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Brief paper

# Reduced order LQG control design for port Hamiltonian systems ☆

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ABSTRACT

The aim of this paper is to propose a reduced order control design method for large scale port Hamiltonian systems. To this end, a structure preserving reduction method and a modified LQG control design are combined to derive a reduced order model suitable for control purposes. We first recall the structure preserving reduction method for port Hamiltonian systems called effort constraint method and characterize the error bound associated to this reduction method. We then give sufficient conditions for non-standard LQG design which allow to design a passive controller equivalent to the control by interconnection of port Hamiltonian systems. This LQG method allows to define an LQG balanced realization by computing the LQG Gramians, the effort-constraint method is then used to derive a reduced order port Hamiltonian system and to design a reduced order passive LQG controller. Finally, the method is illustrated in simulation on a mass–spring–damper system. The performances of the reduced order controller are compared to the results obtained with a full order passive LQG controller.



## 1. Introduction

The port Hamiltonian framework is well adapted to represent a large class of passive systems (van der Schaft, 2000) and is particularly well suited for the compositional modeling of finite and infinite dimensional physical systems (Duindam, Macchelli, Stramigioli, & Bruyninckx H. e, 2009; van der Schaft & Maschke, 2013). Some of these models might be high dimensional or even infinite-dimensional that can be difficult to handle when considering control design, or lead to very high order controllers. This usually motivates the reduction (or discretization) of port Hamiltonian systems prior to their control, however with the constraint of preserving their structure in order to apply passivity-based control techniques for control design (van der Schaft, 2000). Different *open-loop* spatial discretization methods aiming at preserving the passivity and the Hamiltonian structure of the system have been proposed (Baaiu, Couenne, Gorrec, Lefèvre, & Tayakout, 2009; Golo, Talasila, van der Schaft, & Maschke, 2004; Moulla, Lefèvre, & Maschke, 2011). Other structure preserving reduction methods have been developed for high-dimensional port Hamiltonian systems, arising for instance from the discretization of infinite dimensional systems or the modeling of complex systems defined on networks, in Polyuga and van der Schaft (2011, 2012), however without any estimation of the reduction error bounds. The main drawback of the aforementioned reduced order methods is that the reduction step is disconnected to the control objectives. It can lead to spillover effects when the reduction then control design procedure is applied to slightly damped systems (Balas & Jul, 1978). The aim of this paper is to combine a modified LQG control design strategy with a closed loop model reduction method to derive a reduced order controller able to cope with such weakly damped systems.

In this paper, we first present in Section 2 the structure preserving reduction method based on effort constraint (Polyuga & van der Schaft, 2012) and propose an estimate of the reduction error extending (Wu, Hamroun, Gorrec, & Maschke, 2014). Secondly, in Section 3 the LQG control design is adapted in such a way the LQG control is realizable as a control by interconnection of port Hamiltonian systems. The order reduction is then performed by writing the system in a balanced basis with respect to the resulting LQG Gramians and by using the effort-constraint method (Polyuga & van der Schaft, 2012) that preserves the port Hamiltonian structure and the passivity of the system. We illustrate and compare the different LQG controllers on the example of a mass–spring–damper system in Section 4.

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## 2. Effort constraint method for port Hamiltonian systems and its reduction error estimation

In this section we first recall the definition of dissipative port Hamiltonian systems and present the reduction method using the *effort constraint* proposed in Polyuga and van der Schaft (2012). In a second instance we provide an estimation of the error induced by the use of this method.

### 2.1. Port Hamiltonian systems and effort constraint method

Let us first recall the definition of dissipative port Hamiltonian systems.

**Definition 1.** A linear dissipative port Hamiltonian system (PHS) with state variable $x(t) \in \mathbb{R}^n$, input variable $u(t) \in \mathbb{R}^m$, output variable $y(t) \in \mathbb{R}^m$ is defined in his explicit form as follows:

$$\Sigma_{PHS} \begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^T Qx \end{cases} \tag{1}$$

where $J = -J^T \in \mathbb{R}^{n\times n}$ is the skew-symmetric interconnection structure matrix, $R = R^T \in \mathbb{R}^{n\times n}$ is the symmetric positive dissipation matrix, $Q = Q^T \in \mathbb{R}^{n\times n}$ is the symmetric and positive definite energy matrix and $B \in \mathbb{R}^{n\times m}$ is the input matrix. The total energy of the system is given by $H = \frac{1}{2}x^T Qx$. The dissipation inequality which implies the passivity of the system is naturally derived from this Hamiltonian structure:

$$\frac{dH}{dt} = -x^T QRQx + y^T u \leq y^T u. \tag{2}$$

The systems we consider stemming from the modeling of interconnected systems need to be reduced in order to derive a model suitable for control purposes. The effort-constraint reduction method has the advantage of preserving the port Hamiltonian structure and the passivity of the system. It is based on the alternative definition of implicit port Hamiltonian systems (Duindam et al., 2009; van der Schaft & Maschke, 1995, chap. 2) derived from Dirac structures (Dalsmo & van der Schaft, 1999).

