A Simple Robust Controller for Port–Hamiltonian Systems

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Abstract: We consider robust output regulation of passive infinite-dimensional linear port-Hamiltonian systems. As the main result, we present a Lyapunov-based proof to show that a passive internal model based low-gain controller solves the control problem for stable port-Hamiltonian systems. The theoretic results are used to construct a controller controller for robust output tracking of a piezoelectric tube model.

Keywords: Port-Hamiltonian system, robust output regulation, controller design.

1. INTRODUCTION

In this paper we study robust output tracking and disturbance rejection for an exponentially stable passive port-Hamiltonian system (Villegas, 2007; Jacob and Zwart, 2012)

$$\dot{x}(t) = (J-R)Qx(t) + Bu(t) + B_d w_{dist}(t), \tag{1.1a}$$
$$y(t) = B^*Qx(t) \tag{1.1b}$$

on a Hilbert space X. In the control problem we aim to construct a passive dynamic error feedback controller in such a way that the output $y(t)$ of the system converges to a given reference signal $y_{ref}(t)$, i.e.,

$$\|y_{ref}(t) - y(t)\| \to 0, \qquad t \to \infty,$$

at an exponential rate despite the external disturbance signal $w_{dist}(t)$ (cf. Figure 1). In addition, we require that the controller is robust in the sense that the output tracking and disturbance rejection are achieved even if the parameters of the system (1.1) experience small perturbations.

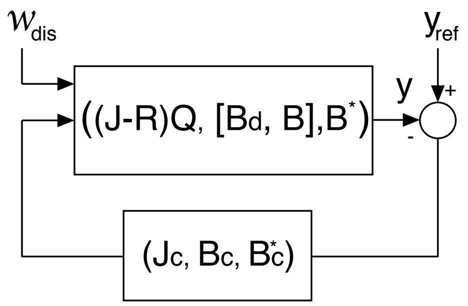

Fig. 1. Tracking control problem.

The robust output regulation problem for infinite-dimensional linear systems has been studied extensively in the literature. Especially the "simple" internal model based controller has been shown to be effective in achieving robust output regulation for stable infinite-dimensional systems (Logemann and Townley, 1997; Hämäläinen and Pohjolainen, 2000; Rebarber and Weiss, 2003). The previous references employ frequency domain methods in the stability analysis of the closed-loop system consisting of (1.1) and the controller. Our main interest in this paper is to consider a similar simple robust controller, but instead use Lyapunov techniques in analysing the closed-loop stability. The motivation for the study is that the Lyapunov techniques provide an ideal starting point for extending results from linear control theory to nonlinear systems and controllers.

In this paper we assume the reference and disturbance signals are finite linear combinations of trigonometric functions with known frequencies $\{\omega_k\}_{k=0}^q \subset \mathbb{R}$ with $\omega_0 = 0$ and unknown amplitudes. The robust controller we construct is a port-Hamiltonian error feedback controller

$$\dot{x}_c(t) = J_c x_c(t) + B_c(y_{ref}(t) - y(t)), \tag{1.2a}$$
$$u(t) = B_c^* x_c(t). \tag{1.2b}$$

As required by the internal model principle (Paunonen and Pohjolainen, 2010), J_c is chosen to contain *an internal model* of the frequencies of $y_{ref}(\cdot)$ and $w_{dist}(\cdot)$, and the controller is finite-dimensional whenever the system (1.1) has a finite number of outputs. The internal model principle implies that the controller (1.2) will solve the robust output regulation problem provided that the closed-loop system consisting of (1.1) and (1.2) is exponentially stable. The specific structure of J_c and B_c is presented in Section 3. In particular, the pair (J_c, B_c) is controllable.

