Effective velocity surfaces for anisotropic elastic composites

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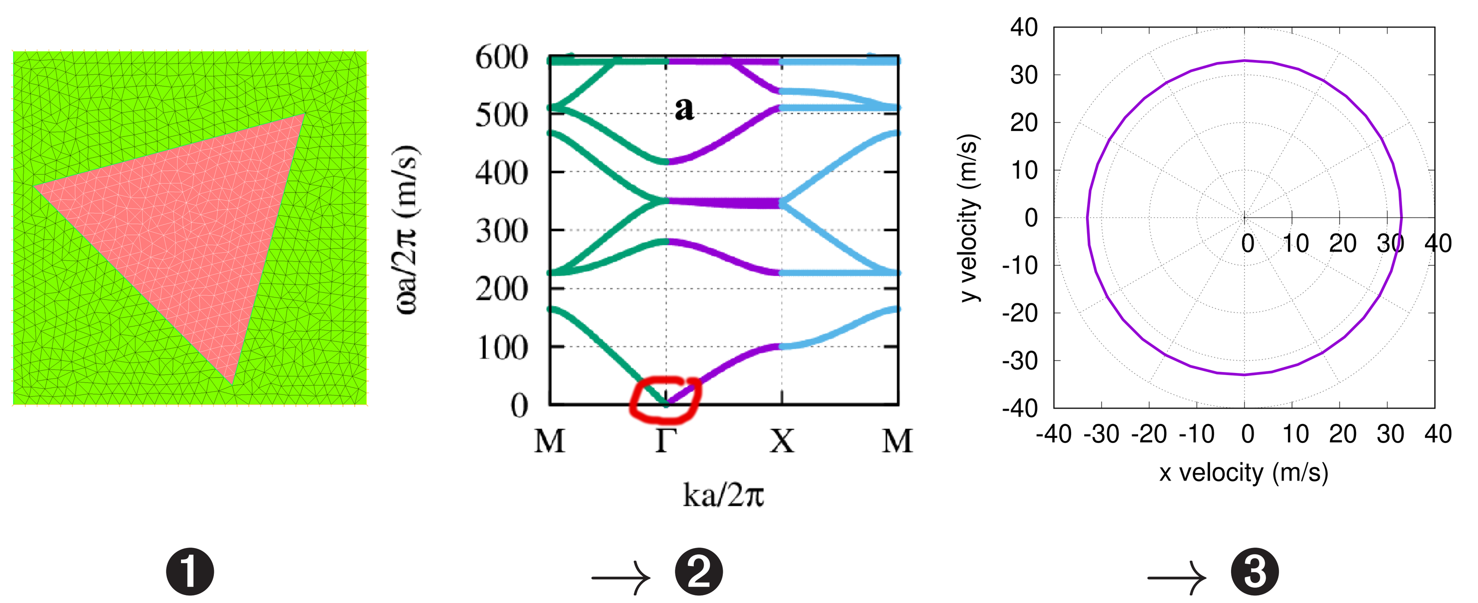
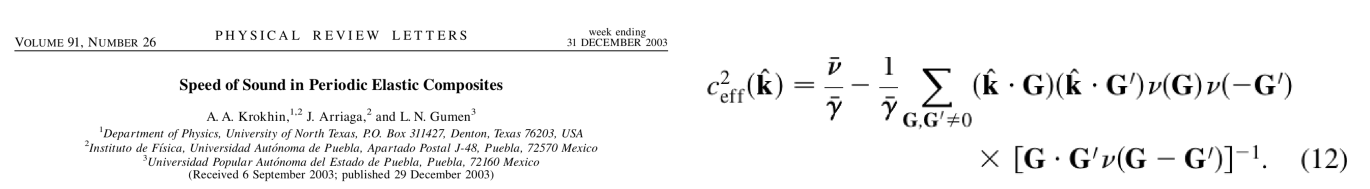
Velocity surfaces for periodic composites

- Non-zero effective elastic constants for periodic composites are dictated by crystal symmetry.
- Numerical values for effective elastic constants can be obtained by fitting the dispersion relation in the limit $k \to 0$ and $\omega \to 0$.
- Our aim: devise an efficient method to obtain velocity surfaces for arbitrary periodic composites, for
  - Sonic crystals (for pressure acoustic waves),
  - Phononic crystals (for vector elastic waves).
- Our method: finite element method, for periodic boundary value problem, with second-order perturbation theory.



