A DEGRADATION TEST PLAN FOR A NON-HOMOGENEOUS GAMMA PROCESS

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There is no longer any need to demonstrate the relevance of the degradation versus the classical life test plans. One of the main advantages is an increasing amount of collected data and its ability of extrapolating the lifetime distributions when degradation models are available. Nevertheless, further developments for classical and accelerated tests in the context of degradation are still required especially when the degradation pattern is clearly non-homogeneous. The aim of this paper is to propose a degradation test plan for non-homogeneous gamma processes for products with cumulative degradation evolution. We first, propose a test plan, which is based on minimizing the asymptotic variance of the true reliability of the products. Next, we apply the proposed test on a specific parametric form of the shape function of a gamma process.

Keywords: Process with independent increments, Non-homogeneous gamma process, Degradation test, Optimal test plan, Reliability estimation.

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1. Introduction

In order to asses the lifetime of products, degradation tests have been widely used to measure the performance characteristics of an unfailed product at different times (see [2], [6], [4] for an overview). Non-homogeneous gamma processes are suitable to describe the cumulative deterioration phenomenon over time (see [9]). Up to our knowledge, degradation test plan for gamma process has not been much studied, except in few works (e.g. [11] for the stationary case, [8]). In this paper, preliminary results are presented for the construction of a classical test plan when the product is subject to non-homogeneous degradation. Some a priori degradation knowledge is assumed to be available such as the shape of the degradation pattern -which is assumed to be a parametric function- and a failure threshold L due to excessive degradation. In case of testing, the degradation measurement interval is usually fixed. We assume that the measurement is perfect in the way that it refers to the real state of the product without any measurement noises. The objective of the test plan is hence to find the best compromise between the number of products to be tested and the test duration for the product qualification at a given reliability.

The remainder of this paper is organized as follows. Section 2 briefly introduces the gamma process. The degradation test plan is proposed in Section 3. A numerical experiment is provided in Section 4 for illustrating the relevance of the model. We finally conclude in Section 5.

2. Definition and properties of a non-homogeneous gamma process

Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ to be a measurable, increasing and right-continuous function with $A(0) = 0$ and $B > 0$. The stochastic process $\boldsymbol{Y} = (Y_t)_{t\geq 0}$ is said to be a non-homogeneous gamma process (written $\boldsymbol{Y} \sim \Gamma_0(A(.), B)$, with $A(.)$ as shape function and B as (constant) scale parameter, if the increments are independent, non-negative and gamma distributed such that $Y_0 = 0$ almost surely. The shape function and scale parameter depend on a parameter vector $\boldsymbol{\theta_0}$ in a parameter space $\Theta \subseteq \mathbb{R}^p$. The probability density function of an increment $Y_t - Y_s$ (with $0 < s < t$) is given by

$$f(x) = \frac{B(\boldsymbol{\theta_0})^{A(t,\boldsymbol{\theta_0})-A(s,\boldsymbol{\theta_0})}}{\Gamma(A(t,\boldsymbol{\theta_0}) - A(s,\boldsymbol{\theta_0}))} x^{A(t,\boldsymbol{\theta_0})-A(s,\boldsymbol{\theta_0})-1} \exp(-B(\boldsymbol{\theta_0})x), \forall x \geq 0 \tag{1}$$

where $\Gamma(A(t,\boldsymbol{\theta_0})) = \int_0^\infty s^{A(t,\boldsymbol{\theta_0})-1} e^{-s} ds$ (e.g. see [1]).

We recall that the mean and, variance of Y_t are given by

$$\mathbb{E}[Y_t] = \frac{A(t, \boldsymbol{\theta_0})}{B(\boldsymbol{\theta_0})};\ \mathbb{V}[Y_t] = \frac{A(t, \boldsymbol{\theta_0})}{B(\boldsymbol{\theta_0})^2};$$

for all $t \geq 0$.

A product is considered as failed when its deterioration level exceeds a given failure threshold L. We denote the time at which the failure occurs $\tau_L = \inf\{t > 0 : X_t > L\}$. The reliability function is given by

$$R(t, \boldsymbol{\theta_0}) = \mathbb{P}[\tau_L \geq t] = \mathbb{P}[X_t \leq L] = 1 - \frac{\Gamma(A(t, \boldsymbol{\theta_0}), LB(\boldsymbol{\theta_0}))}{\Gamma(A(t, \boldsymbol{\theta_0}))} \tag{2}$$

where $\Gamma(A(t, \boldsymbol{\theta_0}), LB(\boldsymbol{\theta_0})) = \int_{LB(\boldsymbol{\theta_0})}^{\infty} s^{A(t,\boldsymbol{\theta_0})-1} e^{-s} ds$ is the incomplete gamma function.

