Dynamics: From analytical principles to architectonical theorems

N. Daher

FEMTO-ST Institute, (University of Franche-Comté, CNRS, ENSMM, UTBM),
15B avenue des Montboucons, F-25030 Besançon Cedex, France

naoum.daher@femto-st.fr

Abstract

A recent article proposed for dynamics a unifying Leibnizian formulation, leading to the quantitative solutions which are usually derived by use of analytical principles (variational, geometrical, dynamical…), each revealing one point of view. Here, we show how to derive not only the solutions but also the formal structures that lead to these solutions. Consequently, with this presentation, the analytical principles, usually postulated separately and independently of one another, appear as theorems.

Keywords and phrases: Analytical formulations, Leibnizian architectonical approach, Theorems versus principles.

Introduction

In a previous article [1], a dynamical Leibnizian formulation accounting simultaneously for a variety of dynamical worlds and an infinity of points of view, has been proposed. This formulation, issued from Huygens procedure with one point of view attached to a single world, has been conceptually extended by Leibniz to multiple worlds and points of view but without being formalized until recently. This formalization includes the different solutions, historically obtained separately by the various analytical principles (variational, geometrical…) [2-5]. We present here a general procedure that allows to generate, in addition to the solutions already developed in Ref.[1], the formal structures that correspond to the analytical principles. These principles, deduced now instead of being postulated, become theorems.

The dynamical Leibnizian formulation [1] which extends Huygens study of frontal elastic collisions, combining, in a general way, the relativity and conservation requirements, before any choice of a parameter for motion, leads to the operator $O = I\, d/dx$, which plays the role of a generator of conserved entities. An indetermination appears in its expression through I which is an arbitrary function of the motion parameter x [$I = I(x)$], specified later on [through x = {v, u or w} = {velocity, celerity or rapidity}]. The operator O is applied twice. The first application allows passing from one conserved entity E to another ($p = OE$) while the second one (O^2) puts a constraint C on the dynamical structure ($C = O^2E$), in order to avoid getting more than the two conserved entities (E, p), required by the dynamical problem.

This method has been applied in [1] to Leibniz infinity of points of view, where Id/dx transforms into $I_\mu d/dv_\mu$. It allowed ordering (iteratively) this infinite multitude thanks to the index μ that takes infinity of values. The strategy, here, is quite different; it remains finite in order to establish links and comparisons with the three analytical principles (or formulations) encountered in the scientific literature.

To this end, we start by recalling the dynamical structure developed in Ref.[1], but limited here to Einstein's dynamical world:

$$C = E/c^2 = O^2E = I d/dx\,[IdE/dx] = I^2 d^2E/dx^2 + I[dI/dx]dE/dx \quad \text{with} \quad p = OE = IdE/dx \qquad (1)$$

This formal structure corresponding to Eq.(14) of Ref.[1] but adapted to the present investigation ($I_\mu d/dv_\mu \rightarrow I\, d/dx$) is under-determinate. It imposes a specific world (Einstein's one: $C = E/c^2$), keeping the points of view unspecified.

As shown explicitly in [1], the indeterminate function I (present in $p = I\, dE/dx$) corresponds to a general unspecified composition law of motion: $x' = T(x, X) = x\, T\, X$. In the particular case: $I = 1$, one recovers Huygens expression of impulse: $p = dE/dx$, corresponding to an additive composition law: $x' = x + X$.

We shall firstly show that it is possible to derive the fundamental equation of Einstein's dynamics, linking together the two conserved entities E and p, by eliminating the entities I and x from Eq.(1), by use of : $O = Id/dx = (IdE/dx)d/dE = pd/dE$, where we have accounted for $p = OE = IdE/dx$. One deduces thus:

$$O^2 = I\, d/dx\, [I\, d/dx] = pd/dE[pd/dE] = p^2\, d^2/dE^2 + p(dp/dE)d/dE$$

Its application to E leads to: $O^2E = pdp/dE$. When combined with (1) we obtain:

$$C = E/c^2 = pdp/dE \qquad (2)$$

Its integration yields the fundamental equation of Einstein's dynamics:

$$E = mc^2\, (1 + p^2/m^2c^2)^{1/2} \qquad (3)$$

Simplification of the general under-determinate dynamical structure

Although conceptually simple ($C = O^2E$), the general structure given in (1) is formally cumbersome and mathematically complicated to handle. It is possible to simplify it with the introduction of two new entities F and G, complementary to E. Precisely, instead of the two entities E and I present in (1), having different dimensions, we manage to eliminate them in favor of the two new entities: F and G, having the same dimension as E. This allows direct comparisons between E, G and F that suggest different identifications, leading to various structures. These turn out to be similar to those corresponding to the different analytical principles (or formulations) used in physics.

