Wave propagation in an auxetic core embedding resonators

M.A. Campana [1], **F. Scarpa** [1], **M. Ouisse** [2], **E. Sadoulet-Reboul** [2]
[1] Bristol Composites institute (ACCIS), CAME, University of Bristol,
University Walk, BS8 1TR Bristol, UK

[2] University Bourgogne Franche-Comté, CNRS/UFC/ENSMM,UTBM,FEMTO-ST, Department of Applied Mechanics,
24 Chemin de l'Epitaphe, Besancon 25000, France

Abstract

Stop bands in auxetic periodic structures including resonators are investigated in this paper. In aerospace field, sandwich panels like honeycombs are mainly used for their structural properties, and can present some advantages for vibroacoustics control. Auxetic structures seem to give enhanced properties for wave propagation control, and have been used here to perform the analysis. Resonators are included to open resonant band gaps in addition to the Bragg band gaps given by the periodicity of the core. Studies on the number of resonators inside the unit cell have been performed for a constant mass to identify the configuration which are able to open stop bands at low frequency range. Also, a parametric analysis on the resonator properties is done to compare the benefit concerning wave propagation while having a reasonable added mass for the core structure.

1 Introduction

Wave propagation in periodic structures has been investigated in many different fields of physics such as optics, acoustics or mechanics. More particularly, the study of crystal lattices known as phononic crystals as been highly investigated, representing a quantum vibrational of energy in an elastic medium, in order to create stop-bands for certain values of frequency, preventing the wave propagation through the periodic media due to an interaction between transmitted and reflected waves. Recently, periodic structure analyses have been extended to engineering structures such as honeycombs [1] in the aerospace field. One of the important aspects of introducing theory of periodic structures in engineering is the ability of describing behaviour of an entire structure just by analysing a part of it, called unit cell, instead of doing the analysis in the whole structure, earning a non negligible amount of computational time. Another interest, as it has been developed in the past with the phononic crystals, is the possibility of analysis of the band gaps linked to the periodicity. Unlike phononic crystals, many types of elastic waves exist inside elastic periodic structures (longitudinal, flexural, shear...) and some techniques can be used to cancel the propagation of specific type of waves. One of them consists in taking advantage of the Bragg band gaps, which are directly linked to the size of the unit cell constituting the whole structure. In this work, another type of band gap is investigated, called resonant band gap, created by adding some inserts such as resonators inside the structure. This approach make possible to improve the vibroacoustics performance of the structure by keeping its geometry and material properties constant, and so find a great interest in sandwich like structure as honeycombs, initially designed to improve the ratio stiffness over the mass of the structure, providing a good resistance for a low weight. Including resonators become on that way interesting since they do not modify the shape of the periodic structure, keeping all the structural characteristic of the honeycomb. Moreover, unlike the Bragg band gap, the resonant effect can open stop bands in the first modes of the structure, making the study more efficient to work in the low frequencies range. Resonators have been mainly investigated for chained spring-mass systems [2], or beams [3] and plates including resonators [4], [5]. Many of these works have been also summarized covering several types of structure [6]. Among all these studies, it has been shown that the location of the band gap is narrowly linked to the eigenfrequency of the resonator while the bandwidth depends on the mass ratio between the resonator and the structure itself, allowing a predictable design of the resonators to target specific frequencies. However, this specificity is not true anymore when the structure has a complex shape like honeycombs. The shape of the media is not continuous as a plate or a beam, and the dynamic of the structure itself has to be taken in account in addition to the effect of the resonator, making the prediction of the cut-off frequencies more complex. Some analyses have been performed in an re-entrant core of a sandwich structure, to investigate how has to be tuned the resonators in order to obtain the widest band gaps in a relatively low range of frequency.

2 Periodic structures

A periodic structure is generally represented by a portion of the structure, called unit cell, representing the whole structure if it is repeated in one or several direction. We call 1D periodic structure if the repetition follows one direction, and so on for 2D and 3D periodic structure. To study the behaviour of the structure in an infinite way, Floquet-Bloch Theorem is applied to have access to the dispersion curve, giving the relation between the frequency and the wavenumber of the whole structure. Also, the Transfer Matrix Method is usually used in 1D case to compute the finite response of the full structure, choosing a fixed number of repetition of the unit cell.

