Testing of Generalized Uncertainty Principle With Macroscopic Mechanical Oscillators and Pendulums

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Recent progress in observing and manipulating mechanical oscillators at quantum regime provides new opportunities of studying fundamental physics, for example to search for low energy signatures of quantum gravity. For example, it was recently proposed that such devices can be used to test quantum gravity effects, by detecting the change in the $[\hat{x}, \hat{p}]$ commutation relation that could result from quantum gravity corrections. We show that such a correction results in a dependence of a resonant frequency of a mechanical oscillator on its amplitude, which is known as amplitude-frequency effect. By implementing of this new method we measure amplitude-frequency effect for 0.3 kg ultra high-Q sapphire split-bar mechanical resonator and for $\sim 10^{-5}$ kg quartz bulk acoustic wave resonator. Our experiments with sapphire resonator have established the upper limit on quantum gravity correction constant of β_0 to not exceed 5.2×10^6, which is factor of 6 better than previously measured. The reasonable estimates of β_0 from experiments with quartz resonators yields $\beta_0 < 4 \times 10^4$. The processing of data sets for physical pendulum from 1936 can lead to even more to much more stringent limitations showing $\beta \ll 1$. However, due to the absence of the evaluation of the pendulum frequency stability the exact upper bound on β_0 can not be established. The pendulum based systems only allow to test a specific form of the modified commutator that depends on the mean value of momentum. The electro-mechanical oscillators to the contrary enable testing of any form of generalized uncertainty principle directly due to the much higher stability and higher degree of control.

Introduction

At present, one of the grandest challenges of physics is to unite its two most successful theories: quantum mechanics (QM) and general relativity (GR), into a single unified mathematical framework. Attempting this unification has challenged theorists and mathematicians for several decades and numerous works have highlighted the seeming incompatibility between QM and GR [1]. It was generally supposed that this required energies at the Planck scale and therefore beyond the reach of current laboratory technology [2]. However in the relatively recent publication, I. Pikovsky et al. [3] proposed a new method of testing a set of quantum gravity (QG) theories [4–8] by using ingenuitive interferometric measurement of an optomechanical system. The prediction of most of the QG theories (such as, *e.g.*, string theory) and the physics of black holes lead to the existence of the minimum measurable length set by the Planck length $L_p = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-35}$ m [4, 7, 8]. This results in the modification of the *Heisenberg uncertainty principle* (HUP) in such a way as to prohibit the coordinate uncertainty, $\Delta x \sim \hbar/\Delta p$, from tending to zero as $\Delta p \to \infty$ [9–13]. The modified uncertainty relation, known as *generalised uncertainty principle* (GUP), is model-independent and can be written for a single degree of freedom of a quantum system as:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\Delta p^2 + \langle p \rangle^2}{M_p^2 c^2}\right], \tag{1}$$

where β_0 is a dimensionless model parameter, $M_p = \sqrt{\hbar c/G} \simeq 2.2 \times 10^{-8}$ kg is Planck mass and $\langle p \rangle$ is the quantum ensemble average of the momentum of the system. The dependence of minimum uncertainty of coordinate on average momentum is questionable but some theories [6, 8] explain as reflects the connection of spacetime curvature and the density of energy and matter manifested in Einstein's equations of general relativity.

Other more intuitive form of the GUP, *e.g.*,

$$\Delta x \Delta p > \frac{\hbar}{2}\left[1 + \gamma_0 \left(\frac{\Delta p}{M_p c}\right)^2\right], \tag{2}$$

which depends only on the uncertainties of the canonical variables of the particle but not on their mean values is predicted in [7]. To test this theory one need to either measure the deviation of the oscillators ground-state energy $E_{\min}$ with respect to its unperturbed value $\hbar\Omega_0/2$ [14], or to test of QG corrections to the dynamics of the quantum uncertainty of the mechanical degree of freedom using pulsed measurement procedure proposed in [3], which requires quantum level of precision.

