Effective velocity surfaces for anisotropic elastic composites
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**Velocity surfaces for periodic composites**

- Non-zero effective elastic constants for periodic composites are dictated by crystal symmetry.
- Numerical values for effective elastic constants can be obtained by fitting the dispersion relation in the limit $k \to 0$ and $\omega \to 0$.
- Our aim: devise an efficient method to obtain velocity surfaces for arbitrary periodic composites, for
  - Sonic crystals (for pressure acoustic waves),
  - Phononic crystals (for vector elastic waves).
- Our method: finite element method, for periodic boundary value problem, with second-order perturbation theory.

**Scalar case (sonic crystals)**

Partial differential equation for Bloch waves

$$ (\rho^{-1}(k) - \omega^2 B^{-1}) p = 0 \quad (1) $$

under periodic boundary conditions. Assume the solution is developed to first order in $k$

$$ p = p_0 + ikp_1(\hat{k}) + o(k) \quad (2) $$

where $p_0$ is a constant field ($\nabla p_0 = 0$) and $\hat{k}$ is a unit vector. First order term:

$$ (\nabla q, \rho^{-1}\nabla p_1) = (\nabla q, \rho^{-1}\hat{k} p_0), \forall q. \quad (3) $$

Second order term:

$$ V_{\text{eff}}^2(\hat{k}) = \frac{\omega^2}{k^2} = \frac{(p_0, \rho^{-1}p_0) - (\hat{k} p_0, \rho^{-1}\nabla p_1)}{(p_0, B^{-1}p_0)} \quad (4) $$

**Vector case (phononic crystals)**

Three possible values for the constant field $u_0$, for three different surfaces:

$$ u \approx \sum_{\alpha=1}^{3} \xi_\alpha(u^{(\alpha)}_0 + iku^{(\alpha)}_1(\hat{k})) + o(k) \quad (6) $$

First-order corrections:

$$ (\nabla q, c\nabla u^{(\alpha)}_0) = (\nabla q, \hat{k} u^{(\alpha)}_0), \forall q. \quad (7) $$

Second-order terms lead to a ‘Christoffel’ equation (3 × 3 generalized eigenvalue problem):

$$ k^2(\hat{k} u^{(\alpha)}_0, c\nabla u^{(\beta)}_0) - (\hat{k} u^{(\alpha)}_0, c\nabla u^{(\beta)}_0) = \omega^2(u^{(\alpha)}_0, \rho u^{(\beta)}_0)\delta(\alpha - \beta) \xi_{\beta} \quad (8) $$

**Krokhin’s formula for sonic crystals (2003)**

- Valid for 2D and 3D sonic crystals
- Plane wave expansion (PWE) type analysis, with second-order perturbation theory
- Spectacular result for air bubbles in water composites (slow sound)
- Small drawback: need to invert a full matrix (no gain with respect to eigenvalue problem)

**Fit to composite theory**

- The velocity surface $V_{\text{eff}}(\hat{k})$ is obtained by solving only one boundary value problem per direction.
- Orthotropic composites:

$$ \hat{\rho} V^2(\phi) = c_{11} \cos^2 \phi + c_{22} \sin^2 \phi \quad (5) $$

**Fit to composite theory**

- 3 boundary value problems per direction.
- Fit for orthotropic composites.

**Figure:** Water/air 1D sonic crystal (orthotropic).

**Figure:** Steel. $C_2$ symmetry and/or 3 symmetry planes.

**Figure:** Uniaxial' water 2D sonic crystal (orthotropic). Can be realized with tubes filled with water.