3. The degradation test plan

The point here is to propose a degradation test plan based on gamma process model. As a first step, we provide the procedure of the testing plan which is designed by minimizing the asymptotic variance of the reliability estimation of the products. As a second step, we work on a specific parametric form of the shape function of the gamma process and estimate the corresponding parameters and the asymptotic variance.

3.1. *Assumptions*

(A_1) The product will not fail during the test plan.
(A_2) The degradation level is measured through periodic and perfect inspections.
(A_3) The shape of the degradation of the product is assumed to be a parametric function.

3.2. *Procedure*

Let n stands for the sample products and t_0 for the test duration of a product. Each unit is subject to an accumulative deterioration over time, which is measured by a non-decreasing stochastic process $\boldsymbol{Y} = (Y_t)_{t \geq 0}$ with $Y_0 = 0$. We here assume that $\boldsymbol{Y}$ is a non-homogeneous gamma process (presented in Section 2) with shape function $A(t, \boldsymbol{\theta}) = at^{\alpha}$ and scale parameter $B(\boldsymbol{\theta}) = \beta$, where $\boldsymbol{\theta} = (a, \alpha, \beta)$ is the unknown parameter vector to be estimated, whose true value is $\boldsymbol{\theta}_0$.

Let now denote by $R^*(t_0)$ the allowed reliability level of a product after t_0 time with $1-\xi$ confidence level, $\xi \in (0,1)$. During test plan, each product n is inspected at time $S_j, j = 1,\ldots,k$ with $S_k \leq t_0$, to measure the level of degradation Y_{S_j}. Then, the parameter vector $\hat{\boldsymbol{\theta}}_i^n = (\hat{a},\hat{\alpha},\hat{\beta})$ is estimated using a statistical estimation technique, which is the Maximum Likelihood Estimation (MLE), presented in Section 3.3, and the corresponding reliability function $R(t,\hat{\boldsymbol{\theta}}_j^n)$ given by (2) is then computed.

An illustration of the degradation level of four products over $[0,t_0]$ is presented in Figure 1.

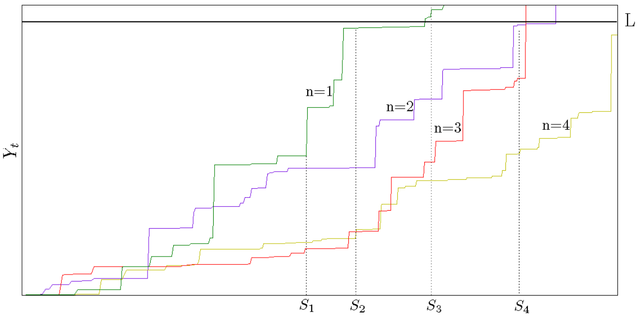

Figure 1. Level of degradation of 4 products over a fixed horizon $[0,t_0]$

Based on Theorem 3.3 from [7], we here have that the MLE estimates are asymptotically normal

$$\sqrt{nj}(\hat{\boldsymbol{\theta}}_{\boldsymbol{j}}^{\boldsymbol{n}} - \boldsymbol{\theta}_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \boldsymbol{J}_{\boldsymbol{\theta}}^{-1})$$

where $\boldsymbol{J}_{\boldsymbol{\theta}}^{-1}$ is the corresponding asymptotic variance. Then by using the functional delta method (see Theorem 2.8 in [5]) we obtain

$$\sqrt{nj}(R(t_0,\hat{\boldsymbol{\theta}}_j^n) - R(t_0,\boldsymbol{\theta}_{\mathbf{0}})) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V(t_0,\boldsymbol{\theta}))$$

where

$$V(t_0,\boldsymbol{\theta}) = \frac{\partial}{\partial\boldsymbol{\theta}} R(t_0,\boldsymbol{\theta}) \boldsymbol{J}_{\boldsymbol{\theta}}^{-1} \frac{\partial}{\partial\boldsymbol{\theta}} R(t_0,\boldsymbol{\theta}).$$

Using this results, we compute the reliability confidence interval by

$$CI_{1-\xi/2}(n,j) = R(t_0,\hat{\boldsymbol{\theta}}_j^n) \pm q_{1-\xi/2}\sqrt{\frac{\hat{V}(t_0,\hat{\boldsymbol{\theta}}_j^n)}{nj}} \qquad (3)$$

where $q_{1-\xi/2}$ is the $1-\xi/2$ quantile of the standard normal distribution and $\hat{V}(t_0, \hat{\boldsymbol{\theta}}_{\boldsymbol{j}}^{\boldsymbol{n}}) = \frac{\partial}{\partial \boldsymbol{\theta}} R(t_0, \hat{\boldsymbol{\theta}}_j^n) \hat{\boldsymbol{J}}_{\hat{\boldsymbol{\theta}}_j^n}^{-1} \frac{\partial}{\partial \boldsymbol{\theta}} R(t_0, \hat{\boldsymbol{\theta}}_j^n)$ where $\hat{\boldsymbol{J}}_{\hat{\boldsymbol{\theta}}_j^n}$ is the estimator of $\boldsymbol{J}_{\boldsymbol{\theta}}$.