2.1 Floquet-Bloch Theorem

The application of the Floquet-Bloch Theorem requires the a periodic structure to be applied. A simple representation for the 2D case is given in the figure 1. The unit cell is repeated along the x_p and y_p direction and its size (a_x, a_y). Each point P of a cell can be linked with the origin of the periodic structure thanks to the relation $r_p = r_{lp} + \boldsymbol{i}x_p + \boldsymbol{j}y_p$, with $\boldsymbol{i}$ and $\boldsymbol{j}$ the number of repetitions in each direction. Floquet Theorem has been introduced for the first time in the analysis of the solutions of periodic differential equations [7] in the 1D case. The Theorem stipulates that if $A(t)$ is a T-periodic matrix and $X(t)$ the fundamental matrix of solution of the ODE

$$\frac{\partial \mathbf{x}}{\partial t} = A(t)\mathbf{x} \tag{1}$$

then $X(t) = P(t)e^{Rt}$ is a solution of equation (1) with $P(t)$ a T-periodic matrix and R a $n \times n$ matrix. From this equation it comes that

$$X(t+T) = X(t)e^{RT} \tag{2}$$

In the field of mechanics, the Floquet theorem has been extended for 2D and 3D cases by the actual Floquet-Bloch Theorem for periodic structures

$$u(P) = u(O)e^{\mathbf{k}(\mathbf{i}x_p + \mathbf{j}y_p)} \tag{3}$$

with $\mathbf{k}$ the wavevector and $u(O), u(P)$ respectively the amplitudes of displacement at the origin and at the current point P.

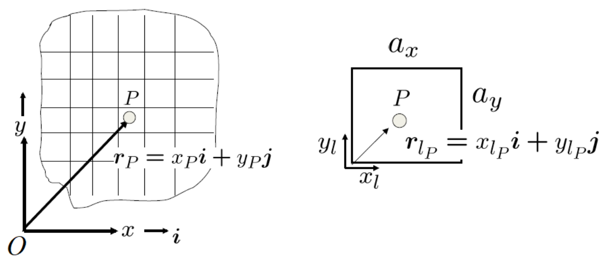


Figure 1: Representation of a 2D unit cell

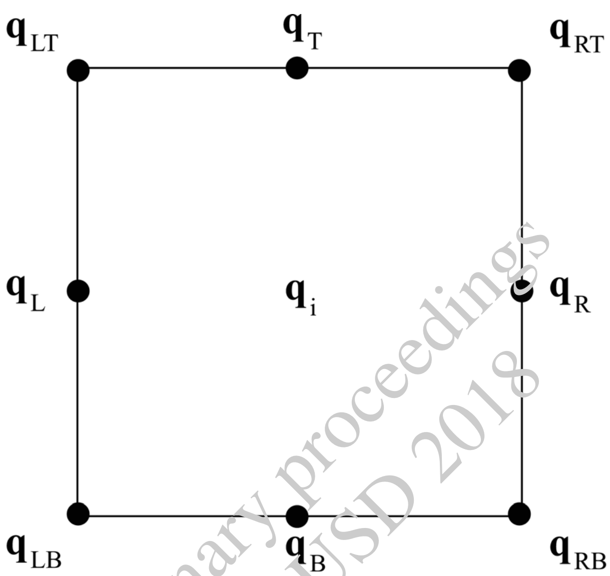


Figure 2: General representation of a unit cell using the finite element method in 2D

2.2 Application to periodic structures

The Floquet-Bloch Theorem can be applied in structures considered as continuous or discretised using the finite element method. Since the analytical expression become complex for honeycombs structures including resonators, the second option is used in this study. One of the most common tool used to study periodic structures involving the Floquet-Bloch theory is the Wave Finite Element Method (WFEM). Considering a discretisation of the structure as shown figure 2, all the boundaries of the unit cell can be linked using the Floquet relations, such as

$$\begin{aligned} &q_R = e^{j\mu_x} q_L,\ q_T = e^{j\mu_y} q_B,\ q_{RT} = e^{j(\mu_x+\mu_y)} q_{LB},\ q_{LT} = e^{j\mu_y} q_{LB},\ q_{RB} = e^{j\mu_x} q_{LB} \\ &f_R = -e^{j\mu_x} f_L,\ f_T = -e^{j\mu_y} f_B,\ f_{RT} = -e^{j(\mu_x+\mu_y)} f_{LB},\ f_{LT} = -e^{j\mu_y} f_{LB},\ f_{RB} = -e^{j\mu_x} f_{LB} \end{aligned} \tag{4}$$