The lowest modal energies measured in large mechanical systems such as AURIGA detector with effective mass of the mode $m_{\text{eff}} \simeq 1000$ kg [14] and in dumbbell sapphire oscillator with $m_{\text{eff}} \simeq 0.3$ kg [15] set the limit on the QG model parameter $\gamma_0 \lesssim 3 \times 10^{33}$, which is still too large compared to the predicted values of the order of unity [16].

Theory

From the GUP (1) one can derive the new canonical commutation relation:

$$[\hat{x}, \hat{p}]_{\beta_0} = i\hbar\left[1 + \beta_0\left(\frac{\hat{p}}{M_p c}\right)^2\right], \tag{3}$$

that is deformed by the QG correction defined by the model parameter β_0. As shown by Kempf et al. [6], parameter β_0 defines the scale of the absolutely smallest coordinate uncertainty $\Delta x_{min} = \hbar\sqrt{\beta_0}/(M_p c)$. In this work, we experimentally set an upper limit on the value of the model parameter β_0 using the dynamical implications of the contorted commutator on the oscillations of a high-Q mechanical resonator of mass m and (unperturbed) resonance frequency Ω_0.

We start our consideration with the simple premise that the modification of the fundamental commutator for a harmonic oscillator is equivalent to the nonlinear modification of the Hamiltonian by means of the perturbative transformation of momentum, $\hat{p} \to \hat{p} - \beta_0 \hat{p}^3/(3M_p^2 c^2)$, which restores the canonic commutator, $[\hat{x}, \hat{p}] = i\hbar$, at the expense of adding the non-linear term to the Hamiltonian of the resonator: $\hat{H} \to \hat{H}_0 + \Delta\hat{H} = \Big(\hat{p}^2/2m + m\Omega_0^2\hat{x}^2/2\Big) + \beta_0\hat{p}^4/\big(3m(M_p c)^2\big)$. Such non-linear correction results in the dependence of the oscillator resonance frequency on its energy [6, 8, 17]. The dynamics of the system can be described by a well known Duffing oscillator model characterized by amplitude dependence of the resonance frequency, i.e. so called amplitude-frequency effect [18, 19]. The necessary frequency resolution in order to sense subtle QG effects can be estimated by using the following expression:

$$\delta\Omega(A)/\Omega_0 = \beta_0\big(m_{\text{eff}}\Omega_0 A/M_p c\big)^2, \tag{4}$$

where $\delta\Omega = \Omega(A) - \Omega_0$ is the deviation of the amplitude-dependent resonance frequency $\Omega(A)$ from the unperturbed value Ω_0, m_{eff} is the effective mass of the mode and A is the oscillation amplitude. So, the experimentally measured dependence of the resonance frequency on the amplitude, particularly its null result, may be used to set an upper limit for the model parameter β_0.

The above mentioned theoretical considerations do not specify, which degree of freedom is subject to the QG corrections. If one considers a center of mass mode, then the scale of perturbation is strongly enhanced for the heavier than the Planck mass oscillators, as compared to individual atoms and molecules in the lattice. For instance, the precise measurement of the Lamb shift in hydrogen yielded an upper bound for the model parameter $\beta_0 < 10^{36}$ [16]. Although, the recent experiments with microscopic high-Q oscillators with effective masses ranging from 10^{-11} kg to 10^{-5} kg, established the new upper bound for $\beta_0 < 3 \times 10^7$ [19]. The intrinsic acoustic nonlinearity of micro oscillators prevented to test quantum gravity corrections with the greater precision.

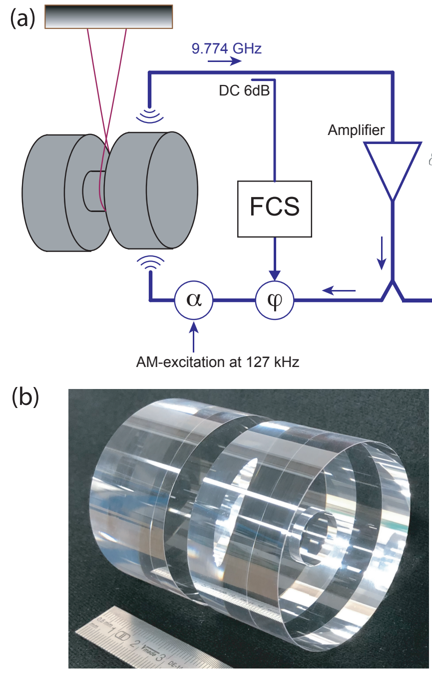

FIG. 1: (Color online) (a) Simplified experiment schematic. See the text for details. (b) Picture of the sapphire SB resonator. The ruler shows the scale of the system.