As the main result of this paper we will introduce a Lyapunov-based argument to prove that the closed-loop system is stable and the controller (1.2) achieves robust output tracking and disturbance whenever that $\|B_c\|$ is sufficiently small. Because of the condition on $\|B_c\|$, (1.2) is a *low-gain* controller. The passivity of the system (1.1) brings the advantage that the controller can be constructed without any knowledge of the values $P(\pm i\omega_k)$ of the transfer function of (1.1) at the frequencies of $y_{ref}(t)$ and $w_{dist}(t)$, as is the case of general linear systems (Logemann and Townley, 1997; Hämäläinen and Pohjolainen, 2000). Earlier research using frequency domain methods has demonstrated that for passive systems (1.1) and (1.2) the condition on the smallness of $\|B_c\|$ is not necessary for closed-loop stability and robust regulation (Rebarber and Weiss, 2003). Instead, in our main result this condition is only required because of the Lyapunov function argument in used in the proof.

Robust output regulation of infinite-dimensional linear systems has been studied previously in (Pohjolainen, 1982; Logemann and Zwart, 1992; Logemann and Townley, 1997; Hämäläinen and Pohjolainen, 2000; Rebarber and Weiss, 2003; Boulite et al., 2009; Hämäläinen and Pohjolainen, 2010; Paunonen and Pohjolainen, 2010; Paunonen, 2016, 2017). In particular, the construction of a robust low-gain controllers for stable systems has been studied in (Logemann and Townley, 1997; Hämäläinen and Pohjolainen, 2000; Rebarber and Weiss, 2003), and also specifically for port-Hamiltonian systems (Humaloja and Paunonen, 2018) and for passive systems (Rebarber and Weiss, 2003; Paunonen, 2017).

Notation: If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{L}(X)$ on a Hilbert space X we define $\operatorname{Re} T = \frac{1}{2}(T + T^*)$.

2. A MOTIVATING EXAMPLE

As a motivating example we consider the output tracking trajectory problem for a piezoelectric tube used in positioning systems for Atomic Force Microscopy (see Figure 2). This actuator provides the high positioning resolution and the large bandwidth necessary for the trajectory control during scanning processes.

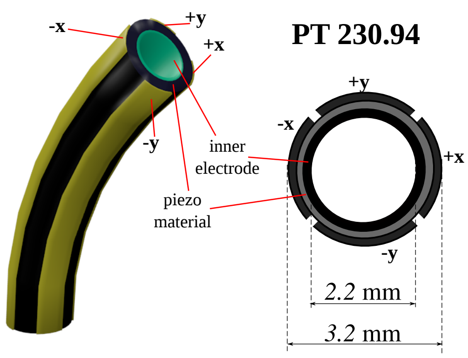

Fig. 2. The piezoelectric tube.

For the sake of simplicity we consider the motion of the piezotube in one direction. In this case the structure of the system behaves as a clamped-free beam, represented by the Timoshenko beam model and actuated through homogeneous distributed control stemming from the piezoelectric action over the last section of the beam (the first section being passive). By choosing as state variables the energy variables, namely the shear displacement $x_1 = \frac{\partial w}{\partial z}(z,t) - \phi(z,t)$, the transverse momentum distribution $x_2 = \rho(z)\frac{\partial w}{\partial t}(z,t)$, the angular displacement $x_3 = \frac{\partial \phi}{\partial z}(z,t)$ and the angular momentum distribution $x_4 = I_\rho \frac{\partial \phi}{\partial t}(z,t)$ for $z \in (a,b)$, $t \geq 0$, where $w(z,t)$ is the transverse displacement and $\phi(z,t)$ the rotation angle of the beam, the port Hamiltonian model of the uncontrolled Timoshenko beam is given by (**?**):

$$\dot{x}(t) = (\mathcal{J} - \mathcal{R})\mathcal{Q}x(t) \tag{2.1}$$

with $\mathcal{Q} = \operatorname{diag}\left(K, \frac{1}{\rho}, EI, \frac{1}{I_\rho}\right)$, $\mathcal{J} = P_1 \frac{\partial}{\partial z} + P_0$, $\mathcal{R} = G_0$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, G_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_\phi \end{bmatrix}$$

and ρ, I_ρ, E, I and K the mass per unit length, the angular moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively, b_w, b_ϕ the frictious coefficients. The energy of the beam is expressed in terms of the energy variables,

$$\begin{aligned} E(t) &= \frac{1}{2}\int_a^b (Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2)dz \\ &= \frac{1}{2}\int_a^b x(z)^T(\mathcal{Q}x)(z)dz = \frac{1}{2}\|x\|_{\mathcal{Q}}^2 \end{aligned}$$