where $\mu_x = k_x L_x$ (respectively $\mu_y = k_y L_y$) is the reduced wavenumber, with k_x, k_y the wavenumber and L_x, L_y the length of the unit cell in the x or y direction, q is the vector of displacements on each node and f the vector of efforts. The letters L, B, R, T relate respectively to left, bottom, right and Top. The effect of damping is neglected here since some works have been done including it [8], so the dynamic equation is given by

$$([K] - \omega^2 [M])q = f \tag{5}$$

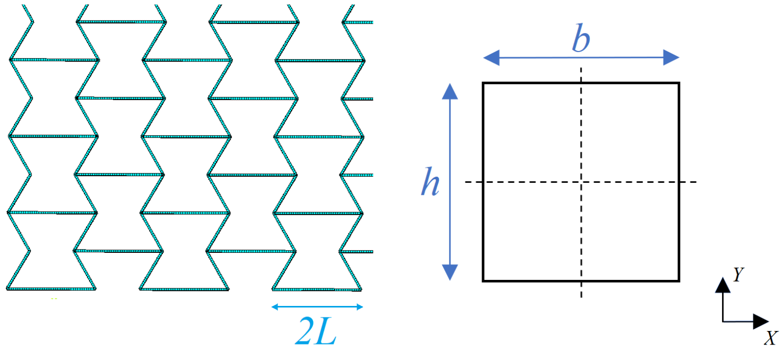


Figure 3: Part of the re-entrant structure

with $[K]$ and $[M]$ respectively the stiffness and mass matrix of the unit cell, q the vector of displacements on each node and f the vector of efforts. Including the Floquet relations into the equation (5), and assuming that all the internal efforts are equal to 0, it leads to a new dynamic relation given by

$$([K_r(\mu_x, \mu_y)] - \omega^2[M_r(\mu_x, \mu_y)])q_r = 0 \tag{6}$$

where $[K_r]$, $[M_r]$ and q_r are the reduced vectors. The equation (6) is an eigenvalue problem in ω^2 which leads, once solved for input values of μ_x and μ_y, to the dispersion curve giving the value of the group velocity inside the full structure, and also to the locations of the band gaps. The values of reduced wavenumber are chosen according to the First Brillouin Zone [9]. In this study, the First Brillouin Zone is given by $\mu_x = [0, \pi]$, $\mu_y = [0, \pi]$, and only the contour of the dispersion surface will be plot in order to identify the band gaps.

2.3 Model of the unit cell

For this study, a re-entrant and an hexagonal structure are studied as the core of the periodic structure. A Finite Element Model has been considered here, using Timoshenko beam elements with squared cross section. A representation of the Finite Element structure is shown in figure 3 and the geometric/material properties are listed in the table 1 for the auxetic configuration, where L refers to the length of one beam constituting the unit cell, b the thickness in X and h the thickness in Y direction. The hexagonal one has exactly the same properties, excepted the internal angle going from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$.

The resonator used is also modelised as a Timoshenko beam, attached to the corners of the re-entrant unit cell in a fixed-free configuration. The properties of the resonators are the same as the unit cell, excepted for the length given by $L_r = 6.8$ mm. A representation of the unit cell with the resonator is shown in figure 4. This design of unit cell has already been used in the past [10], and is very convenient to include resonators. Indeed, in this configuration, it is possible to include two resonators by unit cell, which corresponds to exactly one resonator by "butterfly" pattern. Also, configurations with four resonators by unit cell will be analysed to check the impact on vibroacoustics performances.

The Floquet conditions are applied to the border of the unit cell of the figure 4, leading to the following relations:

$$\begin{aligned} q_R &= e^{j\mu_x} q_L, \ q_{T_1} = e^{j\mu_y} q_{B_1}, \ q_{T_2} = e^{j\mu_y} q_{B_2} \\ f_R &= -e^{j\mu_x} f_L, \ f_{T_1} = -e^{j\mu_y} f_{B_1}, \ f_{T_2} = -e^{j\mu_y} f_{B_2} \end{aligned} \tag{7}$$

