Confidence Intervals for Three-Cornered Hat and Groslambert Covariance estimates

François Vernotte[1] and Éric Lantz[2]

***Abstract*— The three-cornered hat / Groslambert Covariance methods are widely used to estimate the stability of each individual clock in a set of three but no method gives reliable confidence intervals for large integration times.**

We propose a new method which takes into account all the measurements between the pairs of clocks in a Bayesian way. The result is the Cumulative Density Function which yields confidence intervals for each clock AVAR. This CDF provides also a stability estimator which is always positive.

Checked by massive Monte-Carlo simulations, this method proves to be perfectly reliable even for two degrees of freedom.

***Keywords*— Clock stability; Allan variance; three-cornered hat; covariances; confidence interval; Bayesian analysis**

I. INTRODUCTION

Although the three-cornered hat [1] and the Groslambert Covariance [2] methods are widely used to measure the stability of each individual clock in a set of three, the only methods which exists to compute error bars are limited to the smallest integrations times, i.e. when the number of Equivalent Degrees of Freedom (EDF) is high [3], [4], [5]. However, there are no reliable method to assess confidence intervals over the estimates if their number of Equivalent Degrees of Freedom (EDF) is low. However, since this case occurs for the largest integration times, it is an important issue for all applications dealing with long term stability (e.g. time keeping).

In a previous paper, we performed a first Bayesian attempt to estimate confidence intervals from the three-cornered estimates but we observed that this method was only valid beyond 5 EDF [5]. We propose then a new method, which is also based on Bayesian statistics, but which takes into account all the measurements between the pairs of clocks rather than the three-cornered hat estimates. The results is the Cumulative Density Function (CDF) which yields the lower and upper bounds of the 95 % confidence interval or the 95 % upper limit when the lower bound is 0.

The performances of this method have been checked by using massive Monte-Carlo simulations. The principle of these simulations is described in this paper and the comparisons with the theoretical confidence intervals given by our new method are discussed.

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II. PRINCIPLE OF THE METHOD

A. Clock comparisons

Let us consider 3 independent clocks: A, B and C. It is possible to compare these clocks by pairs and to estimate the corresponding AVARs. Let us denote $\bar{y}_{ABk}$ the k^{th} frequency deviation sample between A and B and $\bar{z}_{ABk} = (\bar{y}_{ABk+1} - \bar{y}_{ABk})$. For clocks A and B, the AVAR is $\sigma^2_{AB} = \frac{1}{2}\langle \bar{z}^2_{ABk} \rangle$. Since $\sigma^2_{AB} = \sigma^2_A + \sigma^2_B$, the three-cornered hat is based on the following property: $\hat{\sigma}^2_A = \frac{1}{2}\left(\sigma^2_{AB} - \sigma^2_{BC} + \sigma^2_{CA}\right)$ where the symbol $\hat{}$ means that $\hat{\sigma}^2_A$ is an estimate of the true AVAR value σ^2_A.

On the other hand, the Groslambert covariance is based on this other property: $\text{GCov}_A = \frac{1}{2}\langle \bar{z}_{ABk} \cdot \bar{z}_{ACk} \rangle = \hat{\sigma}^2_A$.

The only difference between these two approaches concerns the measurement noise due to the counters, since GCov rejects it [4]. However, since we deal only with high τ values, the measurement noise is negligible regarding the clock noise and we do not distinguish these two approaches.

B. Bayesian inference

In Bayesian analysis, we have to consider the model parameters $\vec{\Theta} = (\theta_1, \ldots, \theta_m)^T$ which are m real values and the measurements $\vec{X} = (x_1, \ldots, x_n)^T$ which are n random variables (r.v.). In our case, the parameters are the 3 true AVAR values of the 3 clocks that we want to know and the measurements are either the 3 estimates obtained with the three-cornered hat method (KLTG method for Karhunen-Loéve Transform with Gaussian approximation, see [5]) or the $3N$ differences between the clock pairs $\bar{z}_{ABk}$ with $k \in \{1, \ldots, N-1\}$ (the present method).