The beam being clamped at point a, i.e., $\frac{1}{\rho}x_2(a,t) = \frac{1}{I_\rho}x_4(a,t) = 0$ $\forall t \geq 0$ and free at point b, i.e., $Kx_1(b,t) = EIx_3(b,t) = 0$ $\forall t \geq 0$ the domain of the operator $\mathcal{J}$ is

$$\mathcal{D}(\mathcal{J}) = \left\{ x \in H_1(0,1;\mathbb{R}^n) \,\middle|\, \begin{array}{l} x_2(a,t) = 0 \\ x_4(a,t) = 0 \\ x_1(b,t) = 0 \\ x_3(b,t) = 0 \end{array}, \forall t \geq 0 \right\}.$$

Control through piezoelectric actuation is modeled as a homogeneous distributed torque over the segment $[b-\eta, b]$, and thus the controlled version of (2.1) becomes

$$\begin{aligned} \dot{x}(t) &= (\mathcal{J} - \mathcal{R})\mathcal{Q}x(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{B}^*\mathcal{Q}x(t) \end{aligned}$$

where

$$\mathcal{B}(z)u(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1_{[b-\eta,b]}(z) \end{bmatrix} u(t),$$

$$\mathcal{B}^*\mathcal{Q}x(t) = \int_{b-\eta}^b \frac{\partial \phi}{\partial t}(z,t)dz.$$

3. A PASSIVE ROBUST CONTROLLER

In this section we will present a dynamic error feedback controller of the form (1.2) to achieve robust output tracking and disturbance rejection of the signals (3.1). We assume $B \in \mathcal{L}(U,X)$ and $B_d \in \mathcal{L}(U,X)$ in the system (1.1). The input and output spaces $U = Y$ and U_d are Hilbert spaces. The reference signal $y_{ref}(t)$ and disturbance signal $w_{dist}(t)$ are assumed to be of the form

$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} \left[a_k^1 \cos(\omega_k t) + a_k^2 \sin(\omega_k t)\right], \quad (3.1a)$$

$$w_{dist}(t) = b_0 + \sum_{k=1}^{q} \left[b_k^1 \cos(\omega_k t) + b_k^2 \sin(\omega_k t)\right], \quad (3.1b)$$

where the frequencies $\{\omega_k\}_{k=0}^q \subset \mathbb{R}$ are known, and $\omega_0 = 0$ and $\omega_k > 0$ for $k \in \{1, \dots, q\}$. The main control problem is defined in the following.

The Robust Output Regulation Problem. *Choose a controller* (1.2) *in such a way that the following hold.*

(a) *The closed-loop consisting of the plant* (1.1) *and* (1.2) *is exponentially stable.*
(b) *For all $y_{ref}(t)$ and $w_{dist}(t)$ of the form* (3.1) *and for all initial states of the plant and the controller*
$$\|y(t) - y_{ref}(t)\| \to 0, \qquad as \quad t \to \infty$$
at an exponential rate.
(c) *If (J, R, B, B_d) are perturbed to $(\tilde{J}, \tilde{R}, \tilde{B}, \tilde{B}_d)$ in such a way that the perturbed closed-loop system is exponentially stable, then* (b) *continues to hold.*