A test plan (n, S_j) is therefore one of the couples that leads to the validation of the specification. This specification could be determined in addition, e.g., with the customer risk η which can then give an acceptable minimal reliability level or a specific confidence interval CI^*.

3.3. *Estimation*

We now need to estimate the parameters of a gamma process. With that goal, we use a MLE technique to estimate the gamma process. The estimates are obtained by maximizing the following log-likelihood function (see [3], [9])

$$\begin{aligned} l(Y_{S_2} - Y_{S_1}, \ldots, Y_{S_j} - Y_{S_{j-1}} | \boldsymbol{\theta}) &= \ln \left(\prod_{j=1}^{k} \prod_{i=1}^{n} \frac{\beta^{a d_j(\alpha)}}{\Gamma(a d_j(\alpha))} \Delta_{i,j}^{a d_j(\alpha)-1} e^{-\beta \Delta_{i,j}} \right) \\ &= na \ln(\beta) \sum_{j=1}^{k} d_j(\alpha) - n \sum_{j=1}^{k} \ln(\Gamma(a d_j(\alpha))) \\ &+ \sum_{j=1}^{k} \sum_{i=1}^{n} (a d_j(\alpha) - 1) \ln(\Delta_{i,j}) - \beta \sum_{j=1}^{k} \sum_{i=1}^{n} \Delta_{i,j} \end{aligned} \tag{4}$$

where $d_j(\alpha) = S_j^{\alpha} - S_{j-1}^{\alpha}$ and $\Delta_{i,j} = Y_{S_j}^{(i)} - Y_{S_{j-1}}^{(i)}$ refers to the increment interval between two inspections of product $Y^{(i)}, i = 1, \ldots, n$.

The asymptotic variance matrix of the MLE estimates is given by

$$\boldsymbol{J}_{\boldsymbol{\theta}}^{-1} = -\mathbb{E} \left[\begin{pmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a \partial \alpha} & \frac{\partial^2 l}{\partial a \partial \beta} \\ \frac{\partial^2 l}{\partial \alpha \partial a} & \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} \\ \frac{\partial^2 l}{\partial \beta \partial a} & \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} \end{pmatrix} \right]^{-1}$$

where

$$\frac{\partial^2 l}{\partial a^2} = -n \sum_{j=1}^{k} d_j(\alpha)^2 \psi(1, a d_j(\alpha)); \frac{\partial^2 l}{\partial \beta^2} = -\frac{na}{\beta^2} \sum_{j=1}^{k} d_j(\alpha);$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= na\ln(\beta)\sum_{j=1}^{k}\left[\ln(S_j)^2 S_j^{\alpha} - \ln(S_{j-1})^2 S_{j-1}^{\alpha}\right] \\
&- n\sum_{j=1}^{k}\left[a^2(\ln(S_j)S_j^{\alpha} - \ln(S_{j-1})S_{j-1}^{\alpha})^2\psi(1, ad_j(\alpha)) + a(\ln(S_j)^2 S_j^{\alpha} - \ln(S_{j-1})^2 S_{j-1}^{\alpha})\psi(ad_j(\alpha))\right] \\
&+ a\sum_{j=1}^{k}\sum_{i=1}^{n}\left[\ln(S_j)^2 S_j^{\alpha} - \ln(S_{j-1})^2 S_{j-1}^{\alpha}\right]\ln(\Delta_{i,j});
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \alpha \partial a} &= n\ln(\beta)\sum_{j=1}^{k}\left[\ln(t_j)S_j^{\alpha} - \ln(S_{j-1})S_{j-1}^{\alpha})\right] \\
&- n\sum_{j=1}^{k}\left[a\psi(1, ad_j(\alpha))d_j(\alpha) + \psi(ad_j(\alpha))\right](\ln(S_j)S_j^{\alpha} - \ln(S_{j-1})S_{j-1}^{\alpha}) \\
&+ \sum_{j=1}^{k}\sum_{i=1}^{n}\left[\ln(S_j)S_j^{\alpha} - \ln(S_{j-1}))S_{j-1}^{\alpha}\right]\ln(\Delta_{i,j});
\end{aligned}
$$

$$
\frac{\partial^2 l}{\partial \beta \partial a} = \frac{n}{\beta}\sum_{j=1}^{k} d_j(\alpha); \; \frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{na}{\beta}\sum_{j=1}^{k}\left[\ln(S_j)S_j^{\alpha} - \ln(S_{j-1})S_{j-1}^{\alpha}\right].
$$

