Reduced order controller design for Timoshenko beam: A port Hamiltonian approach ⋆

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Abstract: This paper deals with the structure and passivity preserving model reduction and the reduced order controller design for a class of distributed controlled port Hamiltonian systems – Timoshenko beam. The boundary conditions of the beam lead to physical constraints which are hardly considered in the reduction procedure. In this work we propose to use the descriptor system realization of port Hamiltonian system to conserve the physical constraints. A passive LQG control design method is proposed for this type of system. This LQG method defines a balanced coordinate which allows us to reduce the system. Using the obtained reduced model, a reduced order passive controller which stabilizes the full order system is designed using the LQG method. At last we give the numerical simulations to show the effectiveness of the proposed reduced passive controller.

Keywords: Port Hamiltonian systems, distributed control, Hyperbolic PDEs, LQG method, passivity preserving reduction, reduced order control design

1. INTRODUCTION

Port Hamiltonian approach is a very powerful framework for modeling and control of large class of mechanical, electro-mechanical and multi physical systems (Duindam et al., 2009). This approach has been generalized to the distributed parameters systems which are described by partial differential equations (PDEs). However the modeling of complex or multi-physical systems leads to the high dimensional models or even the infinite dimensional (distributed parameters systems). Thus for the simulation or control design objective, it is necessary to consider the reduction of such kind of systems. The reduction or approximation of the port Hamiltonian systems in the finite dimensional case (Polyuga and van der Schaft, 2011, 2012) and infinite dimensional case (Golo et al., 2004; Baaiu et al., 2009; Moulla et al., 2011) have been proposed. These methods have the advantage to preserve the passivity and Hamiltonian structure in the reduced order system. However, all these methods only consider the open loop behaviors of the system and cannot be applied on a large class of distributed parameters systems – the power preserving systems which described by Hyperbolic partial differential equations because all the eigenvalues are on the imaginary axis. Hence the states variables have the same weight, it is difficult to find a reduction model for the control design objective. Secondly modeling of the multi-physical systems often leads to the algebraic physical constraints. The reduction methods presented before have not considered this issue.

The model reduction of port Hamiltonian systems considering the control design problem is firstly introduced in (Wu et al., 2014b). The authors proposed a modified LQG balanced reduction method to relate the LQG control design problem and the balanced reduction problem together. This method provide a reduction scheme to achieve the reduced order controller which stabilizes the full order system. The Hamiltonian structure is also preserved in the closed loop system. The reduction of the constrained port Hamiltonian system is introduced in (Wu et al., 2014a). The descriptor system framework has been used to present the constrained system and the authors propose a reduction scheme which not only reduce the system but also conserve the physical constraints of the original systems.

In this paper, we are interested in the model reduction and reduced order control design for the distributed controlled infinite dimensional port Hamiltonian systems. We develop a power preserving closed loop reduction scheme to derive a reduced order controller. The treated physical

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constraints in this paper are mainly due to the boundary interconnection relations of distributed port Hamiltonian system. These constraints are preserved in the reduction scheme using descriptor formulation.

This paper is organized as follows. In Section 2 we introduce port Hamiltonian formulation of the distributed controlled Timoshenko beam which interconnect with a finite dimensional mechanical system. Next the model reduction scheme and the reduced order control design method is discussed in Section 3. A numerical example is presented and the simulation results are illustrated to show the effectiveness of the proposed method in the Section 4. At last, we give some final remarks and introduce the future work in Section 5.

2. PORT HAMILTONIAN MODELING OF TIMOSHENKO BEAM

In this section, we introduce the modeling of the mechanical system shown in Figure 1. The beam is actuated by the distributed actuators cling on the beam. One side is clamped and other side is interconnected with the manipulated object which can be simplified to a mass-spring-damper system. The modeling can be separated to two parts. The distributed actuated mechanical beam and the mass-spring-damper system.