In the following we describe an experiments with the sub-kilogram split-bar (SB) sapphire mechanical oscillator, where we demonstrate improvement for the upper value of the correction parameter β_0 compared to the previous work with intermediate range mechanical oscillators [19] by nearly an order of magnitude. In addition to the that, we provide the reasonable estimates of β_0 from experiments with bulk acoustic wave (BAW) quartz resonators yields the limit of 4×10^4. As the consequence of mean value entering in the right-hand side of the Eq.(1) the systems with higher mass and larger amplitude are preferred. As an example one may take measurements of the period of the physical pendulum in 1936 [20], where much lower upper bound of $\beta_0 \lesssim 10^{-4}$ can be established from the deviation of the period dependence of amplitude from the well known Bernoulli non-linearity. However, due to the absence of the evaluation of the pendulum frequency stability the exact upper bound on β_0 can not be obtained.

Measurements of correction strength β_0 with sapphire dumbbell oscillator

Microwave oscillators based on electromagnetic Whispering Gallery Mode (WGM) sapphire crystals offer excellent short- and middle-term frequency stability [21] due WGM high quality factors exceeding 10^8 and existence of frequency-temperature turnover points. For these reasons these devices found applications in fundamental tests [22–24]. The mechanical modes of sapphire resonators may attain $Q_M \simeq 10^8 - 10^9$ [25–27]. The resonance frequencies of WGMs are very sensitive to changes in circumference, height of the cylinder resonator and to strain in the crystal lattice thus yielding the necessary coupling between mechanical and electromagnetic degrees of freedom for the observation of mechanical motion. Yet, no acoustic nonlinearities have been detected for the large sapphire mechanical resonators making these devices an excellent platform for QG tests.

The experimental setup, shown in Fig. 1(a), is based on a cylindrical dumbbell shape or split-bar (SB) sapphire resonator, which is fabricated out of a single crystal HEMEX-grade sapphire fabricated by GT Advanced Technologies Inc., USA. The rotation symmetry axis of the resonator is parallel to the c-axis of the crystal. The SB resonator consists of two bars with diameter 55 mm and height 28 mm, which are separated by the neck of diameter 15 mm and length 8 mm, see Fig. 1(b). Two electromagnetic WGM resonators are formed in each bars and undergo the same mechanical motion, i.e. they oscillate in phase for the breathing mode, which is similar to the fundamental longitudinal mode of the conventional cylindrical resonator of the same length and diameter. The resonance frequency of this mode is $\Omega_0/2\pi = 127.071$ kHz and its effective mass is calculated by using finite element modelling $m_{\text{eff}} = 0.3$ kg. In order to maximize mechanical Q-factor, the resonator is suspended via a niobium wire-loop around the neck. The whole construction is placed inside temperature stabilized vacuum chamber at 300 K. The vacuum chamber in turn is placed on vibration isolation platform and kept at a pressure of $\sim 10^{-2}$ mBar.

A parametric transducer is used to detect the mechanical vibrations of the SB resonator, see ref. [15] for the details. For that purpose, the WGM sapphire resonator serves as a dispersive element inside a closed electronic loop, which together with an amplifier and a phase shifter constitute a microwave oscillator operating at the resonance frequency of the chosen WGM mode [28]. An interferometric frequency control system (FCS) suppresses spurious phase fluctuations and locks the microwave oscillator to the frequency of the $\text{WGE}_{15,1,1}$ mode at $\omega_{\text{WGE}}/2\pi \simeq 9.774$ GHz [29]. The in-loop voltage-controlled attenuator α is used for the parametric excitation of the mechanical vibrations at 127 kHz. Approximately a half of the generated power inside the microwave sapphire oscillator is diverted to the interferometric frequency discriminator (IFD). The output signal of IFD is a linear function of its input frequency and is measured with HP 89410A spectrum analyzer. All instruments are time referenced to the hydrogen maser frequency standard VCH-103.