The formally complicated operator: $O^2 = OO = Id/dx\, [Id/dx] = I^2\, d^2/dx^2 + I\, [dI/dx]d/dx$, is composed of two groups of terms, mixing together second and first order differentiations. It will be replaced by two simpler ones, each composed of only one group of terms : $O_2 = Od/dx = I\, d^2/dx^2$ (second-order operator) and $O_1 = (1/x)O = (I/x)d/dx$ (first-order operator), having the same dimension.

This reorganization of the initial formal structure leads to a mathematical form which is much simpler to handle and to integrate. Since O_2 and O_1 have been constructed in such a way that they keep the same dimension as O^2, the introduction of two new entities F and G associated respectively with O_2 and O_1 (O_2F and O_1G) will necessarily have the same dimension as energy E. This formal simplification that leads to :

$$C = O^2E = O_2F = O_1G \qquad (4)$$

corresponds explicitly to:

$$C = I^2\, d^2E/dx^2 + I[dI/dx]dE/dx = Id^2F/dx^2 = (I/x)dG/dx \qquad (5)$$

As shown in [1], O^2 and E express respectively an operator and an entity having a clear physical meaning, directly linked to the relativity and conservation requirements. As to the new operators O_2 and O_1 with their corresponding entities F and G, they have no direct physical interpretations. They aim at simplifying the initial complicated differential structure that becomes more convenient to solve.

The introduction of the entity F also simplifies the expression of impulse: $p = OE = IdE/dx$ that reduces to $p = dF/dx$ (up to an additive constant) since one has: $C = O^2E = OOE = Op = Idp/dx = I\, d^2F/dx^2$.

Thanks to the identity: $xd^2F/dx^2 = d/dx[xdF/dx - F]$, one deduces from (5): $dG/dx = d/dx[xdF/dx - F]$ that leads to the integral form: $G = xdF/dx - F$ (up to an additive constant). On gathering the different results where neither E nor I appear, one gets:

$$p = dF/dx \quad \text{and} \quad G = x\, dF/dx - F \qquad \text{(up to additive constants)} \tag{6}$$

Identification procedures and corresponding points of view

Since G has the same dimension as E and F, it is quite natural to determine the structure by identifying G with E then with F (up to additive constants that one does not need to account for here).

When $G = E$, if we set $x = v$, one is left with:

$$p = dF/dv, \quad G = E = vdF/dv - F \tag{7}$$

When $G = F$, if we set $x = u$, one is left with:

$$p = dF/du, \quad G = F = udF/du - F \tag{8}$$

The two identifications (or projections): $G = E$ and $G = F$ lead to two points of view that may be expressed in a compact way by : $v = dE/dp$ and $u = p/m$. Indeed, the differentiation of $G = E$, given in (7), accounting for $p = dF/dv$ and $E = vdF/dv - F$, leads to $dE = vdp$ or equivalently: $v = dE/dp$. The differentiation of $G = F$, given in (8), accounting for $p = dF/du$, leads to $du/u = dp/p$ from which one deduces $u = p/\mu$, where μ is a constant of integration that may be identified with the mass ($\mu = m$) without loss of generality, in virtue of the conservation properties, getting thus: $u = p/m$.