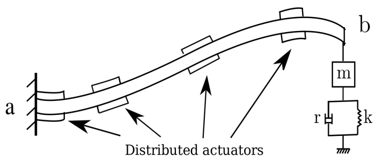

Fig. 1. Distributed controlled Timoshenko beam

2.1 Timoshenko beam with boundary and distributed ports

Let first consider Timoshenko beam described as a boundary controlled port Hamiltonian system (Macchelli and Melchiorri, 2004; Jacob and Zwart, 2012):

$$\dot{x} = \underbrace{(P_1 \frac{\partial}{\partial z} + P_0)}_{\mathcal{J}} \mathcal{L}x \tag{1}$$

with the operator and matrices:

$$\mathcal{L} = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

with the state (energy) variables: the shear displacement $x_1 = \frac{\partial w}{\partial z}(z,t) - \phi(z,t)$, the transverse momentum distribution $x_2 = \rho(z)\frac{\partial w}{\partial t}(z,t)$, the angular displacement $x_3 = \frac{\partial \phi}{\partial z}(z,t)$ and the angular momentum distribution $x_4 = I_\rho \frac{\partial \phi}{\partial t}(z,t)$ for $z \in (a,b)$, $t \geq 0$, where $w(z,t)$ is the transverse displacement and $\phi(z,t)$ is the rotation angle of the beam. The coefficients ρ, I_ρ, E, I and K are the mass per unit length, the angular moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively, and the state space $X = L_2(a;b;\mathbb{R}^4)$. The operator $\mathcal{J} = P_1 \frac{\partial}{\partial z} + P_0$ defined by the matrices $P_1 = P_1^T$ and $P_0 = -P_0^T$ is a first order skew symmetric differential operator acting on the state space X. The energy of the beam is expressed in terms of the energy variables,

$$\begin{aligned} H &= \frac{1}{2}\int_a^b (Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2)dz \\ &= \frac{1}{2}\int_a^b x(z)^T(\mathcal{L}x)(z)dz = \frac{1}{2} \parallel x \parallel_{\mathcal{L}}^2 \end{aligned} \tag{3}$$

In order to define an extended Dirac structure including the boundary (Le Gorrec et al., 2005), the boundary variables are desired by using integration by part:

$$\begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix} = \begin{bmatrix} (\rho^{-1}x_2)(b) - (\rho^{-1}x_2)(a) \\ (Kx_1)(b) - (Kx_1)(a) \\ (I_\rho^{-1}x_4)(b) - (I_\rho^{-1}x_4)(a) \\ (EIx_3)(b) - (EIx_3)(a) \\ (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\ (Kx_1)(b) + (Kx_1)(a) \\ (I_\rho^{-1}x_4)(b) + (I_\rho^{-1}x_4)(a) \\ (EIx_3)(b) + (EIx_3)(a) \end{bmatrix} = \begin{bmatrix} v(b) - v(a) \\ F(b) - F(a) \\ w(b) - w(a) \\ T(b) - T(a) \\ v(b) + v(a) \\ F(b) + F(a) \\ w(b) + w(a) \\ T(b) + T(a) \end{bmatrix} \tag{4}$$

where $F(z), T(z), v(z), w(z)$ are the force, torque, velocity and angular velocity at z point respectively. In additional, we consider some distributed port defined by distributed torques acting on the beam. With the distributed port $\begin{pmatrix} f_{d,\mathcal{L}x} \\ e_{d,\mathcal{L}x} \end{pmatrix}$, the system becomes:

$$\begin{aligned} \dot{x} &= \mathcal{J}\mathcal{L}x + \mathcal{B}e_{d,\mathcal{L}x} \\ f_{d,\mathcal{L}x} &= \mathcal{B}^*\mathcal{L}x \end{aligned} \tag{5}$$

where the $\mathcal{B} : \mathbb{C}^i \mapsto X$ is the distributed input map, $e_{d,\mathcal{L}x} \in \mathbb{C}^i$ are the distributed torques applied on the beam, $f_{d,\mathcal{L}x} \in \mathbb{C}^i$ are the power conjugated variables of $e_{d,\mathcal{L}x}$, i.e. the angular velocities.