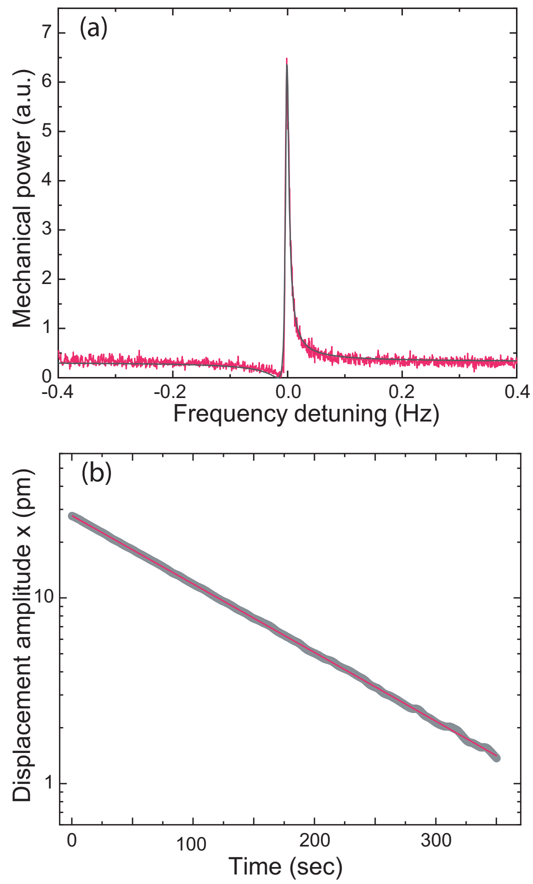

FIG. 2: (Color online) (a) Mechanical response of the SB resonator to the acoustic excitation of the in-phase breathing mode around 127 kHz. The solid curve shows the fit to the quadrature signal due to double-balanced mixing. (b) Typical ringdown measurement of the in-phase breathing mode. The solid curve shows the fit to the exponential decay.

The spatial overlap between microwave and mechanical modes results in the interaction between these degrees of freedom, which can be described by the standard opto-mechanical Hamiltonian $\hat{H}_{int} = -\hbar g_0 \hat{a}^\dagger \hat{a} \hat{x}$, where g_0 is a single photon opto-mechanical coupling, $\hat{a}^\dagger, \hat{a}$ are raising and lowering operators for the WGM and $\hat{x}$ is canonical position operator for the center of mass mechanical motion [30]. The microwave signal modulated at the resonance frequency of mechanical mode Ω_0 induces radiation-pressure force which drives mechanical vibrations. The calibration of amplitude of center of mass motion is made by using the standard expression

$$\delta u(\Omega) = \delta x(\Omega)(du/df)(df/dx), \tag{5}$$

where df/dx is determined from the amplitude of the output IFD signal $\delta u(\Omega)$. That signal is proportional to the applied modulated power δP

$$\delta u(\Omega) = \chi \left(\frac{du}{df}\right)\left(\frac{df}{dx}\right)^2 \delta P, \tag{6}$$

where χ is the constant describing electromagnetic coupling of the signal and mechanical property of the oscillator [15]. The transduction constant is calculated to be $\delta x/\delta u = 526$ nm/mV.

The mechanical response of the SB-resonator to the acoustic excitation in the vicinity of the resonance frequency is shown in Fig. 2(a). The applied excitation signal at 127 kHz is relatively weak resulting in the maximal amplitude of mechanical vibrations of 6 pm. The output signal is measured by using phase-sensitive interferometric setup which results in superposition of dispersive and absorptive quadrature components. The solid curve displays the fit of the experimental data to such composite absorptive-dispersive response and yields the resonance frequency of the mechanical resonator $\Omega_0/2\pi = 127070.9695 \pm 0.0003$ Hz, its FWHM linewidth $\Gamma_M/2\pi = 3.5$ mHz and the 55° degrees mismatch between the arms of the IFD.

