Diffusive systems coupled to an oscillator: a Hamiltonian formulation *

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Abstract: The aim of this paper is to study a conservative wave equation coupled to a diffusion equation : this coupled system naturally arises in musical acoustics when viscous and thermal effects at the wall of the duct of a wind instrument are taken into account. The resulting equation, known as Webster-Lokshin model, has variable coefficients in space, and a fractional derivative in time. The port-Hamiltonian formalism proves adequate to reformulate this coupled system, and could enable another well-posedness analysis, using classical results from port-Hamiltonian systems theory.

First, an equivalent formulation of fractional derivatives is obtained thanks to so-called diffusive representations: this is the reason why we first concentrate on rewriting these diffusive representations into the port-Hamiltonian formalism; two cases must be studied separately, the fractional integral operator as a low-pass filter, and the fractional derivative operator as a high-pass filter.

Second, a standard finite-dimensional mechanical oscillator coupled to both types of dampings, either low-pass or high-pass, is studied as a coupled pHs. The more general PDE system of a wave equation coupled with the diffusion equation is then found to have the same structure as before, but in an appropriate infinite-dimensional setting, which is fully detailed.

Keywords: energy storage, port-Hamiltonian systems, partial differential equations, fractional derivatives, diffusive representation

1. INTRODUCTION

The dissipative model which describes acoustic waves travelling in a duct with viscothermal losses at the lateral walls is a wave equation with spatially-varying coefficients, which involves fractional-order integrals and derivatives with respect to time. This model is first rewritten in a coupled form; then the fractional integrals and derivatives are written in their so-called diffusive representation; essentially, the fractional-order time kernel in the integral is represented by its Laplace transform.

The main idea of the present work is to put the Webster-Lokshin fractional PDE into the port-Hamiltonian framework, in order to take advantage of this setting. To do so, a preliminary work is necessary, that is using diffusive representations of both fractional integrals and derivatives in order to imagine the ad hoc Hamiltonian formulation. The coupling between conservative and dissipative subsystems is then easily tackled in this setting; but for the PDE, as usual, some care must be taken with the functional setting.

The outline of the paper is as follows: in § 2, diffusive representations are introduced in order to replace fractional integral and derivative operators by input-output representations, and state-space representation, which prove compatible with first order dynamical systems. In order to set up a Hamiltonian formulation of both these operators, a finite-dimensional toy-model is first studied in § 3: ad hoc discrete energies are being defined, skew-symmetric and symmetric structural matrices J and R are identified, and the standard port-Hamiltonian stucture of dissipative systems is recovered. Finally, the fully infinite-dimensional case is presented in § 4: the Webster-Lokshin model is recast in the setting of port-Hamiltonian systems with dissipation.

2. A PRIMER ON DIFFUSIVE REPRESENTATION

In this section, we focus on the *causal* solution of a family of first-order ordinary differential equations (ODEs). Hence, the mathematical setting is the convolution algebra $\mathcal{D}'_+(\mathbb{R})$ of causal distributions.

2.1 An Elementary Approach

Consider the numerical identity, valid for $\delta > 1$:
$$\int_0^\infty \frac{\mathrm{d}x}{1 + x^\delta} = \frac{\frac{\pi}{\delta}}{\sin(\frac{\pi}{\delta})}.$$

Letting $s \in \mathbb{R}_*^+$ and substituting $x = (\frac{\xi}{s})^{\frac{1}{\delta}}$ in the above *numerical* identity, we get:

$$\int_0^\infty \frac{\sin(\frac{\pi}{\delta})}{\pi} \frac{1}{\xi^{1-\frac{1}{\delta}}} \frac{1}{s+\xi} \ \mathrm{d}\xi = \frac{1}{s^{1-\frac{1}{\delta}}}$$

