3	18554641
14	9	2	3	2020530
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14	11	1	2	4659
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14	2	11	1	4659
14	3	10	1	69765
14	4	9	1	460245
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14	6	7	1	2760990
14	7	6	1	2760990
14	8	5	1	1533950
14	9	4	1	460245
14	10	3	1	69765
14	11	2	1	4659
14	12	1	1	99
14			sum	3369276867

B.3 Genus 2

d	v	e	f	H
5	1	1	1	4
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6	1	1	2	16
6	1	2	1	16
6	2	1	1	16
6			sum	48
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7	1	2	2	169
7	2	1	2	169
7	1	3	1	67
7	2	2	1	169
7	3	1	1	67
7			sum	708
8	1	1	4	237
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8	2	1	3	1072
8	1	3	2	1072
8	2	2	2	2664
8	3	1	2	1072
8	1	4	1	237
8	2	3	1	1072
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8	4	1	1	237
8			sum	9807
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9	1	3	3	8560
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11	1	4	4	315000
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12	4	3	3	18938994
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12	5	4	1	1610617
12	6	3	1	721382
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13	4	1	6	6764142
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13	3	4	4	172252340
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13	5	4	2	63458654
13	6	3	2	28779051
13	7	2	2	5412883
13	8	1	2	316470
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13	6	4	1	6764142
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13	8	2	1	316470
13	9	1	1	14091
13			sum	1506841872
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14	1	2	9	700045
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14	1	3	8	6235526
14	2	2	8	15012496
14	3	1	8	6235526
14	1	4	7	24417030
14	2	3	7	103175785

14	3	2	7	103175785
14	4	1	7	24417030
14	1	5	6	47238510
14	2	4	6	306159286
14	3	3	6	537417269
14	4	2	6	306159286
14	5	1	6	47238510
14	1	6	5	47238510
14	2	5	5	435785878
14	3	4	5	1173398706
14	4	3	5	1173398706
14	5	2	5	435785878
14	6	1	5	47238510
14	1	7	4	24417030
14	2	6	4	306159286
14	3	5	4	1173398706
14	4	4	4	1799940644
14	5	3	4	1173398706
14	6	2	4	306159286
14	7	1	4	24417030
14	1	8	3	6235526
14	2	7	3	103175785
14	3	6	3	537417269
14	4	5	3	1173398706
14	5	4	3	1173398706
14	6	3	3	537417269
14	7	2	3	103175785
14	8	1	3	6235526
14	1	9	2	700045
14	2	8	2	15012496
14	3	7	2	103175785
14	4	6	2	306159286
14	5	5	2	435785878
14	6	4	2	306159286
14	7	3	2	103175785
14	8	2	2	15012496
14	9	1	2	700045
14	1	10	1	25405
14	2	9	1	700045
14	3	8	1	6235526
14	4	7	1	24417030
14	5	6	1	47238510
14	6	5	1	47238510
14	7	4	1	24417030
14	8	3	1	6235526
14	9	2	1	700045
14	10	1	1	25405
14			sum	14732613116