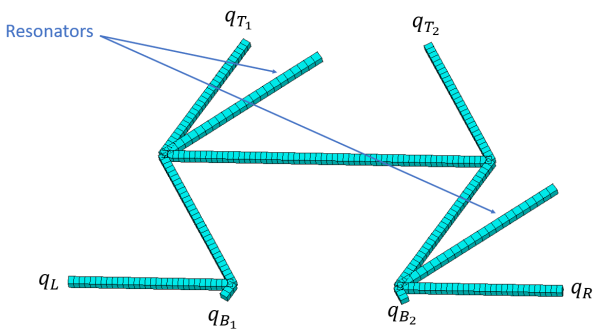


Figure 4: Re-entrant unit cell with resonator

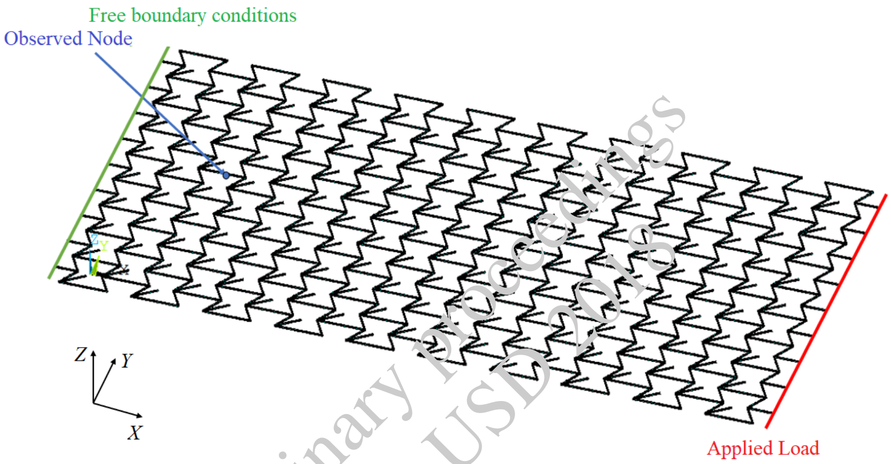


Figure 5: Periodic structure in finite configuration

2.4 Model of the finite structure

In parallel to the Floquet analysis, studies in finite structure are also performed. Harmonic analysis of a panel constituted of finite number of cells is performed, in order to compute the FRF at any node of the structure. A representation of this configuration is shown figure 5. The structure is composed with 9 unit cells in the Y direction, and 20 cells in the X direction. Free-free conditions are used, and, depending the type of analysis performed, an out-of-plane or an in-plane force is applied to excite the different type of waves and thereby observe the different band gaps on the FRF. The amplitude of the displacement is observed in a node picked just before the end of the structure in order to avoid the border effect we could have in the end of the periodic media.

Density, ρ $[kg.m^{-3}]$	1040	L [mm]	6
Young Modulus, E [GPa]	8.1	h [mm]	0.3
Poisson Ratio, ν	0.2	b [mm]	0.3

Table 1: Material and geometrical properties of the core

3 Simulations and Results

3.1 Selection of the core

Some analyses are done in infinite structure to compare several types of configurations of the studied core. The re-entrant and honeycomb configurations are chosen. For each curve, the segments [O-A-B-C-O] represent the values that μ_x and μ_y take along the contour, and are summarized in the table 2. The contour of the dispersion surfaces are represented figures 6 and 7.