**Definition 2.** A *Dirac structure* on $\mathbb{R}^n$ is a linear subspace $\mathcal{D} \subset \mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\mathcal{D} = \{(f, e) \in \mathbb{R}^n \times \mathbb{R}^n | Ff + Ee = 0\} \tag{3}$$

with respect to the two structure matrices $E$ and $F$ in $\mathbb{R}^{n\times n}$ which satisfies

$$EF^T + FE^T = 0, \qquad \text{rank}[E : F] = n. \tag{4}$$

Implicit dissipative port Hamiltonian systems are defined with respect to these Dirac structures.

**Definition 3.** An implicit linear dissipative port Hamiltonian system (PHS) with state variable $x(t) \in \mathbb{R}^n$, flow and effort variables $(f, e) = \left(-\frac{dx}{dt}, Qx\right) \in \mathbb{R}^n \times \mathbb{R}^n$, input and output port variables $(y, u) = \left(f_p, e_p\right) \in \mathbb{R}^m \times \mathbb{R}^m$, dissipation port variables $(f_R, e_R) \in \mathbb{R}^{m_R} \times \mathbb{R}^{m_R}$ is defined with respect to a Dirac structure of the form (3) as follows

$$F \begin{bmatrix} \frac{dx}{dt} \\ f_p \\ f_R \end{bmatrix} + E \begin{bmatrix} Qx \\ e_p \\ e_R \end{bmatrix} = 0$$

with the dissipative closure relation $e_R = -\hat{R} f_R$, $\hat{R} = \hat{R}^T \geq 0 \in \mathbb{R}^{m_R \times m_R}$.

We now assume that the system is given in a specific balanced realization (balancing techniques) such that the state variables of the port Hamiltonian system can be separated into two parts $x^T = (x_1^T, x_2^T)$, $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$ where $x_2$ is the vector of less important state variables with regard to input–output behavior (balanced properties) of the system. Using this separation of state variables, port Hamiltonian system (1) can be expressed as an implicit port Hamiltonian system with the following structure matrices:

$$F = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}, E = \begin{bmatrix} J_{11} & J_{12} & B_1 & g_{r1} \\ J_{21} & J_{22} & B_2 & g_{r2} \\ B_1^T & B_2^T & 0 & 0 \\ g_{r1}^T & g_{r2}^T & 0 & 0 \end{bmatrix}$$

and closure relation $e_R = -\hat{R} f_R$ with $R = g_r \hat{R} g_r^T$. Following the effort constraint method (Polyuga & van der Schaft, 2012), the reduced model is obtained by imposing the constraint $e_2 = 0$.

It may be shown that the reduced order system with state variable $x_1(t)$, can be written as an implicit port Hamiltonian system with respect to the Dirac structure defined by the reduced matrices $F_r = L^e F M_e$ and $E_r = L^e E M_e$ given by:

$$F_r = \begin{bmatrix} I_r & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & -I_{m_R} \end{bmatrix}, E_r = \begin{bmatrix} J_{11} & B_1 & g_{r1} \\ B_1^T & 0 & 0 \\ g_{r1}^T & 0 & 0 \end{bmatrix} \tag{5}$$

where the matrix $M_e \in \mathbb{R}^{(n+m+m_R)\times(r+m+m_R)}$ and the projector matrix $L^e \in \mathbb{R}^{(r+m+m_R)\times(n+m+m_R)}$ are defined by:

$$\begin{aligned} M_e &= \begin{bmatrix} I_r & 0_{r\times m} & 0_{r\times m_R} \\ 0_{(n-r)\times r} & 0_{(n-r)\times m} & 0_{(n-r)\times m_R} \\ 0_{m\times r} & I_m & 0_{m\times m_R} \\ 0_{m_R\times r} & 0_{m_R\times m} & I_{m_R} \end{bmatrix}; \\ L^e &= \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_{m_R} \end{bmatrix} \end{aligned} \tag{6}$$

with $H_r = \frac{1}{2}x_1^T Q_s x_1$, $Q_s = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$ representing the Schur complement of $Q$. The explicit form of the reduced port Hamiltonian system is given by:

$$\begin{cases} \dot{x}_1 = (J_{11} - R_{11})Q_s x_1 + B_1 u \\ y = B_1^T Q_s x_1 \end{cases} \tag{7}$$

with $R_{11} = g_{r1}\hat{R} g_{r1}^T$. Thus the effort constraint based reduction methods preserve the Hamiltonian structure of the original system.

### 2.2. Error bound estimation of the effort constraint method

In this subsection, we provide an error bound for the *effort constraint reduction method*. For that purpose we use a result of Antoulas (2005) proposed in the context of Lyapunov balance truncation methods.

Consider the Lyapunov balanced realization[1] (where the index $b$ stand for balanced coordinates) of the port Hamiltonian system (1):

$$\begin{cases} \dot{x}_b = \underbrace{(J_b - R_b)}_{F_b} Q_b x_b + B_b u \\ y = B_b^T Q_b x_b \end{cases} \tag{8}$$

associated with the following controllability and observability Lyapunov equations:

$$\begin{aligned} A_b \Sigma + \Sigma A_b^T + B_b B_b^T = 0 \\ A_b^T \Sigma + \Sigma A_b + Q_b B_b B_b^T Q_b = 0 \end{aligned} \tag{9}$$

[1] through the change of variables $x_b = T_b x$.

where $A_b = (J_b - R_b)Q_b$ , $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values of the system (Antoulas, 2005). By applying the effort constraint method to (8) we obtain a reduced order system of the form (7)

$$\begin{cases} \dot{x}_{b1} = F_{11}Q_s x_{b1} + B_1 u \\ y = B_1^T Q_s x_{b1} \end{cases} \tag{10}$$

with $F_{11} = J_{b11} - R_{b11}$ and $Q_s = Q_{b11} - Q_{b12}Q_{b22}^{-1}Q_{b21}$.