The ringdown measurements of the mechanical vibrations are made in two steps. In the first step the resonance frequency of mechanical vibrations $\Omega_0/2\pi$ is determined. For that purpose, the radiation pressure force is applied to the resonator for the time sufficient to settle the mechanical vibrations (several minutes). Then, the output signal $\delta u(\Omega)$ is measured for every frequency point in the scanning range of 1 Hz. The resonance frequency corresponds to the point which yields the maximal IFD response $\delta u(\Omega)$. This procedure is repeated for the different excitation amplitudes (20-35 pm) in every single experimental run and detected no resonance frequency shift within accuracy of 10 mHz determined by the resolution bandwidth of FFT analyzer. In the second step, after the mechanical resonance frequency is located, the AM-excitation is turned off, and then the mechanical vibrations are measured as they decrease due to acoustic losses. The amplitude and frequency of the decaying vibrations, i.e. the amplitude and the frequency of the spectral peak, is then tracked and recorded every 0.2 s. For that purpose a marker is placed on the maximum voltage value in the spectra, and its frequency and amplitude is recorded for every time bin. The frequency accuracy of such measurements is determined by the resolution bandwidth of the FFT analyzer, which is set to 5 Hz.

The typical ringdown measurement is presented in Fig. 2(b). For this particular example, the resonant frequency is $\Omega_0/2\pi = 127070.97$ Hz. The solid curve shows the fit of the experimental data to the exponential decay with characteristic time constant $\tau = 173$ sec which yields the mechanical quality factor $Q_M = \Omega_0\tau/2 =$

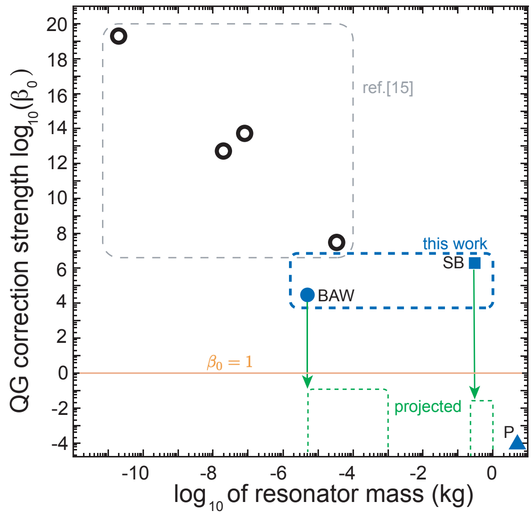

FIG. 3: (Color online) The correction strength β_0 versus mass of mechanical oscillator determined in various experiments. The open circles correspond to β_0 reported in ref. [19]. The closed circle is the estimated β_0 for the quartz BAW at LD-cut, see ref. [18]. The square is the upper limit for β_0 obtained with sapphire split-bar resonator. The triangle shows the estimate of the correction strength from the measurements of the period of the physical pendulum, see ref. [20].

$3.4 \cdot 10^7$ and the same FWHM linewdith Γ_M which is found in mechanical response measurements. In addition to the extracting of the parameters of the exponential decay, the Duffing equation was numerically solved in order to attain the best fit parameters for the ringdown amplitudes. Following this procedure we extract the upper limit for the QG model parameter to be $\beta_0 < 6 \times 10^{11}$.

The frequency measurements yield much more stringent limit on β_0. In all measured ringdown series, there is no evidence of any detectable frequency shift up to the maximum amplitude of mechanical displacement of 75 pm. The null-frequency shift measured in the experiment corresponds to the accuracy of $\delta\Omega/\Omega_0 = 3.9 \times 10^{-5}$ and accordingly to the Eq. 4 yields the upper limit for the QG model parameter $\beta_0 < 5.2 \times 10^6$.