The control objective of system is to control the translation and angular positions of the object connected at the b side of the beam. The beam and the object that we want to manipulate are interconnected by the power conserved manner at the side b. The translation and angular velocity at the side a of the beam are zero because we assume that the beam is clamped at this side. It can be considered as a physical constraints of the system, their power conjugated variables are the reaction force and torque. In order to define the interconnection of the beam and the manipulated object at b side we define the input and output variables by the boundary ports as following:

$$u_b = W \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, \; y_b = \tilde{W} \begin{bmatrix} f_{\partial,\mathcal{L}x} \\ e_{\partial,\mathcal{L}x} \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned} W &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \\ \tilde{W} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

Using this partition the input and output boundary port variables are explicitly given as follows:

$$\begin{aligned} u_b &= \left[\, v(b) \;\; w(b) \;\; F(a) \;\; T(a) \,\right]^T = \left[\, u_{b1} \;\; u_{b2} \,\right]^T, \\ y_b &= \left[\, F(b) \;\; T(b) \;\; -v(a) \;\; -w(a) \,\right]^T = \left[\, y_{b1} \;\; y_{b2} \,\right]^T. \end{aligned} \tag{8}$$

As the beam is clamped an the side a, then the velocity and the angular velocity are zero.

To control the angular position of the beam, we use an active material such as electric active polymer cling on surface of the beam as shown in Fig 1. The port Hamiltonian formulation of the distributed actuator, electric active polymer, can be found in (Nishida et al., 2011). In this paper, we don't consider the physical model of the distributed actuator and apply directly the distributed torques given by actuator over the domain of the beam. The distributed input variables are the distributed torques: $b_i(z)u_{di}(t)$ on the $i-$th small intervals $I_{bi} = [a_i, b_i]$ of the spatial space $[a, b]$, i.e. $b_i(z) = 1$ if $z \in I_{bi}$ and $b_i(z) = 0$ elsewhere. As output, we consider the angular velocity mean values in the same intervals $f_{di} = y_{di} = \int_a^b b_i(z)\frac{1}{I_\rho}x_4 dz$. As consequence the distributed input is:

$$\mathcal{B}e_{d,\mathcal{L}x} = \sum_i \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_i(z) \end{bmatrix} u_{di}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b(z) \end{bmatrix} u_d(t) \tag{9}$$

where $\mathcal{B} : \mathbb{C}^i \mapsto X$, $b(z) = [b_1(z), \cdots, b_i(z), \cdots]$ and $u_d(z) = [u_{d1}(z), \cdots, u_{di}(z), \cdots]$. The output is the power conjugated variable of the input, i.e.,

$$y_d = \mathcal{B}^* \mathcal{L}x \tag{10}$$

The energy balance equation is defined as:

$$\frac{\partial H}{\partial t} = y_b^T u_b + y_d^T u_d. \tag{11}$$

2.2 Mechanical model of mass-spring-damper system

In this work, we first simplify the manipulated object to an ideal mass-spring-damper system and thus admits a port Hamiltonian presentation. We use the sub-index o to present the manipulated object. Then we can write:

$$\begin{aligned} \dot{x}_o &= (J_o - R_o)\frac{\partial H_o}{\partial x_o} + B_o u_o \\ y_o &= B_o^T \frac{\partial H_o}{\partial x_o} \end{aligned} \tag{12}$$

where the state variables $x_o = [q_{o1}, q_{o2}, p_{o1}, p_{o2}]$ are traversal and angular displacements and momentum respectively. $J_o^T = -J_o \in \mathbb{R}^{4\times4}$, $R_o^T = R_o \geq 0 \in \mathbb{R}^{4\times4}$ are the interconnection and dissipation matrices defined as,

$$J_o = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, R_o = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix} \tag{13}$$