For $\lambda \in \rho((J-R)Q)$ we denote the transfer function of the system (1.1) is given by $P(\lambda) = B^*QR(\lambda, (J-R)Q)B$. If we denote $\operatorname{Re} T = \frac{1}{2}(T + T^*)$, then the passivity of the system implies that $\operatorname{Re} P(i\omega) \geq 0$ for all $i\omega \in \rho((J-R)Q) \cap i\mathbb{R}$. To verify this, for any $\lambda \in \rho((J-R)Q) \cap \overline{\mathbb{C}_+}$ we can let $u \in U$ be arbitrary and denote $x = (\lambda Q^{-1} - J + R)^{-1}Bu$. Then

$$\begin{aligned}
\operatorname{Re}\langle P(\lambda)u, u\rangle &= \operatorname{Re}\langle B^*QR(\lambda, (J-R)Q)Bu, u\rangle \\
&= \operatorname{Re}\langle (\lambda Q^{-1} - J + R)^{-1}Bu, Bu\rangle \\
&= \operatorname{Re}\langle x, (\lambda Q^{-1} - J + R)x\rangle \\
&= (\operatorname{Re}\lambda)\langle x, Q^{-1}x\rangle + \langle x, Rx\rangle \geq 0
\end{aligned}$$

since $Q^{-1} \geq 0$, $R \geq 0$, and $\operatorname{Re}\lambda \geq 0$. In order to solve the robust output regulation problem, it is necessary to assume that $\{\pm i\omega_k\}_{k=0}^q \subset \rho((J-R)Q)$ and that the transfer function $P(\lambda)$ is such that $P(\pm i\omega_k)$ are surjective for all $k \in \{0, \dots, q\}$. This necessity can be observed, for example, from the result (Paunonen and Pohjolainen, 2010, Lem. 6.4). In the case of our passive system, we make the following natural assumption.

Assumption 3.1. Assume that $\pm i\omega_k \in \rho((J-R)Q)$ and $\operatorname{Re} P(\pm i\omega_k) > 0$ for all $k \in \{0, \dots, q\}$.

We choose the parameters J_c and B_c of the controller in such a way that the controller (1.2) will incorporate an internal model of the signals $y_{ref}(t)$ and $w_{dist}(t)$ in the sense of (Paunonen and Pohjolainen, 2010; Paunonen, 2016). To this end, we will choose $X_c = Y \times \cdots \times Y = Y^{2q+1}$,

$$J_c = \operatorname{blockdiag}(J_c^0, J_c^1, \dots, J_c^q),$$

$$J_c^0 = 0 \cdot I_Y, \quad J_c^k = \begin{bmatrix} 0 & \omega_k I_Y \\ -\omega_k I_Y & 0 \end{bmatrix},$$

$$B_c = \delta_c \begin{bmatrix} B_{c0}^0 \\ \vdots \\ B_{c0}^q \end{bmatrix}, \qquad B_{c0}^0 = I_Y, \quad B_{c0}^k = \begin{bmatrix} I_Y \\ 0 \end{bmatrix}.$$

Since Y is allowed to be infinite-dimensional, we will use the definition of the internal model expressed in terms of the *$\mathcal{G}$-conditions* (Hämäläinen and Pohjolainen, 2010; Paunonen, 2016).

Lemma 3.2. The controller incorporates an internal model of the signals $y_{ref}(t)$ and $w_{dist}(t)$ in (3.1) in the sense that the $\mathcal{G}$-conditions

$$\mathcal{R}(\pm i\omega_k - J_c) \cap \mathcal{R}(B_c) = \{0\}, \quad \forall k \in \{0, \dots, q\} \quad (3.2a)$$
$$\mathcal{N}(B_c) = \{0\} \quad (3.2b)$$

are satisfied.

Proof. Since B_{c0}^k are injective, the same is true for B_c and thus (3.2b) holds. First let $k = 0$ with $\omega_0 = 0$ and $w \in \mathcal{R}(i\omega_0 - J_c) \cap \mathcal{R}(B_c)$. Then $w = (i\omega_0 - J_c)x_c = B_c y$ for some $w, x_c \in X_c$ and $y \in Y$, and the structure of J_c and B_c imply that in particular $(i\omega_0 - J_c^0)x_c^1 = \delta_c B_c^0 y$. Since $i\omega_0 - J_c^0 = 0$ and B_c^0 is injective, we have $y = 0$, which further implies $w = B_c y = 0$. Thus (3.2a) holds for $k = 0$.