Finally, performing an analytic continuation from $\mathbb{R}^{+*}$ to $\mathbb{C} \setminus \mathbb{R}^-$ for both sides of the above identity in the complex variable s, and letting $\beta := 1 - \frac{1}{\delta} \in (0,1)$, we get the *functional* identity:

$$\begin{aligned} H_\beta : \mathbb{C} \setminus \mathbb{R}^- &\to \mathbb{C} \\ s &\mapsto \int_0^\infty \mu_\beta(\xi) \frac{1}{s+\xi} \ \mathrm{d}\xi = \frac{1}{s^\beta}\,, \end{aligned} \quad (1)$$

with density $\mu_\beta(\xi) = \frac{\sin(\beta\,\pi)}{\pi} \xi^{-\beta}$.

Applying an inverse Laplace transform to both sides gives:

$$\begin{aligned} h_\beta : \mathbb{R}^+ &\to \mathbb{R} \\ t &\mapsto \int_0^\infty \mu_\beta(\xi) \, e^{-\xi\,t} \ \mathrm{d}\xi = \frac{1}{\Gamma(\beta)} t^{\beta-1}\,. \end{aligned} \quad (2)$$

2.2 Input-output Representations

Let u and $y = I^\beta u$ be the input and output of the *causal* fractional integral of order β, defined by the Riemann-Liouville formula $y = h_\beta \star u = \int_0^t h_\beta(t-\tau)\,u(\tau)\,\mathrm{d}\tau$ in the time domain, which reads $Y(s) = H_\beta(s)\,U(s)$ in the Laplace domain.

Using the integral representations above, together with Fubini's theorem, we get:

$$y(t) = \int_0^\infty \mu_\beta(\xi) \, [e_\xi \star u](t)\,\mathrm{d}\xi\,,$$

with $e_\xi(t) := e^{-\xi\,t}$, and $[e_\xi \star u](t) = \int_0^t e^{-\xi\,(t-\tau)}\,u(\tau)\,\mathrm{d}\tau$.

Now for fractional *derivative* of order $\alpha \in (0,1)$ in the sense of distributions of Schwartz, we have $\widetilde{y} = D^\alpha u = D[I^{1-\alpha}u]$, and a careful computation shows that:

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u - \xi\,e_\xi \star u](t)\,\mathrm{d}\xi\,.$$

2.3 State Space Representation

In both input-output representations above, introducing a state, say $\varphi(\xi,.)$ which realizes the classical convolution $\varphi(\xi,.) := [e_\xi \star u](t)$ leads to the following diffusive *realizations*, in the sense of systems theory:

$$\partial_t \varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t),\ \ \varphi(\xi,0) = 0\,, \quad (3)$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi\,; \quad (4)$$

and

$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\,\widetilde{\varphi}(\xi,t) + u(t),\ \ \widetilde{\varphi}(\xi,0) = 0\,, \quad (5)$$

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\widetilde{\varphi}(\xi,t)]\,\mathrm{d}\xi\,. \quad (6)$$

These are first and extended diffusive realizations, respectively. The slight difference between (3)-(4) and (5)-(6), marked by the $\widetilde{\ }$ notation, lies in the underlying functional spaces in which these equations make sense: φ belongs to $\mathcal{H}_\beta := \{\varphi \ s.t. \int_0^\infty \mu_\beta(\xi)|\varphi|^2 \ \mathrm{d}\xi < \infty\}$, whereas $\widetilde{\varphi}$ belongs to $\widetilde{\mathcal{H}}_\alpha := \{\widetilde{\varphi} \ s.t. \int_0^\infty \mu_{1-\alpha}(\xi)|\widetilde{\varphi}|^2\,\xi\,\mathrm{d}\xi < \infty\}$, see e.g. (Haddar et al., 2008, ch. 2).

3. A FINITE-DIMENSIONAL HAMILTONIAN FORMULATION FOR INTEGRAL AND FRACTIONAL DERIVATIVES.

In this section, we first consider a classical mechanical oscillator with fluid damping in § 3.1, then we use the velocity as input of two different types of damping models: a low-pass diffusive subsystem (such as a discretized fractional integral) in § 3.2, or a high-pass diffusive subsystem (such as a discretized fractional derivative) in § 3.3.