with I is the identity matrix with appropriate dimension and $r = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \in \mathbb{R}^{2\times2}$, where $r_1, r_2 \in \mathbb{R}^+$ are the scalars the traversal and angular translation damping coefficients respectively. The Hamiltonian of the system is given by the kinetic and potential energy:

$$H_o = \frac{1}{2}\left(k_1 q_{o1}^2 + k_2 q_{o2}^2 + \frac{p_{o1}^2}{m_o} + \frac{p_{o2}^2}{m_I}\right) \tag{14}$$

where k_1, k_2 are the translational and rotational spring coefficients respectively and m_o , m_I are the mass and moment of inertia respectively. To manipulate the object, we can only apply the transversal force and torque on the contact point with beam, i.e. point b. Hence the input matrix is $B_o = [0, I]^T \in \mathbb{R}^{4\times2}$ and input $u_o \in \mathbb{R}^2$ can be identified by the boundary variables of the beam on the point b:

$$u_o = -\begin{bmatrix} \frac{\partial H}{\partial x_1}(b) \\ \frac{\partial H}{\partial x_3}(b) \end{bmatrix} = -\begin{bmatrix} F(b) \\ T(b) \end{bmatrix} \tag{15}$$

The outputs are the transversal and angular velocity of the mass on the point b which correspond to the input of the beam on the same point b. Thus the port Hamiltonian representation of this finite dimensional system is given as:

$$\begin{aligned} \dot{x}_o &= \begin{bmatrix} 0 & I \\ -I & -r \end{bmatrix} \frac{\partial H_o}{\partial x_o} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_o \\ y_o &= [\,0 \;\; I\,] \frac{\partial H_o}{\partial x_o} \end{aligned}. \tag{16}$$

3. REDUCTION AND CONTROL DESIGN

In this part, we shall consider a closed loop reduction of the system for the control design. To do so, we propose to use the LQG method because this method can relate reduction problem and the control design problem together by balanced method. However, the LQG problem of distributed paramaters systems let us write two operator Riccati equations (filter and control) , which are difficult to solve in the infinite dimensional case. Hence we shall use a spatial discretization of the infinite dimensional port Hamiltonian system of Timoshenko beam in the power conserving way to get a finite dimensional port Hamiltonian approximation in order to solve two operator Riccati equations.

3.1 Power preserving discretization of Timoshenko beam

We use the mixed-finite element discretization method proposed in (Golo et al., 2004). The idea of this method is to approximate flows and efforts with differential forms related to their physical (geometrical) natural. In the case of the Timoshenko beam, defined on a one-dimensional spatial domain, The efforts (torque) correspond to the zero forms (functions) and the flows (angular velocities) correspond to the one forms respectively. This spatial discretization method has been used on the different physical models, the reader can read (Hamroun et al., 2009; Baaiu et al., 2009) for more details and particularly in the Timoshenko beam case can be find in (Macchelli et al., 2009). The explicit finite dimensional port Hamiltonian discretization of the Timoshenko beam is given as follow:

$$\dot{x}_d = J_d \frac{\partial H_d}{\partial x_d} + B_d u \;; \quad y = B_d^T \frac{\partial H_d}{\partial x_d} \tag{17}$$

where $J_d = -J_d^T \in \mathbb{R}^{4N}$ with N infinitesimal subsections for the discretization, H_d is the Hamiltonian function. The following matrices presents the discretized structure operator of the infinite dimensional model :

$$\begin{aligned} J_d &= \underbrace{\begin{bmatrix} 0 & M & 0 & 0 \\ M^T & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & M^T & 0 \end{bmatrix}}_{\bar{P}_1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -\Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi^T & 0 & 0 & 0 \end{bmatrix}}_{\bar{P}_2} \\ B_d &= \begin{bmatrix} 0 & 0 & B_1 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & B_2 & 0 & 0 \end{bmatrix} \end{aligned} \tag{18}$$

where the sub-matrices are:

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{bmatrix} \quad \text{with} \quad M \in \mathbb{R}^{N\times N} \tag{19}$$

$$\Phi = diag(\beta, \cdots, \beta) \ \text{ with } \ \Phi \in \mathbb{R}^{N\times N} \tag{20}$$

$$B_1 = \begin{bmatrix} 0 \\ 0_{N-2} \\ 1 \end{bmatrix} \ \text{ and } \ B_2 = \begin{bmatrix} -1 \\ 0_{N-2} \\ 0 \end{bmatrix} \tag{21}$$

where β is the distance of the infinitesimal section.