On the other hand, two issues are considered: (1) the **direct problem**, which consists in calculating the Probability Density Function (PDF) of the estimates knowing the model parameters $p(\vec{X}|\vec{\Theta})$; (2) the **inverse problem**, which consists in calculating the PDF of the model parameter knowing the estimates $p(\vec{\Theta}|\vec{X})$. The direct problem has been solved in [5] and the inverse problem may be solved thanks to the Bayes theorem: $\begin{cases} p(\vec{\Theta}|\vec{x}) \propto \pi(\vec{\Theta}) \cdot p(\vec{X}|\vec{\Theta}) \\ \int p(\vec{\Theta}|\vec{X}) \mathrm{d}\vec{X} = 1 \end{cases}$ where $\pi(\vec{\Theta})$, called the prior, is the *a priori* probability of the parameter $\vec{\Theta}$ before any measurement.

C. The KLTS method

However, due to the complexity of $p(\vec{X}|\vec{\Theta})$, only an approximation was achieved in [5], the so-called KLTG method, which has proved to be valid for EDFs greater than 5.

We propose then the KLTS method (for Karhunen-Love Transform using Sufficient statistic) which relies on the use of the $\bar{z}_{ABk}$ (or BC or CA), which are Gaussian r.v., instead of the $\hat{\sigma}_A^2$ estimates (or B or C), which are a linear combination of χ^2 laws. The main advantage of this approach lies in the property of the Gaussian estimates which remains Gaussian when they are linearly combined.

However, these estimates are strongly correlated for two reasons: (1) the $\bar{z}_{ABk}$ and $\bar{z}_{ABk+1}$ are not independent (except in the case of White FM and AVAR without overlapping); (2) the $\bar{z}_{ABk}$, $\bar{z}_{BCk}$ and $\bar{z}_{CAk}$ are not independent since their sum is null (if the measurement noise is neglected).

Now, if the estimates are independent, the PDF is easy to calculate since $p(\vec{X}|\vec{\Theta}) = \prod_{j=1}^{3N} p(x_j|\vec{\Theta})$. It is then necessary to transform the $3N$ estimates $\bar{z}_{ABk}$ into $2M$ (with $M \leq N$) uncorrelated estimates z_{Pj}. This is performed by the successive diagonalizations of 2 covariance matrices: (1) to transform the N estimates $\bar{z}_{ABk}$ into M estimates $\bar{w}_{ABj}$; (2) to transform each set of 3 estimates $(\bar{w}_{ABj}, \bar{w}_{BCj}, \bar{w}_{CAj})$ into a set of 2 estimates $(\bar{w}_{Pj}, \bar{w}_{Qj})$.

The last step of the KLTS method concerns the choice of the prior. We have chosen a prior following a $1/\theta$ behavior in such a way that all orders of magnitude have the same probability (total lack of knowledge).

Finally, we obtain the PDF and the CDF of the parameter knowing the estimates which allows us to calculate the bounds of any confidence interval as well as the median value, i.e. the argument giving the CDF equal to 0.5. This value, always positive, may be an alternative estimate of the parameters.

III. VALIDATION OF KLTS METHOD BY MONTE-CARLO SIMULATIONS

A. Principle of the simulation

In order to validate the KLTS method, we have compared its results to Monte-Carlo simulations according to two ways.

The first way concerns the direct problem. It consists in fixing the parameter triplet $(\sigma_A^2, \sigma_B^2, \sigma_C^2)$, randomly drawing the estimates $\bar{z}_{ABk}$ and computing the confidence interval by using the KLTS method. If it is correct, the parameter values should generally remain within the confidence intervals.

The second way concerns the inverse problem. The algorithm is as follows:

1) Select a set of $3N$ measurements $\bar{z}_{ABk}$, N being the chosen EDF number, in such a way that they provide a chosen final estimate triplet $(\hat{\sigma}_A^2, \hat{\sigma}_B^2, \hat{\sigma}_C^2) = (A_0, B_0, C_0)$. We call "reference measurement set" this set of $3N$ measurements $\bar{z}_{ABk}$.
2) Draw at random a parameter triplets $(\sigma_A^2, \sigma_B^2, \sigma_C^2)$ and then randomly draw $3N$ measurements $\bar{z}_{ABk}$ according to these parameters.
   - If this $3N$ measurement set is not equal to the reference measurement set, the corresponding parameter triplet $(\sigma_A^2, \sigma_B^2, \sigma_C^2)$ are thrown.
   - But if the randomly drawn measurement set is close to this reference measurement set within 10 %, the corresponding parameter triplet $(\sigma_A^2, \sigma_B^2, \sigma_C^2)$ are kept.
3) Go to Step 2).