3.1 Harmonic oscillator

We start with the port-Hamiltonian formulation of the single finite dimensional harmonic oscillator. Dynamic equation is usually written in the form:

$$m\ddot{x} + \varepsilon\dot{x} + \kappa x = 0 \quad (7)$$

where $x(t) \in \mathbb{R}$ and m, ε, κ are positive constants. By using as state variables the energy variables (i.e. the position and the momentum) and defining the Hamiltonian H_0 as the total energy of the system, *i.e.*:

$$X := \begin{bmatrix} q = x, \\ p = m\dot{x} \end{bmatrix} \quad \text{and} \quad H_0(X) = \frac{1}{2m}p^2 + \frac{1}{2}\kappa x^2\,;$$

it is possible to rewrite (7) in the form of a port-Hamiltonian system:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \partial_X H_0(X) = (J - R_\varepsilon)\,\partial_X H_0(X)\,.$$

where $\partial_X H_0(X) = \begin{bmatrix} \kappa x = \kappa x \\ \dfrac{1}{m}p = \dot{x} = v \end{bmatrix}$ and:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad R_\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix}$$

J is full rank $n = 2$ and skew-symmetric, whereas R_ε is symmetric positive ($\varepsilon > 0$), with rank equal to 1, thus not positive definite.

3.2 Coupling with a low-pass diffusive system, such as a fractional integral

The damping model is now given by the coupling with another dynamical system, the input of which is the velocity $v := \dot{x}$, and the output of which is y, a positive linear combination of first-order low-pass subsytems, as follows:

$$m\ddot{x} + y + \kappa x = 0, \quad \text{with} \quad y = \sum_{k=1}^K \mu_k \varphi_k$$

$$\text{where} \quad \dot{\varphi}_k = -\xi_k\,\varphi_k + v, \quad \text{for } 1 \le k \le K\,.$$

Hence, with $H_\Phi := \frac{1}{2}\sum_{k=1}^K \mu_k\,\varphi_k^2$, and $\partial_{\varphi_k} H_\Phi = \mu_k\,\varphi_k$, the global system can be described by an extended state $X = (x, p, \Phi)$ and a global Hamiltonian $H := H_0 + H_\Phi$.

$$\frac{d}{dt}X = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\mathbb{1}^T \\ 0 & \mathbb{1} & -\mathrm{diag}(\frac{\xi_k}{\mu_k}) \end{bmatrix} \partial_X H(X) = (J - R)\,\partial_X H(X)\,.$$

In this case, matrices of size $(2+K) \times (2+K)$ are given by:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\mathbb{1}^T \\ 0 & \mathbb{1} & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{diag}(\frac{\xi_k}{\mu_k}) \end{bmatrix}.$$

It can easily be checked that J is skew-symmetric with rank 2 only, and R is symmetric positive ($\xi_k > 0, \mu_k > 0$), but not positive definite (its rank is K); its structure is simply diagonal.

Remark 1. Note that the relation between v and y comes from a possible discretization of a diffusive system, the general structure of which would be given by the following transfer function, namely:

$$H(s) = \int_0^\infty \mu(\xi) \frac{1}{s+\xi} \, \mathrm{d}\xi \quad \text{rather than} \quad H_K(s) = \sum_{k=1}^K \mu_k \frac{1}{s+\xi_k}.$$

As particular and noteworthy case, if $\mu_\beta(\xi) = \frac{\sin(\beta\pi)}{\pi} \xi^{-\beta}$, then $H_\beta(s) = \frac{1}{s^\beta}$ is recovered, which is nothing but the **fractional integral** of order $\beta \in (0,1)$, a low-pass filter.