The inputs and outputs of the system are the velocities in translation v and rotation ω as well as the forces F and torques T at the boundaries a and b shown in the equation (8).

The Timoshenko beam model and the manipulated object presented in the last section are interconnected by the boundary port at the point b using the following power conserving interconnection relations:

$$u_o = -y_{b1} = -\begin{bmatrix} F(b) \\ T(b) \end{bmatrix} \ \text{and} \ u_{b1} = y_o = \begin{bmatrix} v(b) \\ \omega(b) \end{bmatrix} \tag{22}$$

At the other side of the beam, i.e. point a, the beam is clamped, thus the translation and angular velocities are zero, i.e.

$$0 = y_{b2} = [\, -v(a), \ -w(a) \,]^T \tag{23}$$

Thus the ensemble of sub-systems can be represented as:

$$\dot{\tilde{x}} = \underbrace{\begin{bmatrix} J_d & -B_{d1}B_o^T \\ B_oB_{d1}^T & J_o - R_o \end{bmatrix}}_{(J-R)} \frac{\partial \tilde{H}}{\partial \tilde{x}} + \begin{bmatrix} B_{d2} \\ 0 \end{bmatrix} u_{b2} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_d \tag{24}$$

$$y_{b2} = \begin{bmatrix} B_{d2}^T & 0 \end{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{x}} = 0 \tag{25}$$

$$y_d = \begin{bmatrix} B^T & 0 \end{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{x}} \tag{26}$$

where $\tilde{x} = [x_d, x_o]^T \in \mathbb{R}^N$ and $\tilde{H} = H_d + H_o$ are the state variables and the total energy of the ensemble of two systems respectively.

One can observe that the system given by (24)-(26) is a port Hamiltonian systems with the constraints defined on Dirac structures (van der Schaft and Maschke, 1995; Dalsmo and van der Schaft, 1999; Duindam et al., 2009, chap.2) which shows explicitly the constraint equations as well as the associated Lagrangian multipliers. In this system, the output y_{b2} which are the translation and anglar velocities are the physical constraints, and their power conjugated input $_{b2}$ are the Lagrangian multipliers. In the linear case, the reduction of this type of system has been considered in (Wu et al., 2014a) by using the Lyapunov balanced reduction of the descriptor system where the system is reformulated as a descriptor system using particular coordinate transform. Then the reduction model is achieved by the Lyapunov balanced reduction method.

Now we will consider another reduction scheme which can take the control design problem into account during the reduction procedure. The first study of this method for port Hamiltonian system can be found in (Wu et al., 2014b). The authors introduced a model reduction and the reduced order passive control design method for the high order port Hamiltonian systems by using the LQG balanced method. Now we will discuss how to reduce the system given in (24)-(26) for the control design objective.

As the system is linear, thus the Hamiltonian of the system H has a quadratic presentation, i.e.

$$\tilde{H} = \frac{1}{2}\tilde{x}^T Q \tilde{x}.$$

We can transform the system (24)-(26) to its canonical Weierstrass form of descriptor system by using the scheme proposed by (Wu et al., 2014a, Eq. 16). To do so, we shall first eliminate the Lagranger multipliers (u_{b2} in our case) Then the system becomes:

$$\begin{cases} \hat{S}\dot{z} \ = \ \hat{E}\hat{Q}\, z + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u \\ \quad y \ = \ \hat{B}^T \hat{Q}\, z \end{cases} \tag{27}$$

where

$$\hat{S} = \begin{bmatrix} I_{N-k} & 0 \\ 0 & 0 \end{bmatrix}; \quad \hat{E} = \begin{bmatrix} \bar{J}_{11} - \bar{R}_{11} & 0 \\ 0 & I_k \end{bmatrix}; \quad \hat{Q} = \begin{bmatrix} Q_s & 0 \\ 0 & I_k \end{bmatrix} \tag{28}$$

Because the system (24)-(26) has two output constraints, that means $k = 2$.