On the other hand, if $k \in \{1, \dots, q\}$ and $w = (\pm i\omega_0 - J_c)x_c = B_c y$ for $w, x_c \in X_c$ and $y \in Y$, then the structures of J_c and B_c again imply that

$$\begin{bmatrix} \pm i\omega_k I_Y & -\omega_k I_Y \\ \omega_k I_Y & \pm i\omega_k I_Y \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \delta_c B_c^k y = \delta_c \begin{bmatrix} y \\ 0 \end{bmatrix}$$

for some $z_1, z_2 \in Y$. Since $\omega_k > 0$, the second line of the above equation implies $z_1 = \mp i z_2$. Substituting to the first line we get

$$\delta_c y = \pm i\omega_k z_1 - \omega_k z_2 = \pm i\omega_k(\mp i z_2) - \omega_k z_2 = 0,$$

which shows $w = B_c y = 0$. Since $w \in \mathcal{R}(\pm i\omega_k - J_c) \cap \mathcal{R}(B_c)$ was arbitrary, we have that (3.2b) is satisfied. ☐

The internal model principle (Paunonen, 2016, Thm. 7) now states that the controller solves the robust output regulation problem provided that the closed-loop system consisting of the plant and the controller is exponentially stable. If we write

$$\dot{x}_c(t) = J_c x_c(t) + B_c u_c(t) \quad (3.3a)$$
$$y_c(t) = B_c^* x_c(t) \quad (3.3b)$$

then the stability of the closed-loop consisting of (1.1) and (1.2) is equivalent to showing that for $w_{dist}(t) \equiv 0$ the closed-loop consisting of (1.1) and (3.3) under the *power-preserving interconnection* Ramírez et al. (2014)

$$\begin{cases} u(t) = y_c(t) \\ u_c(t) = -y(t) \end{cases}$$

is exponentially stable.

The following is the main result of this paper.

Theorem 3.3. There exists $\delta_c^* > 0$ such that for all $\delta_c \in (0, \delta_c^*)$, the closed-loop system consisting of the plant and the controller is exponentially stable. In this case the controller solves the robust output regulation problem for all reference and disturbance signals (3.1).

The proof of Theorem 3.3 uses the following lemma.

Lemma 3.4. Let $Q = I$ and let $H \in \mathcal{L}(X_c, X)$ be such that $\mathcal{R}(H) \subset \mathcal{D}(J-R)$ and $HJ_c = (J-R)H - BB_c^*$. Then there exist $\delta_0^* > 0$, $M_c > 0$ such that for any $\delta_c \in (0, \delta_0^*)$ we can choose $P_{c0} > 0$ such that $\|P_{c0}\| \leq M_c$ and

$$P_{c0}(J_c + B_c B^* H) + (J_c + B_c B^* H)^* P_{c0} = -\delta_c^2. \quad (3.4)$$

Proof. Applying a block-diagonal similarity transform $T = \text{blockdiag}(T_0, T_1, \dots, T_q)$ where $T_0 = I$ and $T_k = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}$, $T_k^{-1} = \frac{1}{2}\begin{bmatrix} I & -iI \\ I & iI \end{bmatrix}$ we can define $G_1 = T^{-1}J_cT = \text{blockdiag}(i\omega_0 I_Y, i\omega_1 I_Y, -i\omega_1 I_Y, \dots, i\omega_q I_Y, -i\omega_q I_Y)$ and $G_2 = T^{-1}B_c$ and write

$$J_c + B_cB^*H = T(G_1 + G_2B^*HT)T^{-1}.$$

Since $J_c \in \mathcal{L}(X_c)$ with $\sigma(J_c) \subset i\mathbb{R}$ and $\sigma(J-R) \subset \mathbb{C}_-$, the Sylvester equation $HJ_c = (J-R)H - BB_c^*$ has a unique solution $H \in \mathcal{L}(X_c, X)$ satisfying $\mathcal{R}(H) \subset \mathcal{D}(J-R)$ Vũ (1991). The Sylvester equation and the definitions of G_1 and G_2 further imply