3.3 Coupling with a high-pass diffusive system, such as a fractional derivative

The damping model is now given by the coupling with another dynamical system, the input of which is the velocity $v := \dot{x}$, and the output of which is $\widetilde{y}$, a positive linear combination of first-order high-pass subsytems, with a feed-through term, $d := \sum_{l=1}^L \nu_l$, as follows:

$$m\ddot{x} + \widetilde{y} + \kappa x = 0, \quad \text{with} \quad \widetilde{y} = \sum_{l=1}^L \nu_l \dot{\widetilde{\varphi}}_l$$

$$\text{where} \quad \dot{\widetilde{\varphi}}_l = -\xi_l \, \widetilde{\varphi}_l + v, \quad \text{for } 1 \le l \le L.$$

Hence, with $H_{\widetilde{\Phi}} := \frac{1}{2} \sum_{l=1}^L \nu_l \xi_l \, \widetilde{\varphi}_l^2$, and $\partial_{\widetilde{\varphi}_l} H_{\widetilde{\Phi}} = \nu_l \xi_l \, \widetilde{\varphi}_l$, the global system can be described by an extended state $X = (x, p, \widetilde{\Phi})$ and a global Hamiltonian $H := H_0 + H_{\widetilde{\Phi}}$.

$$\frac{d}{dt} X = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -d & \mathbb{1}^T \\ 0 & \mathbb{1} & -\mathrm{diag}(\frac{1}{\nu_l}) \end{bmatrix} \partial_X H(X) = (J-R) \, \partial_X H(X).$$

In this case, matrices of size $(2+L) \times (2+L)$ are given by:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{et} \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sum_{l=1}^L \nu_l & -\mathbb{1}^T \\ 0 & -\mathbb{1} & \mathrm{diag}(\frac{1}{\nu_l}) \end{bmatrix}$$

It can easily be checked that J is skew-symmetric with rank 2 only, and R is symmetric positive ($\xi_l > 0, \nu_l > 0$), but not positive definite (its rank is at most L); its structure is not that simple, but a block computation shows that $X^T R X = \sum_{l=1}^L \left(\sqrt{\nu_l} \, p - \frac{1}{\sqrt{\nu_l}} \, \widetilde{\varphi}_l \right)^2 \ge 0$.

Remark 2. Note that the relation between v and $\widetilde{y}$ comes from a possible discretization of a diffusive system, the general structure of which would be given by the following transfer function, namely:

$$H(s) = \int_0^\infty \nu(\xi) \frac{s}{s+\xi} \, \mathrm{d}\xi \quad \text{rather than} \quad H_L(s) = \sum_{k=1}^L \nu_l \frac{s}{s+\xi_l}$$

As particular and noteworthy case, if $\nu_\alpha(\xi) = \frac{\sin(\alpha\pi)}{\pi} \xi^{\alpha-1}$, then $H_\alpha(s) = s^\alpha$ is recovered, which is nothing but the **fractional derivative** of order $\alpha \in (0,1)$, a high-pass filter.

4. A HAMILTONIAN FORMULATION FOR WEBSTER-LOKSHIN MODEL

Let now consider the Webster-Lokshin (cf. Polak (1991); Hélie et al. (2006)) equation in PHS format. It is given in the usual PDE form:

$$\partial_t^2 w + \left(\varepsilon_z \, \partial_t^{1/2} + \eta_z \, \partial_t^{-1/2} \right) \partial_t w - \frac{1}{r_z^2} \partial_z \left(r_z^2 w \right) = 0 \quad (8)$$

Here, coefficient $\varepsilon_z > 0$ is conversely proportional to the radius r_z, and the proportionality constants involved are linked to the square roots of l_v and l_h, that are the characteristic lengths of visous and thermal effects, respectively.