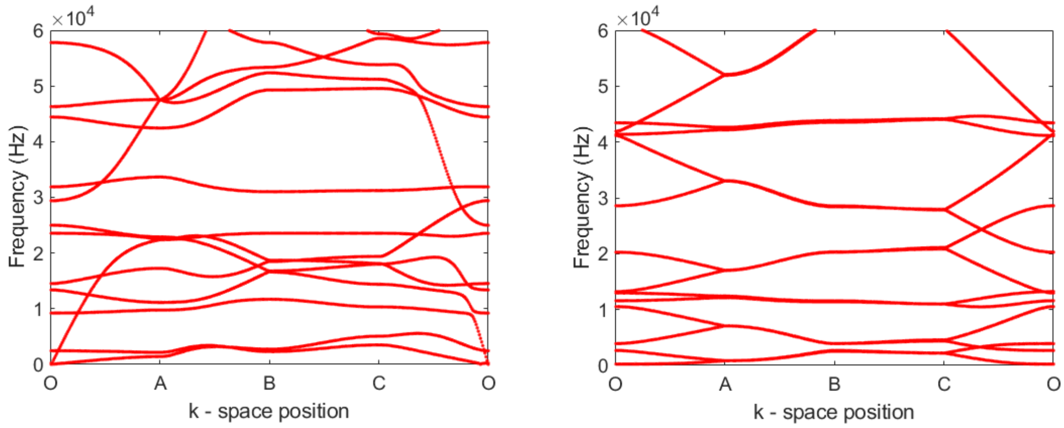
The first one represents the wave propagation in plane (degrees of freedom u_z, θ_x and θ_y blocked) while the second one out of plane (u_x, u_y and θ_z blocked), leading to a total of 4 different configurations. In the 4 configurations, only the fourth one exhibits Bragg band gaps and is selected as the core structure to continue the investigations including the resonators. The Resonators are then added to the re-entrant unit cell as shown in the figure 4, with the same material properties as the core. This resonator emplacement has been chosen for easier manufacturing. The dispersion diagram is plotted figure in 8. Around 20, 50 and 60 kHz, we can observe that some dispersion branches are opened to create a resonant band gap as shown if 8. This result is confirmed in the finite structure analysis, which is obtained by blocking in-plane degree of freedom and applying a force in the Z direction.

Segment	[O-A]	[A-B]	[B-C]	[C-O]
μ_x	$[0, \pi]$	π	$[\pi, 0]$	0
μ_y	0	$[0, \pi]$	π	$[\pi, 0]$

Table 2: Values of μ_x and μ_y for the dispersion contour

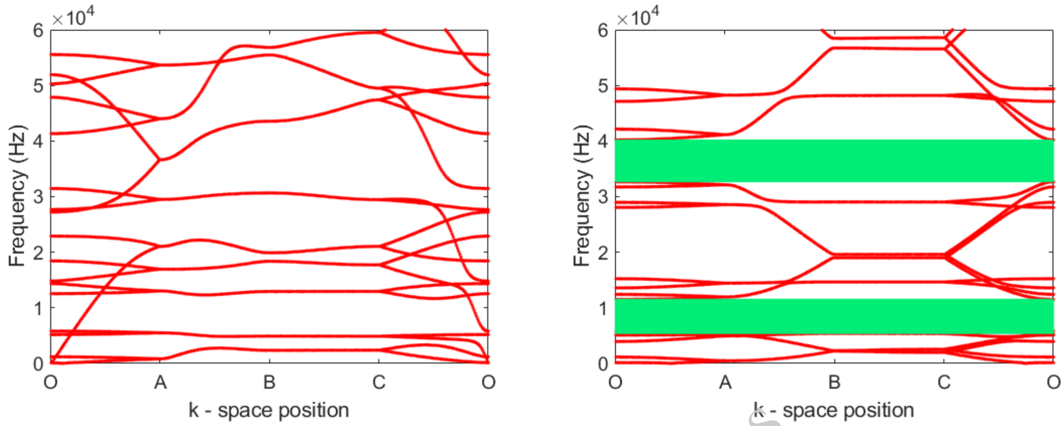
3.2 Number of resonators in the unit cell

To have a first look in the choice of the implementation of the resonator, an investigation has been done to know what is the optimal solution between putting 2 resonators or 4 resonators inside a unit cell, correspond-



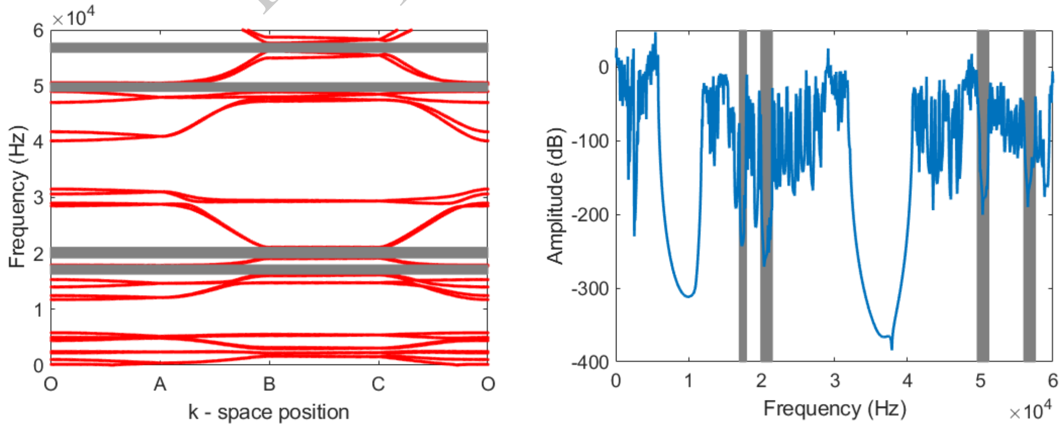
(a) In plane waves (b) Out of plane waves