**Lemma 4.** *Consider the Lyapunov balanced realization of the port Hamiltonian system (8) and its representation in the new coordinates*

$$\begin{cases} \dot{\tilde{x}} = \tilde{F}\tilde{x} + \tilde{B}u \\ y = \tilde{B}^T\tilde{x} \end{cases} \tag{11}$$

*where $\tilde{F} = SF_bS^T$, $\tilde{B} = SB_b$ and $\tilde{x} = Sx_b$ with*

$$S = \begin{bmatrix} Q_s^{\frac{1}{2}} & 0 \\ Q_{b22}^{-\frac{1}{2}}Q_{b21} & Q_{b22}^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \tag{12}$$

*a decomposition of $Q_b$ such that: $Q_b = S^TS$. Then the reduced order system derived from (8) by using the effort constraint and the reduced order system derived from (11) by using the truncation method, given by:*

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{F}_{11}\tilde{x}_1 + \tilde{B}_1 u \\ y = \tilde{B}_1^T\tilde{x} \end{cases} \tag{13}$$

*are input–output equivalent through the change of coordinate $\tilde{x}_1 = S_{11}x_{b1}$.*

**Proof.** To prove this lemma, we first compute the port Hamiltonian system in the coordinate (11) from the Lyapunov balanced realization by $\tilde{x} = Sx_b$. Then we reduce this port Hamiltonian system to system (13) by truncation reduction method. At last we can verify the input–output equivalence of two reduced order systems (10) and (13) by coordinate change $\tilde{x}_1 = Q_s^{\frac{1}{2}}x_{b1}$. □

From Lemma 4, one can relate the error estimation of the reduced order system (10) obtained by using the effort constraint method to the one obtained by the truncation method (13) by

$$\|\Delta(s)\|_{H_\infty} = \|G(s) - G_r^E(s)\|_{H_\infty} = \|G(s) - G_r^T(s)\|_{H_\infty} \tag{14}$$

where $G(s)$ is the transfer function of the full order system:

$$G(s) = B_b^T Q_b (sI - F_bQ_b) B_b = \tilde{B}^T \left(sI - \tilde{F}\right)\tilde{B}, \tag{15}$$

$G_r^E(s)$ is the transfer function of the reduced order system derived by the effort constraint method:

$$G_r^E(s) = B_1^T Q_s (sI - F_{b11}Q_s) B_1, \tag{16}$$

and $G_r^T(s)$ is the transfer function of the reduced order system derived by truncation:

$$G_r^T(s) = \tilde{B}_1^T\left(sI - \tilde{F}_{11}\right)\tilde{B}_1 = G_r^E(s). \tag{17}$$

To characterize the error bound and simplify the demonstration, we introduce the following notation in (11):

$$\tilde{F} = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } \tilde{B} = B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{18}$$

The transfer function of system (11) and the one of the reduced order system (13) can be written as follows:

$$G(s) = B^T(sI - A)^{-1}B \tag{19}$$

and

$$G_r^T(s) = B_1^T(sI - A_{11})^{-1}B_1 \tag{20}$$

respectively. Their controllability and observability Lyapunov equations are:

$$\begin{aligned} AP_c + P_cA^T + BB^T &= 0 \\ A^TP_f + P_fA + BB^T &= 0. \end{aligned} \tag{21}$$

The solutions $P_c$ and $P_f$ can be related to the Hankel matrix $\Sigma$ through the change of coordinate *i.e.* the matrix $S$ (given in (12)) as follows:

$$\begin{cases} P_c = S\Sigma S^T \\ P_f = S^{-T}\Sigma S^{-1}. \end{cases} \tag{22}$$

Let us introduce the following notations:

$$\begin{cases} \phi(s) := (sI - A_{11})^{-1} \\ \psi(s) := sI - A_{22} - A_{21}\phi(s)A_{12} \\ \bar{B}(s) := A_{21}\phi(s)B_1 + B_2 \\ \bar{C}(s) := B_1^T\phi(s)A_{12} + B_2^T. \end{cases} \tag{23}$$

**Proposition 5** (*Error Bound of the Effort Constraint Reduction Method*)**.** *Consider the port Hamiltonian system (8) with transfer function $G(s)$ and its reduced order form by using the effort constraint method (10) with transfer function $G_r^E(s)$, then the maximum of the error $\|\Delta(s)\|_{H_\infty}$ satisfies*[2]

$$\begin{aligned} \|\Delta(s)\|_{H_\infty} \leq \lambda_{\max}^{1/2} \Big( &\left[L + \psi^{-1}(j\omega)L^*\psi^*(j\omega)\right] \\ &\cdot \left[M + \psi^{-*}(j\omega)M^*\psi(j\omega)\right]\Big) \end{aligned} \tag{24}$$