Using the diffusive representation of § 2, Equation (8) can be written:

$$\partial_t^2 w + (\varepsilon_z \, \widetilde{y} + \eta_z \, y) - \frac{1}{r_z^2} \partial_z \left(r_z^2 w \right) = 0 \quad (9)$$

With, for the fractional integral

$$y = \int_0^\infty \mu(\xi) \varphi \, \mathrm{d}\xi$$

where

$$\partial_t \varphi = -\xi \varphi + \partial_t w$$

and, for the fractional derivative

$$\widetilde{y} = \int_0^\infty \nu(\xi) \partial_t \widetilde{\varphi} \, \mathrm{d}\xi = \int_0^\infty \left(-\nu(\xi)\xi\widetilde{\varphi} + \nu(\xi)\partial_t w \right) \, \mathrm{d}\xi \quad (10)$$

with

$$\partial_t \widetilde{\varphi} = -\xi\widetilde{\varphi} + \partial_t w$$

We choose as state variables the **energy variables**:

$$x_1 = r^2 \partial_z w(t,z), \quad x_2 = r^2 \partial_t w(t,z),$$
$$x_3 = \varphi(t,z,\xi), \quad x_4 = \widetilde{\varphi}(t,z,\xi)$$

with

- $x_1, x_2 \in L^2((a,b),\mathbb{R})$,
- $x_3 \in L^2((a,b), H_\mu)$ with $H_\mu : \int_0^\infty \mu(\xi) \, x_3^{\,2}(\xi,.) \, \mathrm{d}\xi < \infty$,
- $x_4 \in L^2((a,b), \widetilde{H}_\nu)$ with $\widetilde{H}_\nu : \int_0^\infty \xi \, \nu(\xi) \, x_4^{\,2}(\xi,.) \, \mathrm{d}\xi < \infty$

The Hamiltonian function $\mathcal{H}(x_1, x_2, x_3, x_4)$ can then be expressed as:

$$\mathcal{H} = \frac{1}{2} \int_a^b \left(x_1^2 + c^2 x_2^2 + \eta_z \int_0^\infty \mu x_3^2 \, \mathrm{d}\xi + \varepsilon_z \int_0^\infty \nu \, \xi x_4^2 \, \mathrm{d}\xi \right) \, \mathrm{d}z$$

In order to define the co-energy variables, we need to define the variational derivative of the Hamiltonian.

Definition 1. (Variational derivative of smooth function). Consider a functional

$$\mathcal{H}[x] = \int_a^b \mathscr{H} \left(z, x, x^{(1)}, \cdots, x^{(n)} \right) \, \mathrm{d}z$$

where $\mathscr{H}$ is a smooth function. The variational derivative of the functional $\mathcal{H}$, denoted by $\frac{\delta \mathcal{H}}{\delta x}$ or $\delta_x \mathcal{H}$ is such that:

$$\mathcal{H}[x + \epsilon \delta x] = \mathcal{H}[x] + \epsilon \int_a^b \frac{\delta \mathcal{H}}{\delta x} \delta x \, \mathrm{d}z + O(\epsilon^2)$$

for every $\epsilon \in \mathbb{R}$ and smooth real function $\delta x(z)$ such that:

$$\delta x^{(i)}(a) = \delta x^{(i)}(b) = 0, i = 0, \cdots, n \,.$$

In the case when $\mathscr{H}$ depends only on x and not on its derivatives:

$$\frac{\delta \mathcal{H}}{\delta x} = \frac{\partial \mathscr{H}}{\partial x}$$

In the case of the Webster-Lokshin model the **co-energy variables** are then defined by:

$$e_1 = \delta_{x_1}\mathcal{H} = \partial_z w, \quad e_2 = \delta_{x_2}\mathcal{H} = \partial_t w,$$
$$e_3 = \delta_{x_3}\mathcal{H} = \eta_z \, \mu \, \varphi \quad e_4 = \delta_{x_4}\mathcal{H} = \varepsilon_z \, \nu \, \xi \, \widetilde{\varphi}$$