$$HTG_1 = AHT - BB_c^*T,$$

where $B_c^*T = \delta_c [I, \dots, I]$. Since G_1 is block-diagonal, it is straightforward to verify that

$$HT = -\delta_c\Big[R(i\omega_0, A)B, R(i\omega_1, A)B, R(-i\omega_q, A)B, \dots, R(i\omega_q, A)B, R(-i\omega_q, A)B\Big],$$

and thus B^*HT is equal to

$$-\delta_c [P(i\omega_0), P(i\omega_1), P(-i\omega_1), \dots, P(i\omega_q), P(-i\omega_q)].$$

Since we have by Assumption 3.1 that $\operatorname{Re} P(\pm i\omega_k) > 0$ for all $k \in \{0, \dots, q\}$, we also have $\sigma(P(\pm i\omega_k)) \subset \mathbb{C}_+$ for all $k \in \{0, \dots, q\}$. Indeed, if $S \in \mathcal{L}(U)$ is such that $\operatorname{Re} S > 0$ and $\operatorname{Re}\lambda \le 0$, then

$$\operatorname{Re}(S - \lambda) = |\operatorname{Re}\lambda| + \operatorname{Re} S > 0,$$

which further implies that $S - \lambda$ is boundedly invertible (see, e.g.,(Paunonen, 2017, Lem. A.1(a))). Write $B^*HT = -\delta_c K$. Since $G_2 = T^{-1}B_c = (\delta_c/2)G_{20}$ where $G_{20} = [I, \dots, I]^*$, the operator $(G_1 + G_2B^*HT)^* = G_1^* - (\delta_c^2/2)K^*G_{20}^*$ is of the form of the operator $A_c(\varepsilon)$ in (Hämäläinen and Pohjolainen, 2011, App. B) with $\varepsilon = \delta_c^2/2$. If we denote by $T_{\delta_c}(t)$ the semigroup generated by $J_c + B_cB^*H = T(G_1 + G_2B^*HT)T^{-1}$, then the proof of Theorem 1 in (Hämäläinen and Pohjolainen, 2011, App. B) shows that there exist $M_0, \omega_0, \delta_0^* > 0$ for all $\delta_c \in (0, \delta_0^*)$ we have that $\|T_{\delta_c}(t)\| \le M_0 e^{-\omega_0\delta_c^2 t}$ for all $t \ge 0$.

Let $\delta_c \in (0, \delta_0^*)$. Since $T_{\delta_c}(t)$ is exponentially stable, we can choose $\tilde{P}_{c0} > 0$ such that

$$(J_c + B_cB^*H)\tilde{P}_{c0} + (J_c + B_cB^*H)^*\tilde{P}_{c0}^* = -I.$$

The operator $\tilde{P}_{c0}$ is given by $\tilde{P}_{c0} = \int_0^\infty T_{\delta_c}(t)^*T_{\delta_c}(t)dt$. Thus

$$\|\tilde{P}_{c0}\| \le \int_0^\infty \|T_{\delta_c}(t)\|^2 dt \le M_0^2 \int_0^\infty e^{-2\omega_0\delta_c^2 t}dt = \frac{M_0^2}{2\omega_0\delta_c^2}.$$

Now the operator $P_{c0} = \delta_c^2\tilde{P}_{c0}$ satisfies the conditions of the lemma. □

Proof of Theorem 3.3. As explained in Section 3 it sufficient to show that the closed-loop system is exponentially stable without the presence of the reference and disturbance signals. By possibly changing variables $x(t)$ to $Q^{1/2}x(t)$ and defining new operators $\tilde{J} = Q^{1/2}JQ^{1/2}$, $\tilde{R} = Q^{1/2}RQ^{1/2}$, and $\tilde{B} = Q^{1/2}B$ we can assume throughout the proof that $Q = I$.