By using this canonical form, the constrained port Hamiltonian system can be separated to two part, slow and fast dynamical sub-systems (Dai, 1989). The fast dynamical part is only the physical constraints of the system, hence we consider only the reduction of the slow dynamical part while conserving the constraints (fast dynamical part) along the reduction.

3.2 LQG control design and reduction

In this part we will use the LQG balancing reduction method (Wu et al., 2014b) to get a reduced order system also a reduced order controller. To do so, we recall the LQG control design method for the beam which is the combination of a Kalman filter and a static state feedback shown in Figure 2.

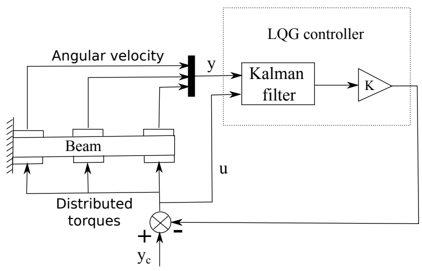

Fig. 2. LQG control design for beam

The LQG controller can be reformulated as:

$$\begin{cases} \ \dot{\hat{z}} \ = \ [(\bar{J}_{11} - \bar{R}_{11})Q_s - \bar{B}_1 K - F\bar{B}_1^T Q_s]\hat{z} + F u_c \\ y_c \ = \ K\hat{z} \end{cases} \tag{29}$$

with $\hat{z}$ is the state variables of the LQG controller and

$$K = \tilde{R}^{-1}\bar{B}_1^T P_c \text{ and } F = P_f Q_s \bar{B}_1 R_w^{-1} \tag{30}$$

The filter and static state feedback gains are obtained by the solutions of the following filter and control Riccati equations:

$$\begin{array}{c}(\bar{J}_{11}-\bar{R}_{11})\bar{Q}_s P_f + P_f \bar{Q}_s(\bar{J}_{11}-\bar{R}_{11})^T \\ -P_f\bar{Q}_s\bar{B}_1 R_w^{-1}\bar{B}_1^T Q_s P_f + Q_v = 0\end{array} \quad (31)$$

$$\begin{array}{c}Q_s(\bar{J}_{11}-\bar{R}_{11})^T P_c + P_c(\bar{J}_{11}-\bar{R}_{11})Q_s \\ -P_c\bar{B}_1\tilde{R}^{-1}\bar{B}_1^T P_c + \tilde{Q} = 0\end{array} \quad (32)$$

where Q_v and R_w are the covariance matrices of the state and output measurement white noises, $P_f = P_f^T > 0$ is the unique solution of the Riccati equation. The matrices $\tilde{Q} = \tilde{Q}^T > 0$ and $\tilde{R} = \tilde{R}^T > 0$ are the weighting matrices of optimal control problem consisting the following cost function:

$$J_c = \lim_{T\to\infty}\left[\int_0^T (z_1^T\tilde{Q}z_1 + u^T\tilde{R}u)dt\right]. \quad (33)$$

However, the LQG controller (29) is not passive and the Hamiltonian structure can not preserved in the closed loop system in general. But we can always reformulate the LQG controller under the port Hamiltonian realization if the weighting matrices and the covariance matrices are chosen in the following way:

Theorem 1. (Q-conjugated LQG control design). (Wu et al., 2014b) Denote the LQG Gramians P_f, solution of the filter Riccati equation (31) and P_c, solution of the control Riccati equation (32). Consider the LQG problem with the following relation between the covariance matrix R_w and the weighting matrix $\tilde{R}$

$$R_w = \tilde{R}. \quad (34)$$

and the relation between the covariance matrix Q_v and weighting matrix $\tilde{Q}$ is given by :

$$Q_v = Q_s^{-1}(2Q_S\bar{J}_{11}^T P_c + 2P_c\bar{J}_{11}Q_s + \tilde{Q})Q_s^{-1} \quad (35)$$

Then the LQG Gramians satisfy the following relation:

$$P_c Q^{-1} = QP_f. \quad (36)$$

Furthermore, assuming that the port Hamiltonian system is stable, the control Riccati equation (32) and the filter Riccati equation (31) admit a unique solution, the LQG controller is passive and the closed loop system can be written as the feedback interconnection of the port Hamiltonian system (27) with the port Hamiltonian realization of the LQG controller.

This theorem can be used to design a passive LQG controller and derive a port hamiltonian closed loop system. It is not the only way to get a passive LQG controller, see (Wu et al., 2014b), but this LQG theorem provides a balanced reduction coordinate which consider the the reduction and control design (LQG) problem in the same time. We define the balanced reduction coordinate for the port Hamiltonian system as follow:

Definition 2. The port Hamiltonian system (27) admits a *Q-conjugated* balanced realization if the Grammians P_f and P_c of the Q-conjugated LQG problem of Theorem 1, are diagonal:

$$P_c = P_f = M = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n) \quad (37)$$

where, denoting by $\lambda_i(P)$ the i-th eigenvalue of a matrix P,

$$\mu_i = \sqrt{\lambda_i(P_cP_f)} \text{ and } \mu_1 > \mu_2 > \cdots > \mu_n > 0 \quad (38)$$

Denoting by T the transformation matrix that diagonalizes the Grammians P_f and P_c of the Q-conjugated LQG problem:

$$TP_cT^T = T^{-T}P_fT^{-1} = M \quad (39)$$

The Q-conjugated LQG balanced realization of the slow dynamical part of port Hamiltonian system (28) shall be denoted as follows

$$\left\{\begin{array}{rcl}\dot{z}_{b1} &=& (J_b - R_b)Q_b z_{b1} + B_b u \\ y &=& B_b^T Q_b z_{b1}\end{array}\right. \quad (40)$$

where $J_b = T\bar{J}_{11}T^T$, $R_b = T\bar{R}_{11}T^T$, $Q_b = T^{-T}Q_sT^{-1}$ and $B_b = T\bar{B}_1$.

Thus the Q-conjugated LQG balanced realization of the port Hamiltonian descriptor systems is

$$\left\{\begin{array}{rcl}\begin{bmatrix} I_{N-k} & 0 \\ 0 & 0\end{bmatrix}\dot{z}_b &=& \begin{bmatrix} J_b - R_b & 0 \\ 0 & I_k\end{bmatrix}\begin{bmatrix} Q_b & 0 \\ 0 & I_k\end{bmatrix} z_b + \begin{bmatrix} B_b \\ 0\end{bmatrix} u \\ y &=& \begin{bmatrix} B_b & \hat{B}_2\end{bmatrix}\begin{bmatrix} Q_b & 0 \\ 0 & I_k\end{bmatrix} z_b\end{array}\right. \quad (41)$$

Then we will use the effort constraint reduction method to reduce this balanced system (41) with preserving the passivity and Hamiltonian structure. The readers can find the detailed technics in (Polyuga and van der Schaft, 2012). By using this structure preserving method we can get the reduced port Hamiltonian descriptor system as follow:

$$\left\{\begin{array}{rcl}\begin{bmatrix} I_{r-k} & 0 \\ 0 & 0\end{bmatrix}\dot{z}_r &=& \begin{bmatrix} J_{b11} - R_{b11} & 0 \\ 0 & I_k\end{bmatrix}\begin{bmatrix} Q_{bs} & 0 \\ 0 & I_k\end{bmatrix} z_r + \begin{bmatrix} B_{b1} \\ 0\end{bmatrix} u \\ y &=& \begin{bmatrix} B_{b1} & \hat{B}_2\end{bmatrix}\begin{bmatrix} Q_{bs} & 0 \\ 0 & I_k\end{bmatrix} z_r\end{array}\right. \quad (42)$$

We can use the above reduced system and the Theorem 1 to design a reduced order LQG controller for the system. We can apply this reduced order LQG controller to the full stable system (27).