*with*

$$L = S_{22}\Sigma_2S_{22} + S_{21}\Sigma_1\left[S_{21}^T + S_{11}\phi^*(j\omega)A_{21}^T\right] \tag{25}$$

*and*

$$M = S_{22}^{-1}\Sigma_2S_{22}^{-1}\left[I - S_{21}S_{11}^{-1}\phi(j\omega)A_{12}\right] \tag{26}$$

*where $\lambda_{\max}^{1/2}$ means the square of the maximal frequency eigenvalue.*

**Proof.** See the details in Wu (2015, Chapter 3, Section 4). □

Proposition 5 shows a clear difference between the error bound of the balanced truncation method which depends only on the neglected singular eigenvalues $\Sigma_2$ and the error bound related to the effort constraint method which depends on both $\Sigma_1$ and $\Sigma_2$.

## 3. LQG control design and structure preserving reduction for port Hamiltonian systems

In this section we first recast the standard LQG control design method in the port Hamiltonian framework. Then we show that the passivity of the controller can be guaranteed by properly choosing the weighting matrices of the LQG problem. Among all the possible choices, we choose the one inspired from the work of Jonckheere and Silverman (1983) that allows to separate the closed loop spectrum before reducing the system.

### 3.1. LQG control design to port Hamiltonian systems

We apply the LQG control design method (Hespanha, 2009) to the linear port Hamiltonian system defined by (1) to derive the dynamic observer based controller:

$$\begin{cases} \dot{\hat{x}} = [(J - R)Q - BK - FB^TQ]\hat{x} + Fu_c \\ y_c = K\hat{x} \end{cases} \tag{27}$$

[2] $M^*$ is the Hermitian transpose of $M$.

where the state of the controller $\hat{x}$ represents the estimation of the state $x$ of the system. The feedback gains are:

$$F = P_f Q B R_w^{-1}, \qquad K = \tilde{R}^{-1} B^T P_c \tag{28}$$

where $P_f = P_f^T > 0$ and $P_c = P_c^T > 0$ are the solutions of the following Riccati equations:

$$(J-R)QP_f + P_f Q(J-R)^T - P_f Q B R_w^{-1} B^T Q P_f + Q_v = 0 \tag{29}$$

$$Q(J-R)^T P_c + P_c(J-R)Q - P_c B \tilde{R}^{-1} B^T P_c + \tilde{Q} = 0 \tag{30}$$

with $Q_v = Q_v^T > 0$ and $R_w = R_w^T \geq 0$ the covariance matrices and $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} = \tilde{R}^T \geq 0$ the optimal control weighting matrices. The cost function of the optimal control is $J_c = \int_0^\infty \left( x^T \tilde{Q} x + u^T \tilde{R} u \right) dt$.

**Remark 6.** Following Möckel, Reis, and Stykel (2011), we call $P_f$ and $P_c$ the *LQG Gramians* of the port Hamiltonian system (1).

The LQG controller (27) may be expressed as follows

$$\begin{cases} \dot{\hat{x}} = [(J-R_c)Q]\hat{x} + P_f Q B R_w^{-1} u_c \\ y_c = \tilde{R}^{-1} B^T P_c \hat{x} \end{cases} \tag{31}$$

with

$$R_c = R + B\tilde{R}^{-1} B^T P_c Q^{-1} + P_f Q B R_w^{-1} B^T. \tag{32}$$

This expression resembles the one of a port Hamiltonian system (1). However the matrix $R_c$ in (32) is in general neither symmetric nor positive. Furthermore the input and output matrices in (31) are not adjoined. There is no reason that the controller admits a passive port-Hamiltonian realization as it is well-known that LQG controllers are in general neither passive nor stable (Halevi, 1994).

### 3.2. Equivalence to control by interconnection of port Hamiltonian systems

To make sure that the port Hamiltonian structure and the passivity are preserved in closed loop by using LQG control design, we shall require that the LQG controller is a port Hamiltonian system. In this case the controller and the system can be considered as coupled by a power-preserving feedback interconnection and the closed loop system is still a port Hamiltonian system. A similar result holds for the feedback interconnection of passive systems (Duindam et al., 2009).

To do so, we consider particular choices of the weighting and covariance matrices of the LQG problem, as stated in the following theorems.

**Theorem 7.** *Denote the LQG Gramians $P_f$, solution of the filter Riccati equation (29) and $P_c$, solution of the control Riccati equation (30). Consider the LQG problem with the following relation between the covariance matrix $R_w$ and the weighting matrix $\tilde{R}$*

$$R_w = \tilde{R} \tag{33}$$

*and with the following relation between the covariance matrix $Q_v$ and the weighting matrix $\tilde{Q}$:*

$$Q_v = Q^{-1}(2QJ^T P_c + 2P_c J Q + \tilde{Q})Q^{-1}. \tag{34}$$

*In this case the LQG Gramians satisfy the following relation:*

$$P_c Q^{-1} = Q P_f. \tag{35}$$

*Furthermore, assuming that the port Hamiltonian system is stable, the control Riccati equation (30) and the filter Riccati equation (29) admit a unique solution, the LQG controller is passive and the closed loop system can be written as the feedback interconnection of the port Hamiltonian system (1) with the port Hamiltonian realization of the LQG controller.*