The sapphire SB resonator demonstrate a large potential for even more stringent test of $\beta_0 \ll 1$. Here, we propose two possible ways to improve the experiment. Firstly, the mechanical response (Fig. 2(a)) could be measured for much larger input power δP. It is possible to excite vibrations in sapphire resonator with amplitude of several nanometers [27]. That would result is much higher signal-to-noise ratio and as a consequence would improve the accuracy of determination of the mechanical resonance frequency Ω_0 to be better than 0.1 mHz. Together with an increase of the oscillation amplitude by two-three orders of magnitude that may result at least in 10^8 fold improvement for the upper limit on β_0. Secondly, one can implement an electromechanical sapphire oscillator by closing a feedback loop with the IFD output signal

$\delta u(\Omega_M)$. In that case the uncertainty in determination of frequency shift will be decreased with the integration time T as $1/\sqrt{\Omega_0 T}$. Assuming the driving amplitude of SB resonator of $A \simeq 1$ nm and average time of $T = 1$ hour, the testable limit $\beta_0 \ll 1$ is within experimental accuracy.

Estimation of the correction strength with BAW

Another mechanical system, namely quartz bulk BAW resonator also constitutes a fruitful platform for precise tests of quantum gravity. This system exhibits high resonance frequency $\Omega_0/2\pi \simeq 10$ MHz, milligram scale of the effective mass of oscillating modes [31], large Q-factor close to 10^{10} at low temperatures [32] and high frequency stability of electromechanical oscillators reaching the level of 5×10^{-14}. The above listed features of quartz BAW are very attractive for fundamental tests such as Lorentz symmetry [33]. However, we note that quartz crystals possess its own quite strong elastic non-linearities that can mimic the quantum gravity effect. These non-linearities lead to a similar frequency shift, quadratic in amplitude and known as amplitude-frequency effect or isochronism, see ref. [34], p. 245. This effect can be made to nearly vanish by means of an optimal choice of the cut angle of the crystal, known as LD-cut [18, 35]. The QG correction strength can be estimated from Eq. 4 and by using the experimental parameters $m_{\text{eff}} = 5$ mg, $\Omega_0/2\pi = 10$ MHz, $\delta\Omega/2\pi \simeq 1$ mHz, $A \simeq 1$ nm. Our estimation yields $\beta_0 \lesssim 4 \times 10^4$, which is still limited by elastic non-linearity. In order to single out quantum gravity frequency from such non-linearity, the amplitude frequency shift shall be measured in dependence on the effective mass of the resonating mode. We also believe that experimenting with kilogram scale quartz BAW [36] will result in much more stringent test of the quantum gravity model parameter in regime $\beta_0 \lesssim 1$, because of weaker non-linearity due to the lower acoustic energy density and much larger effective mass.

Estimation of the correction strength with physical pendulums

At the present time, the most stringent limit on correction strength β_0 can be by using the data set gathered during experiments with mechanical pendulums. Given relatively large amplitude ~ 1 cm, large mass of the pendulum ~ 1 kg and reasonable fractional frequency stability $\sim 10^{-7}$ such system, in principle, seems to be ideal for the testing of generalized commutator described by Eq.(1). The pendulum possesses an intrinsic softening non-linearity which can be calculated exactly. The QG correction is assumed to contribute to the non-linearity of the system and the Eq.(4) can be written in the following form:

$$\frac{\delta T(\theta)}{T_0} = -\beta_0 \left(\frac{m 2\pi L\theta}{M_p c T_0}\right)^2 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4, \qquad (7)$$

where $\delta T(\theta)$ is amplitude dependent deviation of the period of the pendulum T_0, m is the mass of the pendulum, θ is the angular amplitude of a pendulum and L is its length. The last two terms describes the intrinsic non-linearity of the mathematical pendulum. The dependence between the rate and arc (θ) for the free pendulum was measured already in 1936 by using physical pendulum [20] with the length $L = 1$ m and the mass $m \simeq 6$ kg. By taking this data set, we estimate the $\beta_0 \lesssim 10^{-4}$. Other experiments with different kind of pendulums carried at different times shows no evidence of the deviation of the oscillation period from the conventional theory of mathematical pendulums [37–39], and result in similar order of magnitude for the upper limit for β_0. However, absence of the knowledge of the important experimental details such as Allan deviation, i.e. frequency stability, the absence of any information on systematic and statistical errors lead us to the qualitative conclusion only that correction strength $\beta_0 \ll 1$, thus putting quite severe constraints on the GUP described by Eq. 3. For the quantitative measurements of β_0 one has to repeat measurements with pendulums or other systems (SB sapphire resonators or BAWs).