Figure 6: Dispersion curve without resonator with hexagonal unit cell



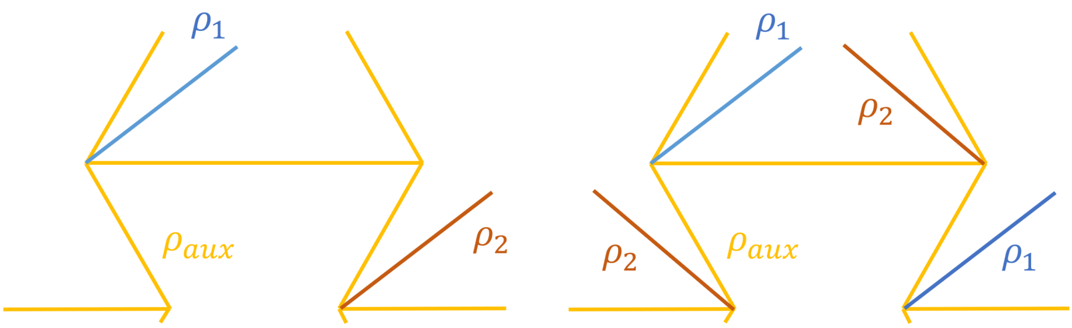
(a) In plane waves (b) Out of plane waves

Figure 7: Dispersion curve without resonator with re-entrant unit cell



(a) Dispersion curve with resonator (b) FRF with resonators

Figure 8: Dispersion curve without resonator with re-entrant unit cell



(a) 2 resonators (b) 4 resonators

Figure 9: Two different case of resonator implementation

ing to include respectively 1 or 2 resonators inside a "butterfly" pattern of the periodic structure. Two studies have been carried out making a variation of the density of each resonator, such as the total mass of the unit cell remain constant. The low frequencies are focused here, since the objective here is to put resonators is in order to open band gap for the first Floquet modes. To understand what are the low frequencies in that situation, we take as a reference the first mode of one of the beam constituting the unit cell, and we compute its first eigenfrequency in supported-supported boundary conditions. The value of the first eigenfrequency is $f_0 = 10505$ Hz. The unit cells are represented figure 9 and the evolution of the band gaps are shown in figure 10. Each configuration corresponding to the values of ρ_1 and ρ_2 are given in the table 3.

The results observed in figure 10 show that the optimal configuration to open band gaps in the lowest frequency correspond to the configuration number 10 for the case with 2 resonators, meaning that having the same density for the 2 resonators is more interesting than having two different densities. For the case with 4 resonators, it is the configuration 1 (or 19) which is the more interesting, and converge to the same result that the configuration number 10 for the case with 2 resonators. The similitude of the results is explained in the way that, for the configuration 1, one has $\rho_1 << \rho_2$ and $\rho_2 \simeq \rho_{aux}$, which tends to the case with 2 resonators. The configuration number 10 correspond to 4 equal densities for each resonator, and corresponds the worst result for a low frequency band gap opening in that situation.

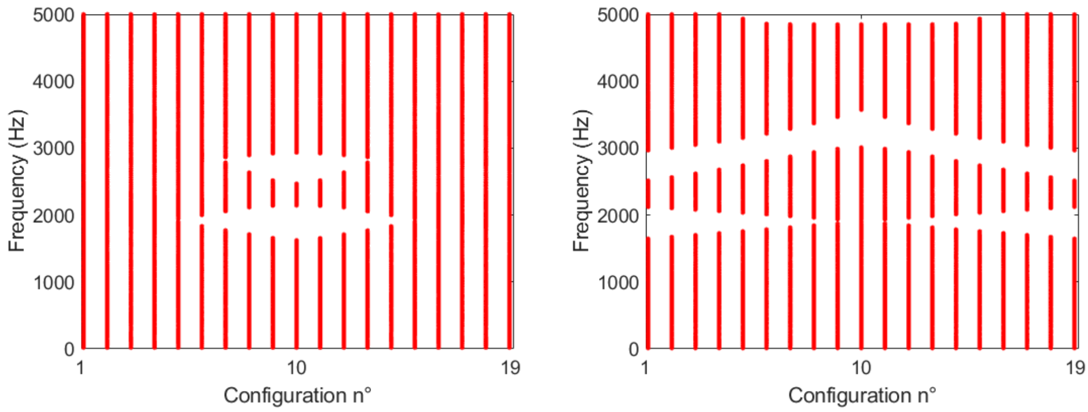
These results show that, for this type of unit cell, having 1 resonator per pattern is the best solution to have an interesting wave behaviour in low frequencies.