Krokhin's formula for sonic crystals (2003)


PHYSICAL REVIEW LETTERS

Speed of Sound in Periodic Elastic Composites

A. A. Krokhin, J. Arriaga, and L. N. Gumen

$$c^2_{\rm eff}(\hat{\mathbf{k}}) = \frac{\bar{\rho}}{\bar{\gamma}} - \frac{1}{\bar{\gamma}} \sum_{\mathbf{G},\mathbf{G}'\neq 0} (\hat{\mathbf{k}}\cdot\mathbf{G})(\hat{\mathbf{k}}\cdot\mathbf{G}')\rho(\mathbf{G})\rho(-\mathbf{G}') \times [\mathbf{G}\cdot\mathbf{G}'\rho(\mathbf{G}-\mathbf{G}')]^{-1}. \quad (12)$$

- Valid for 2D and 3D sonic crystals
- Plane wave expansion (PWE) type analysis, with second-order perturbation theory
- Spectacular result for air bubbles in water composites (slow sound)
- Small drawback: need to invert a full matrix (no gain with respect to eigenvalue problem)

Scalar case (sonic crystals)

Partial differential equation for Bloch waves

$$(\rho^{-1}(k) - \omega^2 B^{-1})p = 0 \quad (1)$$

under periodic boundary conditions. Assume the solution is developed to first order in k

$$p \approx p_0 + ikp_1(\hat{k}) + o(k) \quad (2)$$

where p_0 is a constant field ($\nabla p_0 = 0$) and $\hat{k}$ is a unit vector. First order term:

$$(\nabla q, \rho^{-1}\nabla p_1) = (\nabla q, \rho^{-1}\hat{k}p_0), \forall q. \quad (3)$$

Second order term:

$$V^2_{\rm eff}(\hat{k}) = \frac{\omega^2}{k^2} = \frac{(p_0, \rho^{-1}p_0) - (\hat{k}p_0, \rho^{-1}\nabla p_1)}{(p_0, B^{-1}p_0)} \quad (4)$$

Fit to composite theory

- The velocity surface $V_{\rm eff}(\hat{k})$ is obtained by solving only one boundary value problem per direction.
- Orthotropic composites:

$$\bar{\rho}V_L^2(\phi) = \bar{c}_{11}\cos^2\phi + \bar{c}_{22}\sin^2\phi \quad (5)$$

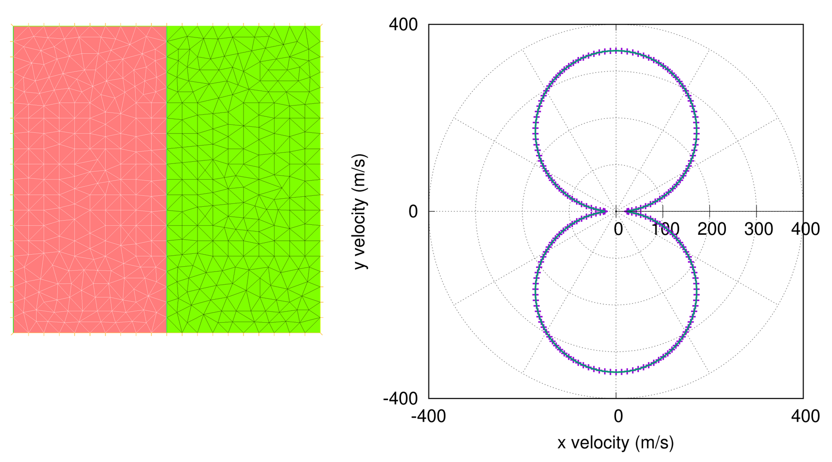


Figure: Water/air 1D sonic crystal (orthotropic).

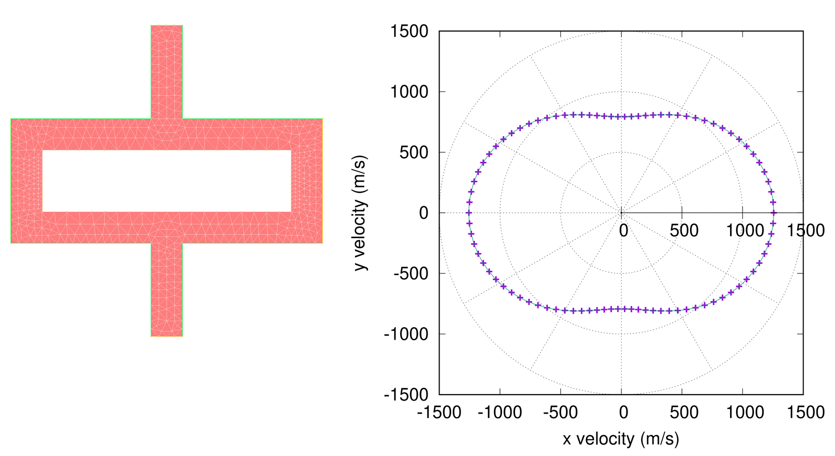


Figure: 'Uniaxial' water 2D sonic crystal (orthotropic). Can be realized with tubes filled with water.