4. SIMULATION RESULTS

In this section, we will consider the Timoshenko Beam model shown on Figure 3. This Timoshenko beam is

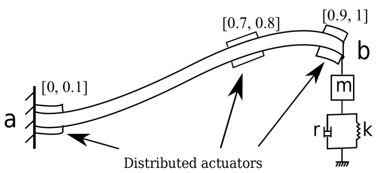

Fig. 3. Distributed controlled Timoshenko beam by three actuators

actuated by the distributed torques in three intervals as the distributed input variables: $b_i(z)u_{di}(t)$ with $i = 1, 2, 3$ on the small intervals $I_{b1} = [0, 0.1]$, $I_{b2} = [0.7, 0.8]$ and $I_{b3} = [0.9, 1]$. Thus $b_1(z) = 1$ if $z \in I_{b1}$ and $b_1(z) = 0$ elsewhere, as same as $b_2(z), b_3(z)$. As consequence the distributed inputs and the input operator are:

$$u_d(t) = \begin{bmatrix} T_{d1}(t) \\ T_{d2}(t) \\ T_{d3}(t)\end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1(z) & b_2(z) & b_3(z)\end{bmatrix} \quad (43)$$

where T_{di} is the distributed torque applied on the i-th interval. Before the closed loop reduction procedure, we take the 10 infinitesimal subsections of the infinite dimensional Timoshenko beam (44 state variables). By using the LQG reduction method, we achieve a low order LQG controller (4 state variables for the controller) which can stabilize the full order system. We apply the unitary step signals as the input and we measure the angular position at the manipulation side of the beam. The Figure

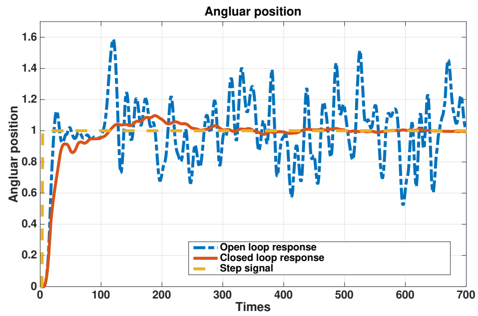

Fig. 4. Angular positions of point b

4 shows the angular positions of the open loop system and the closed loop system. One can observe the angular position of the open loop system is always oscillating over an equilibrium however the closed loop angular position is stabilized in this equilibrium.

5. CONCLUSION

The port Hamiltonian framework has been used to study the model and reduced controller design of the distributed controlled Timoshenko beam. The beam is clamped on one side and other side is interconnected with a manipulated object. The boundary ports of this port Hamiltonian system have been used to connect with the simplified model of the manipulated object. On the other hand, the boundary conditions arise the algebraic physical constraints to the systems. The descriptor system realization of the constrained port Hamiltonian system has been used to describe the system as well as the physical constraints. The main contribution of this work is proposed a model and control reduction scheme for the boundary constrained distributed controlled port Hamiltonian system by using a passivity and structure preserving LQG balancing method.

The future work will deal with modeling of the Timoshenko beam interconnected with the distributed control actuators. In this work we didn't consider the physical model of the actuators. But in practical applications, we can use the smart materials such as electro active polymer (Nishida et al., 2011), piezoelectric actuators, to control the position of the beam. Hence the modeling of the multi-physical systems (beam interconnected with actuators) will be studied in the future.

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