	Configuration n°	**1**	**2**	**3**	...	**10**	...	**19**
2 Resonators	$\frac{\rho_1}{\rho_{aux}}$	0.1	0.2	0.3	...	0.1	...	1.9
	$\frac{\rho_2}{\rho_{aux}}$	1.9	1.8	1.7	...	1	...	0.1
4 Resonators	$\frac{\rho_1}{\rho_{aux}}$	0.05	0.1	0.15	...	0.5	...	0.95
	$\frac{\rho_2}{\rho_{aux}}$	0.95	0.90	0.85	...	0.5	...	0.05

Table 3: Values of ρ_1 and ρ_2 for each configuration

3.3 Parametric Analysis

A parametric analysis is then performed by varying the material of the resonators. Since one of the main interest of creating a sandwich panel is the improvement of the ratio mass over the stiffness, the density of the resonator is chosen as a parameter. A comparison between the improvement band gap positioning and width versus the total added mass of the unit cell is checked. The evolution of the band gaps is represented figure 11 and is again focused on the low frequency range. The percentage of added mass is



(a) Study with 2 resonators

(b) Study with 4 resonators

Figure 10: Band gap evolution analysis by varying the density of the resonators. Configurations are given table 3

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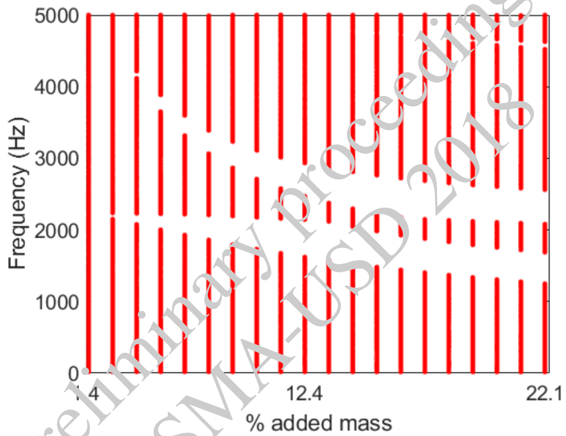

Figure 11: Band gaps investigation with variation of the mass of the resonator

$$a_m = 1 + \frac{2L_r \rho_{reso}}{8L_{aux}\rho_{aux}} \tag{8}$$

where ρ_{reso} represents the density of the resonator. Looking at the figure 11, we can observe that waves are naturally going to the low frequencies as we increase the value of the density of the resonator. For low values of added mass (< 5 %) no band gap is present in this frequency range. However, they start to open after that value in two different branches. Above about 15 %, the width of the band gaps remain almost constant but continue to slowly going to lowest frequencies. By seeing this result and according to the frequency that we want to isolate and the maximum of added mass tolerated, it is possible to chose a value of the density for the resonator to create a stop band.

4 Conclusion

This work gives first investigations in wave behaviour of auxetic honeycombs including resonators. After a first look on different geometry of unit cell, the re-entrant one is selected for the better Bragg band gaps provided, and more specifically for waves travelling out of plane of the periodic structure. Beam resonators have been introduced then, comparing the configurations with 2 resonators or 4 resonators in one unit cell. The results have shown that the best configuration on the analysed ones is having exactly one resonator by butterfly core cell, which corresponds to two resonators in the unit cell with the same density. Finally, since one of the most important parameter in the aerospace domain is the mass, a parametric study on the density of the resonator has been done in order to see what improvement can been obtained in a vibroacoustics versus a structural point of view. On the structure of interest, it has been shown that we need to have at least 5% added mass to open band gaps in low frequency range, while the bandwidth remains nearly constant after 15% added mass.

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