**Proof.** Assume $Q$ is invertible since it is symmetric and positive definite. The filter Riccati equation (29) can be written as:

$$\begin{aligned} &Q\,(J-R)\,QP_f Q + QP_f Q\,(J-R)^T Q \\ &-QP_f Q B R_w^{-1} B^T Q P_f Q + Q Q_v Q = 0. \end{aligned} \tag{36}$$

By using condition (34), the above equation becomes

$$\begin{aligned} &Q\,(J-R)\,QP_f Q + QP_f Q\,(J-R)^T Q \\ &-QP_f Q B R_w^{-1} B^T Q P_f Q + 2QJ^T P_c + 2P_c J Q + \tilde{Q} = 0. \end{aligned} \tag{37}$$

Then subtracting (37) to the control Riccati equation (30) and considering $R_w = \tilde{R}$, we can get

$$\begin{aligned} &QJ\left(QP_f Q - P_c\right) + \left(QP_f Q - P_c\right) J^T Q \\ &-\left(QP_f Q - P_c\right) B R_w^{-1} B^T \left(P_c - QP_f Q\right) = 0 \end{aligned} \tag{38}$$

which is satisfied for

$$P_c Q^{-1} = Q P_f. \tag{39}$$

One can check that this choice allows to satisfy the two Riccati equations (29) and (30).

For an asymptotically stable port Hamiltonian system, the detectability conditions are met for any matrix $\tilde{Q}$ and $Q_v$ and both the filter and control Riccati equations admit a unique solution.

Finally, we show the LQG controller is passive and can be formulated as an interconnected port Hamiltonian system. First by using condition (35) and condition (33), the output of controller (31) becomes:

$$y_c = (\tilde{R}^{-1} B^T P_c Q^{-1}) Q x_c \quad \Leftrightarrow \quad y_c = (R_w^{-1} B^T Q P_f) Q x_c \tag{40}$$

which means that the output $y_c$ of controller (31) is port-conjugated to the input $u_c$. Secondly, considering condition (35) and condition (33), one can check that the matrix $R_c$ defined in Eq. (32) is symmetric. The controller (31) is designed by the LQG design method then the closed loop system is stable and $R_c$ is positive. □

Theorem 7 provides a passive LQG control design method which is equivalent to the control by interconnection of port Hamiltonian systems. We shall call this LQG method *the Q-conjugated LQG problem* since the two LQG Gramians $P_c$ and $P_f$ are related through the energy matrix $Q$ according to (39).

It may be noticed that when using the weighting matrices $\tilde{Q}$ and $\tilde{R}$ as control design parameters, the covariance matrices $Q_v$ and $R_w$ are derived using (34) and the $Q$-conjugated LQG problem is completely specified. The covariance matrices are used as design parameters and have no longer statistical meaning. Such kind of approach is very similar to the one used for LQG/LTR (Loop Transfer Recovery) design in which the covariance matrix is used to recover the good robustness properties of the LQ regulator. A similar passive LQG control design for positive real systems has been proposed in Brogliato, Lozano, Maschke, and Egel (2007).

**Remark 8.** The other advantage of this LQG control design method is that by using this specific choice of weighting and covariance matrices (34) and (33), the two solutions of (29) and (30) are related by

$$P_c Q^{-1} = Q P_f. \tag{41}$$

It means that only one Riccati equation has to be solved for the overall design.

Finally, note that the product of the LQG Gramians obtained from Theorem 7

$$P_f P_c = P_f Q P_f Q \neq I$$

is not equal to the identity which allows to reduce the system through its balanced realization .

### 3.3. Structure preserving LQG reduction

In the sequel we use the $Q$-conjugated LQG method in order to derive, in a first step, a balanced representation of the port Hamiltonian system (1) and in a second step, a reduced order model.

Let us first define the balanced realization with respect to the $Q$-conjugated LQG Gramians $P_f$ and $P_c$ that we shall call for convenience of writing *Q-conjugated balanced realization*.

**Definition 9.** The port Hamiltonian system (1) admits a $Q$-*conjugated* balanced realization if the Gramians $P_f$ and $P_c$ of the $Q$-conjugated LQG problem of Theorem 7, are diagonal and equal:

$$P_c = P_f = M = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) \tag{42}$$

where, denoting by $\lambda_i(P)$ the $i$th eigenvalue of a matrix $P$,

$$\mu_i = \sqrt{\lambda_i(P_c P_f)} \text{ and } \mu_1 > \mu_2 > \cdots > \mu_n > 0. \tag{43}$$

**Remark 10.** Definition 9 proposes a novel balanced realization which differs from the standard LQG balanced realization (Jonckheere & Silverman, 1983) where the covariance matrices are:

$$Q_v = BB^T \quad \text{and} \quad R_w = I, \tag{44}$$

and the matrices defining the optimal control criterion are:

$$\tilde{Q} = C^T C \quad \text{and} \quad \tilde{R} = I. \tag{45}$$

**Remark 11.** Following Proposition 7, we have $P_c = QP_fQ$, then

$$P_c P_f = QP_f QP_f.$$

Hence the $Q$-conjugated balanced realization is derived by diagonalizing of the matrix $QP_f$ .