Let $H \in \mathcal{L}(X_c, X)$ satisfying $\mathcal{R}(H) \subset \mathcal{D}(J-R)$ be the solution of the Sylvester equation $HJ_c = (J-R)H - BB_c^*$. Since $J_c \in \mathcal{L}(X_c)$ with $\sigma(J_c) \subset i\mathbb{R}$ and $\sigma(J-R) \subset \mathbb{C}_-$, the solution H exists and is unique Vũ (1991). Moreover, since $B_c = \delta_cB_{c0}$, also $H = \delta_cH_0$ for a fixed $H_0 \in \mathcal{L}(X_c, X)$. We choose the Lyapunov function for the closed-loop system as

$$V_e = \langle x, Px\rangle + \langle x_c, (P_c + H^*PH)x_c\rangle + 2\operatorname{Re}\langle x, PHx_c\rangle$$

where $x = x(t)$ and $x_c = x_c(t)$ are the states of the plant and the controller, respectively. We have

$$\begin{bmatrix} I & 0 \\ -H^* & I \end{bmatrix}\begin{bmatrix} P & P_0 \\ P_0^* & P_c + H^*PH \end{bmatrix}\begin{bmatrix} I & -H \\ 0 & I \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & P_c \end{bmatrix},$$

which implies that $P_e = \begin{bmatrix} P & P_0 \\ P_0^* & P_c + H^*PH \end{bmatrix} > 0$ whenever $P > 0$ and $P_c > 0$. Thus under these conditions V_e is a valid Lyapunov function candidate.

If we denote $\tilde{A} = J - R - HB_cB^*$, then a direct computation using $u(t) = y_c(t) = B_c^*x_c(t)$, $u_c(t) = -y(t) = -B^*x(t)$, and $HJ_c = (J-R)H - BB_c^*$ can be used to verify that

$$\begin{aligned}\dot{V}_e &= 2\operatorname{Re}\langle (J-R)x + Bu, Px\rangle \\ &\quad + 2\operatorname{Re}\langle J_cx_c + B_cu_c, (P_c + H^*PH)x_c\rangle \\ &\quad + 2\operatorname{Re}\langle (J-R)x + Bu, PHx_c\rangle \\ &\quad + 2\operatorname{Re}\langle x, PH(J_cx_c + B_cu_c)\rangle \\ &= \langle x + Hx_c, (P\tilde{A} + \tilde{A}^*P)(x + Hx_c)\rangle \\ &\quad + \langle x_c, (P_c(J_c + B_cB^*H) + (J_c + B_cB^*H)^*P_c)x_c\rangle \\ &\quad + 2\operatorname{Re}\langle x + Hx_c, (PHB_cB^*H + BB_c^*P_c)x_c\rangle.\end{aligned}$$

Since the plant is exponentially stable, there exists $\varepsilon > 0$, $P > 0$ and $\delta_1^* > 0$ such that for all $\delta_c \in (0, \delta_1^*)$ we have

$$P\tilde{A} + \tilde{A}^*P \le -\varepsilon I.$$

Indeed, for any $\varepsilon > 0$ we can choose a fixed $P > 0$ such that $P(J-R) + (J-R)^*P = -2\varepsilon I$, and

$$\begin{aligned}&P\tilde{A} + \tilde{A}^*P \\ &\quad = P(J-R) + (J-R)^*P + 2\delta_c^2\operatorname{Re}(PH_0B_{c0}) \le -\varepsilon I\end{aligned}$$

when $\delta_c > 0$ is small enough. If we let $P_{c0} > 0$ be as in Lemma 3.4 and $\varepsilon_c > 0$, then for $P_c = \varepsilon_cP_{c0} > 0$ we have

$$P_c(J_c + B_cB^*H) + (J_c + B_cB^*H)^*P_c = -\varepsilon_c\delta_c^2 I$$

and $\|P_c\| \le M_c\varepsilon_c$ for some constant M_c and for all $\varepsilon_c > 0$.