Vector case (phononic crystals)

Three possible values for the constant field u_0, for three different surfaces:

$$u \approx \sum_{\alpha=1}^{3} \xi_\alpha (u_0^{(\alpha)} + iku_1^{(\alpha)}(\hat{k})) + o(k) \quad (6)$$

First-order corrections:

$$(\nabla q, c\nabla u_1^{(\alpha)}) = (\nabla q, c\hat{k}u_0^{(\alpha)}), \forall q \quad (7)$$

Second-order terms lead to a 'Christoffel' equation (3×3 generalized eigenvalue problem):

$$k^2[(\hat{k}u_0^{(\alpha)}, c\hat{k}u_0^{(\beta)}) - (\hat{k}u_0^{(\alpha)}, c\nabla u_1^{(\beta)})]\xi_\beta = \omega^2(u_0^{(\alpha)}, \rho u_0^{(\alpha)})\delta(\alpha-\beta)\xi_\beta \quad (8)$$

Fit to composite theory

- 3 boundary value problems per direction.
- Fit for orthotropic composites.

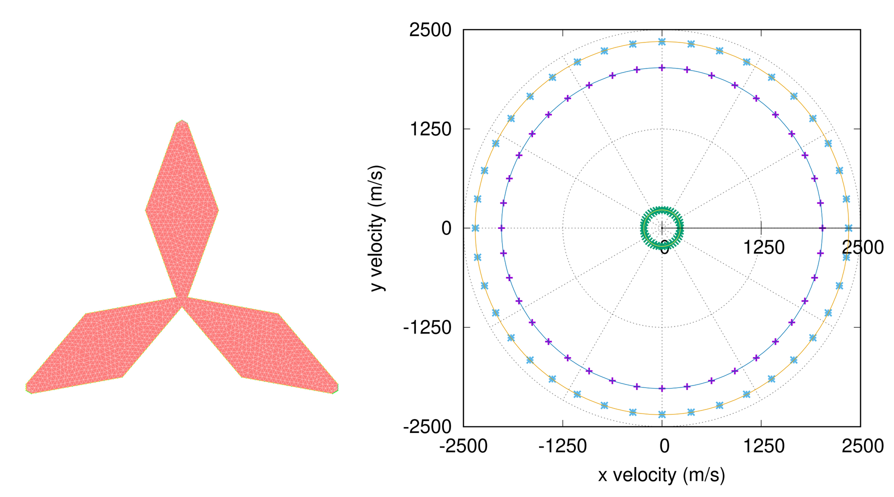


Figure: Steel. C_3 symmetry and/or 3 symmetry planes.

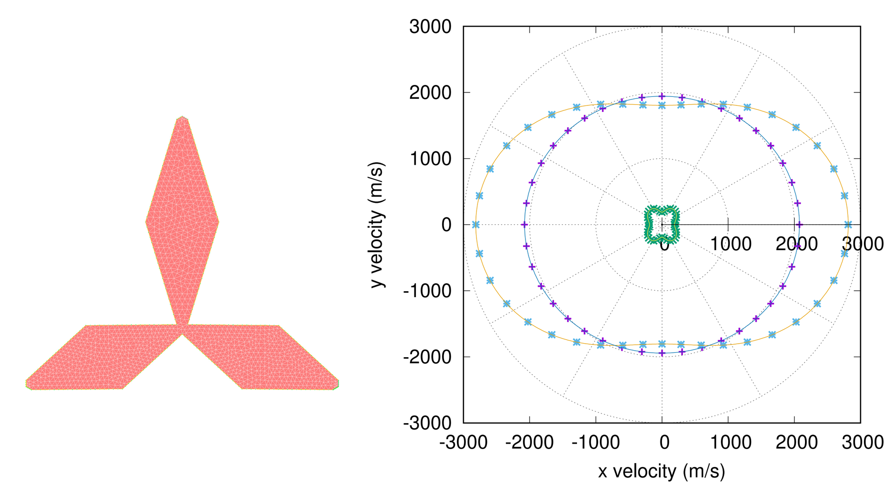


Figure: Steel. One symmetry plane.