Each simulation run stops when 10,000 achievements have been obtained.

This ensemble of 10,000 parameter triplets giving $(\hat{\sigma}_A^2, \hat{\sigma}_B^2, \hat{\sigma}_C^2) = (A_0, B_0, C_0)$ is then compared to the confidence interval obtained by the KLTS method.

Thanks to the sufficient statistic properties of the final estimate triplets, it turns out that any $3N$ measurement set providing the given (A_0, B_0, C_0) final estimate triplet leads to the same statistical distribution of the parameter triplet.

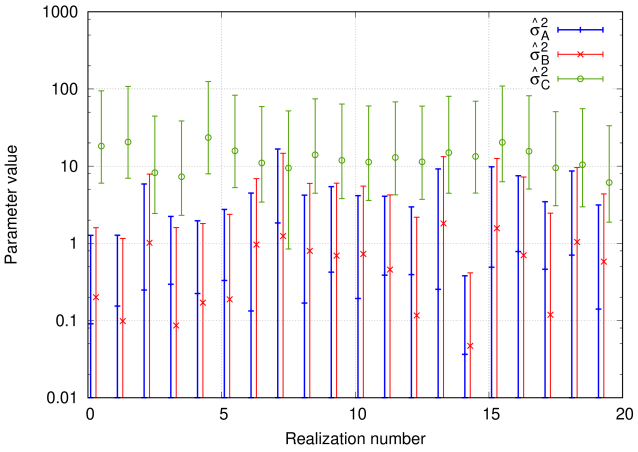

Fig. 1. Validation by the direct problem with the following fixed parameters: $\sigma_A^2 = 0.1$, $\sigma_B^2 = 1$, $\sigma_C^2 = 10$. The EDF number is 2.

B. Results and discussion

Figure 1 shows an example of validation by the first way for estimates with 2 EDF. We can remark that the parameter values (0.1 for the blue error bars, 1 for the red and 10 for the green) are well in the confidence intervals except in 1 case (realization number 14, red error bar, where the parameter value $\sigma_B^2 = 1$ is clearly above the upper limit). Although this test seems conclusive, it remains somewhat qualitative.

Table I uses the second validation way and shows if the confidence interval obtained by the KLTG method of [5] or by the new KLTS method are really at 95 % of confidence. It compares also the bounds to the empirical bounds given by 10,000 Monte-Carlo simulations. Unlike the results of KLTG, the KLTS bounds show a good agreement with the empirical bounds and the confidence intervals are almost equal to 95 %. Other tests performed with other choices of parameters and with higher EDF show the same concordance with the simulations.

IV. CONCLUSIONS

This study shows that this new KLTS method is fully reliable even with 2 EDF. However, its computation can not been achieved for EDF greater than a few 100s. We recommend then to use this method below 300 EDF and to use the KLTG method described in [5] above.

TABLE I

COMPARISON OF THE CONFIDENCE INTERVALS OBTAINED BY THE MONTE-CARLO SIMULATIONS (EMPIRICAL), BY THE KLTG METHOD OF [5] AND BY THE NEW KLTS METHOD. THE (A_0, B_0, C_0) ESTIMATE TRIPLET IS $(0.1, 1, 10)$ AND THE NUMBER OF EDF IS 2.

	Bound	$\hat{\sigma}_A^2 = 0.1$		$\hat{\sigma}_B^2 = 1$		$\hat{\sigma}_C^2 = 10$	
Emp.	2.5 %					0.0065	(2.5 %)
	95 %	20	(95 %)	21	(95 %)		
	97.5%					370	(97.5 %)
KLTG	2.5 %					3	(8.8 %)
	95 %	2.2	(74.5 %)	2.4	(73.9 %)		
	97.5%					44	(80.2 %)
KLTS	2.5 %					0.0055	(2.4 %)
	95 %	20	(95.0 %)	21	(94.9 %)		
	97.5%					280	(96.6 %)

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