Finally, the asymptotic variance of the delta method presented in Section 3.2 is given below.

$$
V(t_0, \boldsymbol{\theta}) = \left(\frac{\partial R(t_0, \boldsymbol{\theta})}{\partial a}, \frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \alpha}, \frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \beta}\right) \boldsymbol{J}_{\boldsymbol{\theta}}^{-1} \left(\frac{\partial R(t_0, \boldsymbol{\theta})}{\partial a}, \frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \alpha}, \frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \beta}\right)^T
$$

with

$$
\frac{\partial R(t_0, \boldsymbol{\theta})}{\partial a} = \frac{{t_0}^{\alpha}}{\Gamma(a{t_0}^{\alpha})}\left[\int_{L\beta}^{\infty} \ln(x)\, x^{at_0^{\alpha}-1} e^{-x} dx - \Gamma(a{t_0}^{\alpha}, L\beta) t^{\alpha} \psi(at_0^{\alpha})\right];
$$

$$
\frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \alpha} = \frac{a\ln(t_0){t_0}^{\alpha}}{\Gamma(a{t_0}^{\alpha})}\left[\int_{L\beta}^{\infty} \ln(x)\, x^{at_0^{\alpha}-1} e^{-x} dx - \Gamma(a{t_0}^{\alpha}, L\beta)\psi(at_0^{\alpha})\right];
$$

$$
\frac{\partial R(t_0, \boldsymbol{\theta})}{\partial \beta} = \frac{(L\beta)^{at_0^{\alpha}-1} e^{-L\beta}}{\Gamma(a{t_0}^{\alpha})}.
$$

4. Numerical experiments

In this section we provide a numerical illustration of the combined effects of the number of the tested products and the testing duration. We here assume that all of the provided degradation trajectories come from a non-homogeneous gamma process. With that goal, we consider $A(t, \boldsymbol{\theta}) = t^{0.6}$ and $B(\boldsymbol{\theta}) = 1.4$ and set $L = 2$ and $t_0 = 10$. The qualification specifications are the objective reliability $R^*(t_0) \simeq 86.5\%$ with a confidence level of 95% and $CI^* = [81\%, 91\%]$. The inspections times S_j are chosen as multiple of $\delta = 1$.

We first independently simulate $R = 5000$ sets of n independent trajectories. This provides R estimations $\hat{\boldsymbol{\theta}}^{\boldsymbol{n}}_{\boldsymbol{j,r}}$ with $r = 1, \dots, R$, $j = 1, \dots, 10$, and $n = 1, \dots, 10$. Once the parameters are estimated, we compute the corresponding reliability and confidence intervals from (2) and (3), respectively. The results are displayed in Table 4. In this example, we can notice the greater influence of the test duration on the quality of the estimation compared to the number of products to be tested, compare for instance the 1% estimated reward to add 4 products at S_6 to the 7% reward with 4 additionnal time units (S_6 to S_{10}) for $n = 6$. This fact is irrelevant in the case of homogeneous gamma degradation processes where only the number of observations should be important in the quality of the estimation.

Table 1. Confidence intervals for different couples (n, S_j)

(n, S_j)	6	7	8	9	10
6	[75.62%, 91.94%]	[78.82%, 90.27%]	[80.64%, 89.28%]	[82.06%, 88.92%]	[82.89%, 88.49%]
7	[76.06%, 90.87%]	[79.27%, 89.83%]	[81.03%, 89.02%]	[82.33%, 88.59%]	[82.90%, 88.00%]
8	[76.33%, 90.04%]	[79.57%, 89.24%]	[81.00%, 88.42%]	[82.21%, 88.08%]	[83.14%, 87.93%]
9	[76.75%, 89.37%]	[79.25%, 88.29%]	[81.01%, 87.93%]	[82.40%, 87.89%]	[83.17%, 87.67%]
10	[76.90%, 88.67%]	[79.51%, 88.07%]	[81.37%, 87.99%]	[82.50%, 87.71%]	[83.48%, 87.71%]

5. Conclusion

In this paper, we have presented the materials for the design of degradation test plans for a non-homogeneous gamma process using a specific parametric form. The numerical application highlights the predominant effect of the test duration time versus the number of products to be tested. This points out the validity of the assumption that no failures are observed during the tests, especially if the degradation growth rate is rather high.

Removing this assumption could be of interest if the cost of a product is high regarding the testing time. This work should be seen as an initiative of a future work, which is about extending the proposed test plan to a semi-parametric test plan where the parametric form is in general unknown. The semi-parametric estimation technique to estimate the gamma processes, developed by [10], could be a promising approach even if it is integration in the test plan design process seems to be challenging.

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