Let us denote $T$ the transformation matrix that diagonalizes the Gramians $P_f$ and $P_c$ of the $Q$-conjugated LQG problem *i.e.*:

$$TP_cT^T = T^{-T}P_fT^{-1} = M. \tag{46}$$

The $Q$-conjugated LQG balanced realization of the port Hamiltonian system (1) is derived as follows

$$\begin{cases} \dot{x}_b = (J_b - R_b)Q_b x_b + B_b u \\ y = B_b^T Q_b x_b \end{cases} \tag{47}$$

where $J_b = TJT^T$, $R_b = TRT^T$, $Q_b = T^{-T}QT^{-1}$ and $B_b = TB$.

Using this balanced realization, the singular values (43) are ordered and split into two set $(\mu_i)_{i=1,\ldots,r}$ and $(\mu_i)_{i=r+1,\ldots,n}$ with the objective of reducing the system in such a way to retain only the states associated with the first $r$ singular values *i.e.* the states that have a significant contribution with respect to the desired closed loop performances. The state variables are then decomposed as $x_b^T = [x_{b1}^T, x_{b2}^T]^T$ where $x_{b1} \in \mathbb{R}^r$ and $x_{b2} \in \mathbb{R}^{n-r}$. Thus we use the effort constraint method to reduce this $Q$-conjugated LQG balanced realization in order to keep only the first $r$ states while preserving the Hamiltonian structure. The reduced order port Hamiltonian system is presented as follows:

$$\begin{cases} \dot{x}_{b1} = (J_{b11} - R_{b11})Q_s x_{b1} + B_{b1} u \\ y = B_{b1}^T Q_s x_{b1} \end{cases} \tag{48}$$

where $Q_s = Q_{b11} + Q_{b12}Q_{b22}^{-1}Q_{b21}$.

The reduced order controller is then derived by applying the $Q$-conjugated LQG design procedure given in Theorem 7 to the reduced order system (48). It is important to notice that the important states of the original system have been selected by considering the closed loop achievable performances and not the open loop ones. The overall control design is summarized in Algorithm 1. The reduced order controller can be applied to the given full order port Hamiltonian system (1) in order to stabilize the system (Jonckheere & Silverman, 1983). Hereafter (Algorithm 1) we give an algorithm which summarizes the proposed Q-LQG reduced controller design method.

**Algorithm 1** Q-conjugated LQG reduced controller design

Input: $J \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\tilde{Q} \in \mathbb{R}^{n\times n}$, $\tilde{R} \in \mathbb{R}^{m\times m}$.
Output: Reduced order LQG controller;

1. Choose $R_w$ and $Q_v$ accordingly to (33) and (34) respetctively and compute $P_c$ and $P_f = Q^{-1}P_cQ^{-1}$ solving (30);
2. Compute the transformation matrix $T \in \mathbb{R}^{n\times n}$ defined by (46);
3. Find the balanced realization of the system (47) with $J_b = TJT^T$ , $R_b = TRT^T$ , $Q_b = T^{-T}QT^{-1}$ and $B_b = TB$;
4. Choose the order of reduction from the analysis of the singular values.
5. Proceed to the reduction by using the effort constraint method accordingly to (48).
6. Compute the reduced order LQG controller.

## 4. Illustration on a 1D mass–spring–damper chain

In this section, we consider the benchmark example of a 1D mass–spring–damper chain treated in Polyuga and van der Schaft (2012) which can be interpreted as the spatially discretized model of a robotic flexible link of a robot or a vibration absorber. We first compare the closed loop performances obtained with the full order $Q$-conjugated LQG controller to the one obtained with the full order standard LQG controller designed and applied on the full order system. Secondly, we compare the performances of the reduced order $Q$-conjugated LQG controller, designed on the reduced model and applied to the full order system, to the performances of the full order $Q$-conjugated LQG controller.

The mass–spring–damper chain represented in Polyuga and van der Schaft (2012) can be formulated as a port Hamiltonian system of the form (1) :

$$\begin{cases} x = [d_1,\ p_1,\ d_2,\ p_2,\ d_3,\ p_3, \ldots,\ d_N,\ p_N]^T \\ u = F \\ y = v_1. \end{cases}$$

The state vector $x \in \mathbb{R}^{2N}$ contains the relative displacement $d$ and the momentum $p$ of the $N$ masses, the input of system is the force $F$ applied to the mass $m_1$ and its dual output is the velocity of the same mass.

The physical parameters and their numerical values are: the masses $m_i = 2$, the elasticity coefficients of the springs $k_i = 4$, the friction coefficients $c_i = 0.01$. The structure matrix $J = -J^T \in \mathbb{R}^{2N\times 2N}$, the dissipation matrix $R = R^T \geq 0 \in \mathbb{R}^{2N\times 2N}$, the energy matrix $Q > 0 \in \mathbb{R}^{2N\times 2N}$ and the input matrix $B \in \mathbb{R}^{2N}$.

In the numerical simulation, we choose $N = 200$, leading to a state space of size $2N = 400$, *i.e.* $x \in \mathbb{R}^{400}$.