Equation (9) is then "formally" equivalent to:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & \partial_z & 0 & 0 \\ \partial_z & -\int_0^\infty \varepsilon_z \nu(\xi).\mathrm{d}\xi & -\int_0^\infty .\mathrm{d}\xi & \int_0^\infty .\mathrm{d}\xi \\ 0 & 1 & -\frac{\xi}{\eta_z \mu(\xi)} & 0 \\ 0 & 1 & 0 & -\frac{1}{\varepsilon_z \nu(\xi)} \end{pmatrix} \begin{pmatrix} \partial_z w \\ \partial_t w \\ \eta \mu \varphi \\ \varepsilon \nu \xi \widetilde{\varphi} \end{pmatrix}$$

(11)

Remark 3. In equation (11), we split the integral of equation (10) into two terms that are not well defined, in fact one must understand the term $(\varepsilon_z \int_0^\infty (\xi \widetilde{\varphi} - \partial_t w) \, \nu(\xi) \, \mathrm{d}\xi)$ as non separable.

From a geometrical point of view, the dynamical system (11) can be then written in the form:

$$f = (\mathcal{J} - \mathcal{R}) \, e \tag{12}$$

with $e \in \mathcal{E} = H^1([a,b],\mathbb{R}) \times H^1([a,b],\mathbb{R}) \times L^2([a,b],\mathbf{H}) \times L^2([a,b],\widetilde{\mathbf{H}})$, $f \in \mathcal{F} = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbf{H}) \times L^2([a,b],\widetilde{\mathbf{H}})$ and operators $\mathcal{J}$ and $\mathcal{R}$ defined as follows:

$$\mathcal{J} = \begin{pmatrix} 0 & \partial_z & 0 & 0 \\ \partial_z & 0 & -\int_0^\infty .\mathrm{d}\xi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \int_0^\infty \varepsilon_z \nu(\xi).\mathrm{d}\xi & 0 & -\int_0^\infty .\mathrm{d}\xi \\ 0 & 0 & \frac{\xi}{\eta_z \mu(\xi)} & 0 \\ 0 & -1 & 0 & \frac{1}{\varepsilon_z \nu(\xi)} \end{pmatrix}.$$

Remark 3 applies to the second line of operator $\mathcal{R}$.

The bond space $\mathcal{B}$ defined as $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ is equipped with the natural power product:

$$\langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4) \rangle =$$
$$\int_a^b \left(e_1 f_1 + e_2 f_2 + \int_0^\infty (e_3 f_3 + e_4 f_4) \, \mathrm{d}\xi \right) \mathrm{d}z. \tag{13}$$

Lemma 1. $\mathcal{J}$ is formally skew-symmetric and $\mathcal{R}$ is symmetric *i.e.*:

$$\mathcal{J} = -\mathcal{J}^* \quad \text{and} \quad \mathcal{R} = \mathcal{R}^*, \mathcal{R} \geq 0$$

Proof 1. Let's first consider the skew-symmetry of $\mathcal{J}$:

$$\langle e', \mathcal{J}e \rangle = \langle \left(e'_1 \ e'_2 \ e'_3 \ e'_4 \right), \begin{pmatrix} 0 & \partial_z & 0 & 0 \\ \partial_z & 0 & -\int_0^\infty .\mathrm{d}\xi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \rangle$$
$$= \int_a^b \left(e'_1 \partial_z e_2 + e'_2 \partial_z e_1 + \int_0^\infty e'_3 e_2 \, \mathrm{d}\xi - e'_2 \int_0^\infty e_3 \, \mathrm{d}\xi \right) \mathrm{d}z$$
$$= \int_a^b \left(-\partial_z e'_1 e_2 - \partial_z e'_2 e_1 + \int_0^\infty e'_3 e_2 \, \mathrm{d}\xi - e'_2 \int_0^\infty e_3 \, \mathrm{d}\xi \right) \mathrm{d}z$$
$$= \langle -\mathcal{J}e', e \rangle.$$