B.4 Genus 3

d	v	e	f	H
7	1	1	1	30
7			sum	30
8	1	1	2	385
8	1	2	1	385
8	2	1	1	385
8			sum	1155
9	1	1	3	2900
9	1	2	2	7070
9	2	1	2	7070
9	1	3	1	2900
9	2	2	1	7070
9	3	1	1	2900
9			sum	29910
10	1	1	4	15308
10	1	2	3	65972
10	2	1	3	65972
10	1	3	2	65972
10	2	2	2	159608
10	3	1	2	65972
10	1	4	1	15308
10	2	3	1	65972
10	3	2	1	65972
10	4	1	1	15308
10			sum	601364
11	1	1	5	63355
11	1	2	4	418810
11	2	1	4	418810
11	1	3	3	740100
11	2	2	3	1779193
11	3	1	3	740100
11	1	4	2	418810
11	2	3	2	1779193
11	3	2	2	1779193
11	4	1	2	418810
11	1	5	1	63355
11	2	4	1	418810
11	3	3	1	740100
11	4	2	1	418810
11	5	1	1	63355
11			sum	10260804
12	1	1	6	220244
12	1	2	5	2054974
12	2	1	5	2054974
12	1	3	4	5567550
12	2	2	4	13314231
12	3	1	4	5567550
12	1	4	3	5567550
12	2	3	3	23377106
12	3	2	3	23377106
12	4	1	3	5567550
12	1	5	2	2054974
12	2	4	2	13314231
12	3	3	2	23377106
12	4	2	2	13314231
12	5	1	2	2054974
12	1	6	1	220244
12	2	5	1	2054974
12	3	4	1	5567550
12	4	3	1	5567550
12	5	2	1	2054974
12	6	1	1	220244
12			sum	156469887
13	1	1	7	668591
13	1	2	6	8342532
13	2	1	6	8342532
13	1	3	5	31916775
13	2	2	5	76003367
13	3	1	5	31916775
13	1	4	4	48937240
13	2	3	4	203571133
13	3	2	4	203571133
13	4	1	4	48937240
13	1	5	3	31916775
13	2	4	3	203571133
13	3	3	3	355620834
13	4	2	3	203571133
13	5	1	3	31916775
13	1	6	2	8342532
13	2	5	2	76003367
13	3	4	2	203571133
13	4	3	2	203571133
13	5	2	2	76003367
13	6	1	2	8342532
13	1	7	1	668591
13	2	6	1	8342532
13	3	5	1	31916775
13	4	4	1	48937240
13	5	3	1	31916775
13	6	2	1	8342532
13	7	1	1	668591
13			sum	2195431068
14	1	1	8	1824323
14	1	2	7	29267487
14	2	1	7	29267487
14	1	3	6	149721473
14	2	2	6	355267058
14	3	1	6	149721473
14	1	4	5	324171185
14	2	3	5	1338142324
14	3	2	5	1338142324
14	4	1	5	324171185
14	1	5	4	324171185
14	2	4	4	2041388556
14	3	3	4	3551405485
14	4	2	4	2041388556
14	5	1	4	324171185
14	1	6	3	149721473
14	2	5	3	1338142324
14	3	4	3	3551405485
14	4	3	3	3551405485
14	5	2	3	1338142324
14	6	1	3	149721473
14	1	7	2	29267487
14	2	6	2	355267058
14	3	5	2	1338142324
14	4	4	2	2041388556
14	5	3	2	1338142324
14	6	2	2	355267058
14	7	1	2	29267487
14	1	8	1	1824323
14	2	7	1	29267487
14	3	6	1	149721473
14	4	5	1	324171185
14	5	4	1	324171185
14	6	3	1	149721473
14	7	2	1	29267487
14	8	1	1	1824323
14			sum	28897471080

B.5 Genus 4

d	v	e	f	H
9	1	1	1	900
9			sum	900
10	1	1	2	19344
10	1	2	1	19344
10	2	1	1	19344
10			sum	58032
11	1	1	3	207876
11	1	2	2	496224
11	2	1	2	496224
11	1	3	1	207876
11	2	2	1	496224
11	3	1	1	207876
11			sum	2112300
12	1	1	4	1510846
12	1	2	3	6268712
12	2	1	3	6268712
12	1	3	2	6268712
12	2	2	2	14872428
12	3	1	2	6268712
12	1	4	1	1510846
12	2	3	1	6268712
12	3	2	1	6268712
12	4	1	1	1510846
12			sum	57017238
13	1	1	5	8417332
13	1	2	4	52864504
13	2	1	4	52864504
13	1	3	3	91902888
13	2	2	3	216973192
13	3	1	3	91902888
13	1	4	2	52864504
13	2	3	2	216973192
13	3	2	2	216973192
13	4	1	2	52864504
13	1	5	1	8417332
13	2	4	1	52864504
13	3	3	1	91902888
13	4	2	1	52864504
13	5	1	1	8417332
13			sum	1269067260
14	1	1	6	38547144
14	1	2	5	338317960
14	2	1	5	338317960
14	1	3	4	890383128
14	2	2	4	2093639428
14	3	1	4	890383128
14	1	4	3	890383128
14	2	3	3	3622371084
14	3	2	3	3622371084
14	4	1	3	890383128
14	1	5	2	338317960
14	2	4	2	2093639428
14	3	3	2	3622371084
14	4	2	2	2093639428
14	5	1	2	338317960
14	1	6	1	38547144
14	2	5	1	338317960
14	3	4	1	890383128
14	4	3	1	890383128
14	5	2	1	338317960
14	6	1	1	38547144
14			sum	24635879496

B.6 Genus 5

d	v	e	f	H
11	1	1	1	54990
11			sum	54990
12	1	1	2	1588218
12	1	2	1	1588218
12	2	1	1	1588218
12			sum	4764654
13	1	1	3	22482432
13	1	2	2	52815168
13	2	1	2	52815168
13	1	3	1	22482432
13	2	2	1	52815168
13	3	1	1	22482432
13			sum	225892800
14	1	1	4	211558928
14	1	2	3	853365360
14	2	1	3	853365360
14	1	3	2	853365360
14	2	2	2	1995345826
14	3	1	2	853365360
14	1	4	1	211558928
14	2	3	1	853365360
14	3	2	1	853365360
14	4	1	1	211558928
14			sum	7750214770

B.7 Genus 6

d	v	e	f	H
13	1	1	1	5263764
13			sum	5263764
14	1	1	2	192834612
14	1	2	1	192834612
14	2	1	1	192834612
14			sum	578503836