### 4.1. Comparisons of the two LQG control design methods

In Fig. 1 are plotted the open loop step response (green dashed curve), and the closed loop step responses (the output variable is

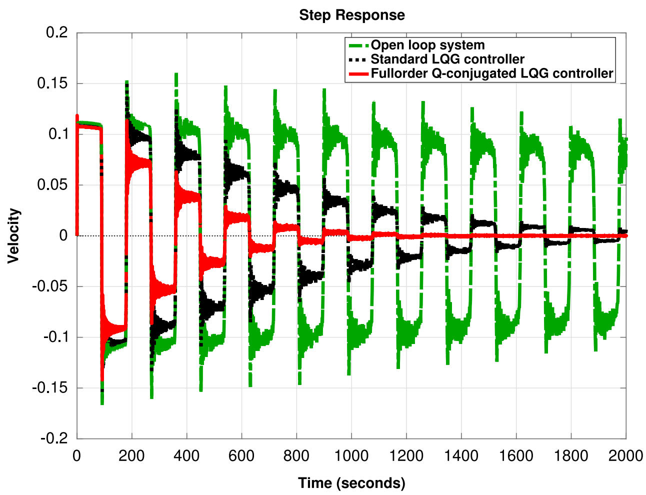



**Fig. 1.** Step responses of the closed loop (full order) system with standard (full order) LQG controller and (full order) $Q$-conjugated LQG controller.

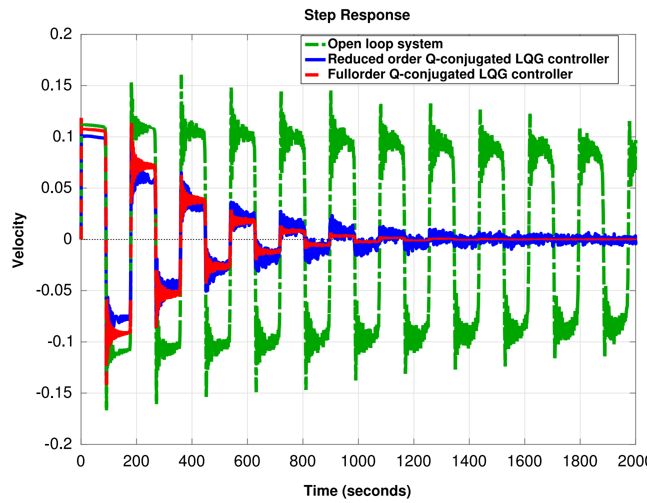



**Fig. 3.** Step responses of the closed loop (full order) systems with $Q$-conjugated full order and reduced order controllers.

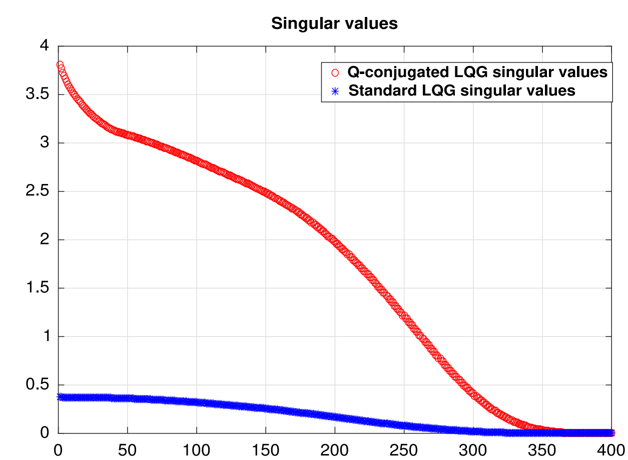



**Fig. 2.** Singular values associated with different LQG problems.

the velocity) by using the $Q$-conjugated LQG controller (red solid curve) and the standard LQG controller (black dashed curve).

### 4.2. Comparisons of the two LQG balanced reduction methods

It should be reminded that the closed loop system with the $Q$-conjugated LQG controller is still a port Hamiltonian system and that the weighting matrices can be freely chosen. The closed loop performances are similar to the ones obtained by using the standard LQG controller with modulated states weighting matrix while preserving the closed loop structure of the system and guaranteeing the passivity of the controller. In the $Q$-conjugated LQG controller design, the form of $\tilde{Q}$ is chosen to be $\tilde{Q} = \gamma CC^T$ with $\gamma > 0$ ($\gamma = 5$ is used for the simulation).

Before proceeding to the reduction of the system, the singular values obtained from the standard LQG balancing and the $Q$-conjugated balancing are compared in Fig. 2.

One may observe that the singular values (depicted by red circles in Fig. 2) obtained by the $Q$-conjugated LQG balancing have a bigger decay rate than the ones obtained with standard LQG balancing method (depicted by blue crosses in Fig. 2). The decay rate of the singular values for the $Q$-conjugated LQG balancing can be increased by increasing the value of $\gamma$ in the expression of $\tilde{Q}$. It should be recalled that in the $Q$-conjugated LQG balancing method, the choice of the weighting matrices is defined by the relation of Theorem 7.

The $Q$-conjugated LQG method does not only allow us to design a passive Hamiltonian controller, but also allows us to reduce both the system and the controller while preserving the Hamiltonian structure of the system. In Fig. 3, we show the step responses of the closed loop system with the full order $Q$-conjugated LQG controller (red dashed curve) and with the reduced order $Q$-conjugated controller (blue solid curve).