The adjoint operator of $\mathcal{J}$ is equal to $-\mathcal{J}$ and then $\mathcal{J}$ is formally skew-symmetric. In a similar way, one can prove that $\mathcal{R}$ is symmetric *i.e.*

$$\langle e', \mathcal{R}e \rangle = \left(e'_1 \ e'_2 \ e'_3 \ e'_4 \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \int_0^\infty \varepsilon_z \nu.\mathrm{d}\xi & 0 & -\int_0^\infty .\mathrm{d}\xi \\ 0 & 0 & \frac{\xi}{\eta_z \mu} & 0 \\ 0 & -1 & 0 & \frac{1}{\varepsilon_z \nu} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$
$$= \int_a^b \left(e'_2 \int \varepsilon_z \nu e_2 \, \mathrm{d}\xi - e'_2 \int_0^\infty e_4 \, \mathrm{d}\xi + \int_0^\infty e'_3 \frac{\xi e_3}{\eta_z \mu} \right.$$
$$\left. -e'_4 e_2 + e'_4 \frac{1}{\varepsilon_z \nu} e_4 \, \mathrm{d}\xi \right) dz$$
$$= \langle \mathcal{R}e', e \rangle.$$

Moreover, the positivity of $\mathcal{R}$ can be proved as follows:

$$\langle e, \mathcal{R}e \rangle = \int_a^b \int_0^\infty \left(e_2 \varepsilon_z \nu e_2 - e_2 e_4 + e_3 \frac{\xi e_3}{\eta_z \mu} - e_4 e_2 + e_4 \frac{1}{\varepsilon_z \nu} e_4 \, \mathrm{d}\xi \right) \mathrm{d}z$$
$$= \int_a^b \int_0^\infty \left(\frac{\xi}{\eta_z \mu} e_3^2 + (\sqrt{\varepsilon_z \nu} e_2 - \frac{1}{\sqrt{\varepsilon_z \nu}} e_4)^2 \, \mathrm{d}\xi \right) \mathrm{d}z \geq 0.$$

Of course, $\mathcal{R}$ is not even positive definite, thus never coercive. $\square$

System (12) can be written in the form of an extended system with closure equation related to the dissipation by using the extended operator $\mathcal{J}_e$ with $\begin{pmatrix} f \\ f_e \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_e \end{pmatrix}$ with $e_e = S f_e$, where:

$$\mathcal{J}_e = \begin{pmatrix} 0 & \partial_z & 0 & 0 & 0 & 0 & 0 \\ \partial_z & 0 & -\int_0^\infty .\mathrm{d}\xi & 0 & -\int_0^\infty .\mathrm{d}\xi & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$S = \begin{pmatrix} \varepsilon_z \nu & 0 & 1 \\ 0 & \frac{\xi}{\eta_z \mu} & 0 \\ 1 & 0 & \frac{1}{\varepsilon_z \nu} \end{pmatrix},$$

and $e_e \in \mathcal{E}_e = H^1([a,b],\mathbb{R}) \times L^2([a,b],\mathbf{H}) \times L^2([a,b],\widetilde{\mathbf{H}})$, $f_e \in \mathcal{F}_e = L^2([a,b],\mathbb{R}) \times L^2([a,b],\mathbf{H}) \times L^2([a,b],\widetilde{\mathbf{H}})$ and operators $\mathcal{J}$ and $\mathcal{R}$.