If $0 < \delta_c < \min\{\delta_0^*, \delta_1^*\}$, we can use the inequality $2\operatorname{Re}\langle x, y\rangle \le \alpha^2\|x\|^2 + \frac{1}{\alpha^2}\|y\|^2$ for $\alpha > 0$ to estimate

$$\begin{aligned}\dot{V}_e &= \langle x + Hx_c, (P\tilde{A} + \tilde{A}^*P)(x + Hx_c)\rangle \\ &\quad + \langle x_c, (P_c(J_c + B_cB^*H) + (J_c + B_cB^*H)^*P_c)x_c\rangle \\ &\quad + 2\operatorname{Re}\langle x + Hx_c, (PHB_cB^*H + BB_c^*P_c)x_c\rangle \\ &\le -\varepsilon\|x + Hx_c\|^2 - \varepsilon_c\delta_c^2\|x_c\|^2 + \|H^*P(x + Hx_c)\|^2 \\ &\quad + \|B_cB^*Hx_c\|^2 + \alpha^2\|B^*(x + Hx_c)\|^2 + \frac{1}{\alpha^2}\|B_c^*P_cx_c\|^2 \\ &= \left[-\varepsilon + \delta_c^2\|H_0^*P\|^2 + \alpha^2\|B\|^2\right]\|x + Hx_c\|^2 \\ &\quad + \delta_c^2\left[-\varepsilon_c + \delta_c^2\|B_{c0}B^*H_0\|^2 + \frac{1}{\alpha^2}M_c^2\varepsilon_c^2\|B_{c0}\|^2\right]\|x_c\|^2.\end{aligned}$$

Here $\varepsilon > 0$ is fixed, and we can choose $\alpha^2 = \varepsilon/(2\|B\|^2)$. Then we can choose a sufficiently small fixed $\varepsilon_c > 0$ and $\delta_2^* > 0$ such that if $0 < \delta_c < \delta_c^* := \min\{\delta_0^*, \delta_1^*, \delta_2^*\}$, then

$$\dot{V}_e \le -\tilde{\varepsilon}_e\|Tx\|^2 \le -\tilde{\varepsilon}_e\|T^{-1}\|^2\|Q_e^{-1/2}\|\langle x_e, Q_ex_e\rangle =: -\varepsilon_eV_e,$$

where ε_e depends on the choice of $\delta_c > 0$. □

4. ROBUST CONTROLLER FOR THE PIEZOELECTRIC TUBE

We can now construct a controller to achieve robust output tracking for the piezotube model presented in Section 2. We consider the reference signal

$$y_{ref}(t) = a\cos(\omega t) + b\sin(\omega t), \qquad a, b \in \mathbb{R} \setminus \{0\}.$$

with a single pair of frequencies $\pm\omega$ where $\omega > 0$. Since the piezotube is a single-input single-output system, we can use a controller

$$\dot{x}_c(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x_c(t) + \delta_c \begin{bmatrix} 1 \\ 0 \end{bmatrix} (y_{ref}(t) - y(t)),$$
$$u(t) = \delta_c \, [1 \;\, 0] \, x_c(t)$$

on $X_c = \mathbb{R}^2$. Theorem 3.3 implies that whenever $\omega > 0$ is such that $\operatorname{Re} P(i\omega) \neq 0$, the above controller achieves asymptotic output tracking of the reference signal $y_{ref}(t)$ for all sufficiently small $\delta_c > 0$. Moreover, due to robustness the output tracking is achieved even if the physical parameters of the piezotube model in $\mathcal{Q}$, $\mathcal{J}$, $\mathcal{R}$, or $\mathcal{B}$ contain uncertainty or experience changes, as long as the closed-loop system stability is preserved.

5. CONCLUSIONS

In this paper we have considered the robust output regulation of passive stable infinite-dimensional port-Hamiltonian systems. As our main result we have presented a new Lyapunov proof to show that a passive internal model based controller achieves exponential closed-loop stability and robust output regulation. The use of Lyapunov techniques in the proof opens new possibilities in design of robust controllers for nonlinear port-Hamiltonian systems.

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