The order of reduction is chosen equal to $r = 40$, since it appears in Fig. 2, that the first 40 $Q$-conjugated LQG singular values are bigger than the remaining ones which is not the case for the standard LQG singular values (the first 150 standard LQG singular values are almost equal). One can observe that performances obtained with both the full order and reduced order $Q$-conjugated LQG controllers are quite similar.

## 5. Conclusion

In this paper is proposed a reduced order LQG control design technique for port Hamiltonian systems. It is based on equivalent port Hamiltonian formulation of the traditional LQG control design and effort constraint structure preserving model reduction. In a first instance an error estimate of the effort constraint reduction method for port Hamiltonian systems is provided. In a second instance it is shown that an appropriate choice of the LQG control design matrices allows to formulate the control problem as a control by interconnection problem. This method, called $Q$-conjugated LQG method, allows to find a balanced realization defined by separable singular values. The reduced order system is derived by using the aforementioned effort constraint reduction method, preserving the structure and the passivity of the system during the reduction. The reduced order LQG controller is then derived using this reduced order system. This approach can be seen as an adaptation to the LQG balanced reduction method proposed for finite dimensional systems in Jonckheere and Silverman (1983). Finally the effectiveness of the proposed method is illustrated on a classical mechanical mass–spring–damper chain system. The extension of this work to infinite dimensional systems is under study.

## References


Antoulas, A. (2005). A new result on passivity preserving model reduction. *Systems & Control Letters*, *54*(4), 361–374.

Baaiu, A., Couenne, F., Gorrec, Y. L., Lefèvre, L., & Tayakout, M. (2009). Structure-preserving infinite dimensional model reduction application to adsorption processes. *Journal of Process Control*, *19*(3), 394–404.

Balas, M. J. (1978). Active control of flexible systems. *Journal of Optimization Theory and Applications*, *25*(3), 415–436.

Brogliato, B., Lozano, R., Maschke, B., & Egel, O. (2007). *Dissipative systems analysis and control: theory and applications , communications and control engineering* (second ed.). Springer.

Dalsmo, M., & van der Schaft, A. (1999). On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal on Control and Optimization*, *37*(1), 54–91.

Duindam, V., Macchelli, A., Stramigioli, S., & Bruyninckx H. e, . (2009). *Modeling and control of complex physical systems - the port-hamiltonian approach*. ISBN: 978-3-642-03195-3, Springer.

Golo, G., Talasila, V., van der Schaft, A., & Maschke, B. (2004). Hamiltonian Discretization of Boundary Control Systems. *Automatica*, *40*, 757–771.

Halevi, Y. (1994). Stable LQG controllers. *Automatic Control. IEEE Transactions on*, *39*(10), 2104–2106.

Hespanha, J. (2009). *Linear systems theory*. Princeton University Press.

Jonckheere, E., & Silverman, L. (1983). A New Set of Invariants for Linear Systems-Application to Reduced Order Compensator Design. *IEEE Transactions on Automatic Control*, *28*, 953–964.

Möckel, J., Reis, T., & Stykel, T. (2011). Linear-quadratic gaussian balancing for model reduction of differential–algebraic systems. *International Journal of Control*, *84*, 1627–1643.

Moulla, R., Lefèvre, L., & Maschke, B. (2011). Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws. *Journal of Computational Physics*, *231*, 1272–1292.

Polyuga, R., & van der Schaft, A. (2011). Structure preserving moment matching for port-Hamiltonian systems: Arnoldi and Lanczos. *Automatic Control. IEEE Transactions on*, *56*(6), 1458–1462.

Polyuga, R., & van der Schaft, A. (2012). Effort- and Flow-constraint Reduction Methods for Structure Preserving Model Reduction of Port-Hamiltonian Systems. *Systems & Control Letters*, *61*(3), 412–421.

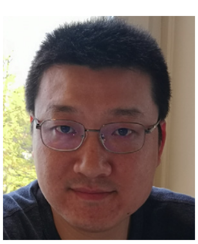
van der Schaft, A. (2000). *Communications and control engineering series. $L_2$-Gain and passivity techniques in nonlinear control*. Springer-Verlag.

van der Schaft, A., & Maschke, B. (1995). The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik Und ÜBertragungstechnik*, *49*(5/6), 362–371.

van der Schaft, A., & Maschke, B. (2013). Port-Hamiltonian systems on graphs. *SIAM Journal on Control and Optimization*, *51*(2), 906–937.

Wu, Y. (2015). *Passivity preserving balanced reduction for the finite and infinite dimensional port hamiltonian systems*. (Ph.D. thesis), Université Claude Bernard Lyon 1.

Wu, Y., Hamroun, B., Gorrec, Y. Le., & Maschke, B. (2014). Structure preserving reduction of port Hamiltonian system using a modified LQG method. In *Proceeding of the 33rd chinese control conference , Nanjing, 2014* (pp. 3528–3533). IEEE.

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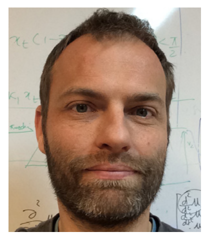

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