One can check that $\mathcal{J}_e$ is formally skew-symmetric and S positive *i.e.* :

$$\mathcal{J}_e^* = -\mathcal{J}_e \text{ ans } S = S^* \geq 0$$

We now consider systems with non zero boundary flow. One can naturally extend the effort and the flow spaces to include the boundary, by defining:

$$\mathscr{E} = \mathcal{E} \times \mathcal{E}_e \times \mathbb{R}^2$$
$$\mathscr{F} = \mathcal{F} \times \mathcal{F}_e \times \mathbb{R}^2$$

We define a symmetric pairing from the power product by:

$$\langle (e, e_\partial, f, f_\partial), (\tilde{e}, \tilde{e}_\partial, \tilde{f}, \tilde{f}_\partial) \rangle_+ = \langle (e, \tilde{f}) \rangle + \langle (\tilde{e}, f) \rangle$$
$$- \langle (\tilde{e}_\partial, f_\partial) \rangle - \langle (e_\partial, \tilde{f}_\partial) \rangle \quad (14)$$

with $(e, e_\partial, f, f_\partial)$ and $(\tilde{e}, \tilde{e}_\partial, \tilde{f}, \tilde{f}_\partial) \in \mathscr{B} = \mathscr{F} \times \mathscr{E}$.

In order to define a Dirac structure we need to define appropriate boundary port variables with respect to the considered differential operator and symmetric pairing. In Haddar et al. (2008) is given a parametrization of boundary port variables in the case of non full rank linear differential operators. This parametrization can be adapted to our case study as follows:

Definition 2. Considering the following decomposition of $\mathcal{J}_e$:

$$\mathcal{J}_e = \begin{pmatrix} \Sigma_2 & 0_{2,5} \\ 0_{5,2} & 0_{5,5} \end{pmatrix} \partial_z + P_0$$

with $\Sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the boundary port variables associated with the differential operator $\mathcal{J}_e$ are the vectors $f_\partial, e_\partial \in \mathbb{R}^2$ given by:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma_2 & -\Sigma_2 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0_{2,5} & 0_{2,2} & 0_{2,5} \\ 0_{2,2} & 0_{2,5} & I_2 & 0_{2,5} \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix}$$

The definition of the boundary port variables gives rise to the definition of the associated Dirac structure.

Theorem 1. The subspace $\mathcal{D}_{\mathcal{J}_e}$ of $\mathscr{B}$ defined as:

$$\mathcal{D}_{\mathcal{J}_e} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} | f = \mathcal{J}e \text{ and} \right.$$
$$\left. \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma_2 & 0_{2,5} & -\Sigma_2 & 0_{2,5} \\ I_2 & 0_{2,5} & I_2 & 0_{2,5} \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} \right\}$$

is a Dirac structure.

Proof 2. We used the parametrization proposed in Villegas et al. (2006) to define some boundary port variables such that the symmetric pairing (14) is non degenerate and $\mathcal{D}_{\mathcal{J}_e}$ is a Dirac structure, *i.e.*

$$\mathcal{D}_{\mathcal{J}_e} = \mathcal{D}_{\mathcal{J}_e}^\perp$$

Such parametrization arises from the integration by part of the skew differential operator, the projection of the image space and the definition of the inner product. $\square$

5. CONCLUSION

In this paper, we propose a port-Hamiltonian formulation of systems arising from the coupling of a wave equation with a diffusion equation related to acoustic phenomena. The considered diffusion equation contains a fractional derivative in time and physical coefficients variable in space. First we consider the finite dimensional approximation of the integral and fractional derivatives. It is based on a diffusive representation of integral and fractional derivatives. In a second instance, we consider the Webster-Lokshin equation that is made up by the coupling of the wave equation and the aforementionned diffusion term. From the definition of the energy variables, Hamiltonian function and power conjugate flow and effort vectors, we propose the definition of some appropriate boundary port variables in order to define a Dirac structure. This Dirac structure allows to connect the internal energetic behavior of the system with the power flow at the boundary. This first work on the geometrical formulation of such system will open to the use of functional analysis tools that have been previously derived in the context of differential systems with dissipation in Zwart et al. (2011). Nevertheless a particular care will have to be taken on the characterization of functional spaces, particularly in the case of the diffusion function for which the domain is not separable, as already taken care of in Haddar et al. (2008).

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