Conclusion

To conclude, we have presented measurement of the upper limit on QG correction strength by using ultra-high-Q mechanical sapphire resonator with sub-kilogram mass of the resonating mode. In the original work [3], a light pulse is proposed to reflect off an oscillator four times, separated by one quarter the oscillation period - before having the its phase measured. Our analysis and experiment shows, that one can attain the same goal in continuous RF measurement, which makes the experiment much simpler and reliable because the oscillation frequency can be measured with more precision compared to any other physical parameter. The overview of results of testing β_0 with mechanical oscillators is presented in Fig. 3, which shows the measured constraints on the QG correction strength in dependence of the effective mass of the mechanical resonator. The heavier oscillators allows for the better determination of the correction strength. In principle, old data sets gathered with mechanical pendulums can be used for the immediate estimation of correction strength which yield qualitative results that $\beta_0 \ll 1$ and put severe constraints on GUP relation described by Eq.(1). However, the remarkable high-Q and frequency stability of state of the art quartz BAW resonators and SB sapphire resonator in conjunction with low acoustic non-linearities have a great potential for its further applications in precise tests of minimal length scale scenarios for the quantum gravity theories [40], where the ultimate limit on β_0 can be measured very precisely.

The great advantage of the electromechanical (sapphire split-bar) or opto-mechanical systems is that potentially they offer the feasible path towards quantum-limited experiments. In this case, other forms of GUP(described by Eq.(2)), which solely depend on quantum-mechanical momentum uncertainty Δp and have no boost from the momentum amplitude, can be tested. In that respect, the perspectives of utilizing of low-frequency (< 1 Hz) mechanical oscillators remains unclear compared to the high-frequency system (kHz-GHz), where nearly quantum regime [41, 42] or even quantum limit [43–45] can already be accessed.

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[1] S. Hossenfelder, *Experimental Search for Quantum Gravity* (Springer, Cham, 2018).
[2] G. Amelino-Camelia, Nature **410**, 1065 (2001).
[3] I. Pikovski, M. R. Vanner, M. Aspelmeyer, M. S. Kim, and Č. Brukner, Nat. Physics **8**, 393 (2012).
[4] M. Maggiore, Phys. Lett. B **304**, 65 (1993).
[5] M. Maggiore, Phys. Rev. D **49**, 5182 (1994).
[6] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995).
[7] L. J. Garray, Int. J. Mod. Phys. A **10**, 145 (1995).
[8] A. F. Ali, S. Das, and E. C. Vagenas, Phys. Rev. D **84**, 044013 (2011).
[9] S. W. Hawking, Nuc. Phys. B **144**, 349 (1978).
[10] F. Scardigli, Phys. Lett. B **452**, 39 (1999).
[11] P. Jizba, H. Kleinert, and F. Scardigli, Phys. Rev. D **81**, 084030 (2010).
[12] G. Amelino-Camelia, Phys. Rev. D **62**, 024015 (2000).
[13] Y. J. Ng and H. van Dam, Found. Phys. **30**, 795 (2000).
[14] F. Marin, F. Marino, M. Bonaldi, M. Cerdonio, L. Conti, P. Falferi, R. Mezzena, A. Ortolan, G. A. Prodi, L. Taffarello, et al., Nat. Physics **9**, 71 (2012).
[15] J. Bourhill, E. Ivanov, and M. E. Tobar, Phys. Rev. A **92**, 023817 (2015).
[16] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008).
[17] P. Bosso, S. Das, and R. B. Mann, Phys. Rev. D **96**, 066008 (2017).
[18] N. Gufflet, F. Sthal, J.-J. Boy, R. Bourquin, and M. Mourey, IEEE Trans. Ultrasonics, Ferroelectrics, and Frequency control **48**, 1681 (2001).
[19] M. Bawaj, C. Biancioforе, M. Bonaldi, F. Bonfigli, A. Borrielli, G. D. Giuseppe, L. Marino, R. Natali, A. Pontin, G. A. Prodi, et al., Nature Comm. **6**, 7503 (2015).
[20] E. C. Atkinson, Proc. Phys. Soc. **48**, 606 (1936).
[21] E. N. Ivanov and M. E. Tobar, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control **56**, 263 (2009).
[22] P. L. Stanwix, M. E. Tobar, P. Wolf, C. R. Locke, and E. N. Ivanov, Phys. Rev. D **74**, 081101(R) (2006).
[23] V. Giordano, S. Grop, B. Dubois, P.-Y. Bourgeois, Y. Kersal, G. Haye, V. Dolgovskiy, N. Bucalovic, G. Di Domenico, S. Schilt, et al., Review of Scientific Instruments **83**, 085113 (2012).
[24] M. Nagel, S. Parker, E. Kovalchuck, P. Stanwix, J. Hartnett, E. Ivanov, A. Peters, and M. Tobar, Nature Comm. **6**, 8174 (2015).
[25] V. B. Braginsky, V. P. Mitrofanov, and V. I. Ivanov, *Systems with small dissipation* (The University of Chicago Press, Chicago, 1985).
[26] M. Tobar, E. Ivanov, D. Oi, B. Cuthbertson, and D. Blair, Appl. Phys. B **64**, 153 (1997).
[27] C. R. Locke, M. E. Tobar, and E. N. Ivanov, Rev. Sci. Instruments **71**, 2732 (2000).
[28] D. B. Leeson, Proc. IEEE **56**, 329 (1966).
[29] E. N. Ivanov, M. E. Tobar, and R. A. Woode, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control **45**, 1526 (1998).
[30] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[31] J. Bon, L. Neuhaus, S. Deléglise, T. Briant, P. Abbé, P. Cohadon, and S. Galliou, J. Appl. Phys. **124**, 073104 (2018).
[32] M. Goryachev, D. L. Creedon, S. Galliou, and M. E. Tobar, Phys. Rev. Lett. **111**, 085502 (2013).
[33] A. Lo, P. Haslinger, E. Mizrachi, L. Anderegg, H. Müller, M. Hohensee, M. Goryachev, and M. E. Tobar, Phys. Rev. X **6**, 011018 (2016).
[34] W. P. Manson and R. N. Thurston, *Physical Acoustics: principles and methods vol. XI* (Academic Press, New York, 1975).
[35] S. Galliou, F. Sthal, J. Boy, R. Bourquin, and M. Mourey, 2004 IEEE International Ultrasonics, Ferroelectrics, and Frequency Control Joint 50th Anniversary Conference p. 475 (2004).
[36] J. Vig and D. Howe, IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control **60**, 428 (2013).
[37] M. K. Smith, Am. J. Phys **32**, 632 (1964).
[38] L. P. Fulcher and B. F. Davis, Am. J. Phys. **44**, 51 (1976).
[39] S. Gil, A. E. Lagarreta, and D. E. D. Gregorio, Am. J. Phys. **76**, 843 (2008).
[40] S. Hossenfelder, Living Rev. Relativity **16**, 2 (2013).
[41] P. Bushev, G. Hétet, L. Slodiĉka, D. Rotter, M. A. Wilson, F. Schmidt-Kaler, J. Eschner, and R. Blatt, Phys. Rev. Lett. **110**, 133602 (2013).
[42] D. V. Martynov, E. D. Hall, B. P. Abbott, R. Abbott, T. D. Abbott, C. Adams, R. X. Adhikari, R. A. Anderson, S. B. Anderson, K. Arai, et al., Phys. Rev. D **93**, 112004 (2016), URL `https://link.aps.org/doi/10.1103/PhysRevD.93.112004`.
[43] A. Schliesser, O. Arcizet, R. Riviére, G. Anetsberger, and T. J. Kippenberg, Nature Phys. **5**, 509 (2009).
[44] J. Teufel, T. Donner, M. Castellanos-Beltran, J. Harlow,

and K. Lehnert, Nature Nanotech. **4**, 820 (2009).
[45] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, Nature **475**, 359 (2011).