On combining (3) with $v = dE/dp$ and $u = p/m$, it becomes possible to express impulse and energy, in terms of v and u. After some calculations and formal manipulations, one is left with:

$$p = mv/(1 - v^2/c^2)^{1/2} \qquad E = mc^2/(1 - v^2/c^2)^{1/2} \tag{9}$$

$$p = mu \qquad E = mc^2(1 + u^2/c^2)^{1/2} \tag{10}$$

Having derived p and E in terms of v and u, one may deduce, from the definition of impulse: $p = IdE/dx$ or more explicitly: $p = I_v\, dE/dv$ and $p = I_u\, dE/du$, the expressions:

$$I_v = 1 - v^2/c^2 \quad \text{and} \quad I_u = (1 + u^2/c^2)^{1/2} \tag{11}$$

These can be linked to historical and/or conventional writings, associated with the so-called "relativistic mass" M and Lorentz factor Γ. On combining (11) with (9) and (10), one deduces:

$$p = Mv \text{ and } E = Mc^2 \qquad \text{with } M = C = m/I_v^{1/2} = m/(1 - v^2/c^2)^{1/2} \qquad (12)$$

$$p = mu \text{ and } E = mc^2\Gamma \qquad \text{with } \Gamma = I_u = (1 + u^2/c^2)^{1/2} \quad \Rightarrow \quad c^2\Gamma^2 - u^2 = c^2 \qquad (13)$$

The determinations of v and u (velocity and celerity), related to M and Γ, given in (12) and (13) result from the two different projections expressed by the dynamical constraints: $G = E$ and $G = F$. This violently contrasts with their usual kinematical (spatiotemporal) definitions [attached to coordinate time for $v = dr/dt$ and invariant time for $u = dr/d\tau$]. The point of view attached to v is usually derived from the variational formulation while the point of view attached to u is usually derived from the geometrical formulation, expressed in a compact way through:

$$\mathbf{p} = m\mathbf{u} \qquad \text{with } \mathbf{u.u} = c^2 \qquad (14)$$

The equivalence between (13) and (14) is obtained thanks to the introduction of the conventional notations $\mathbf{u} = (c\Gamma, u)$ an $\mathbf{p} = (E/c, p)$, with a Minkowskian signature applied to the scalar product $\mathbf{u.u} = c^2$. The expression of $\mathbf{p} = m\mathbf{u}$ (or its derivative with respect to τ: $\mathbf{F} = m\mathbf{a}$) reflects the vector version of the geometrical formulation. In another work, we shall show how one may also derive a scalar version apt to deduce kinematics (the space-time metrical structure) from dynamics and to reveal a certain unity inaccessible to the vector version.

Comments: Notice that the "relativistic mass" M which coincides with C and the Lorentz factor Γ which coincides with I_u [see (12) and (13)] are determined here by purely dynamical considerations without any relation to space-time physics. Dynamics is here autonomous, it contrasts with the variational and geometrical points of view, both usually founded on the spatiotemporal constraints imposed by the Lorentz transformations.

Historically M is introduced in Einstein's spatiotemporal physics through $p = mv/(1 - v^2/c^2)^{1/2} = Mv$, so that one obtains a relation directly comparable to the one corresponding to Newtonian physics: $p = mv$. Here, M and v result from the particular projection $G = E$ where M coincides with the constraint C (defined by $C = O^2E$), imposed to get two and only two conserved entities.
As to the Lorentz factor Γ, historically defined by the ratio between the infinitesimal variations of coordinate time and invariant time [$dt/d\tau = (1 + u^2/c^2)^{1/2}$], it results from the particular projection $G = F$ where Γ coincides with $I_u = (1 + u^2/c^2)^{1/2}$, intimately related to the non-additive composition of motion attached to u.

Let us finally note that in addition to $G = E$ and $G = F$ developed above, there is a third solution that corresponds to $F = E$, having the peculiarity of not depending on $G = x\, dF/dx - F$, thanks to which the points of view attached to v and u have been determined.

If one sets now: $F = E$ with $x = w$, then $p = dF/dx$ transforms into: $p = dE/dw$, so that its combination with (3) allows to express Einstein's dynamics in terms of w, obtaining thus:

$$p = mc \sinh(w/c) \qquad E = mc^2 \cosh(w/c) \qquad (15)$$

This point of view attached to w (called the rapidity), through $p = dE/dw$, corresponds to the one developed, in the second-half of the 20th century, by use of group theory [3-5] which provides to Huygens procedure, also based on $p = dE/dw$, a better rationality and a stronger foundation.

Conclusion

The three points of view, derived here in a unified way from the Leibnizian formulation, have been developed separately and progressively in the course of physical history. They also have been the subject of numerous investigations some of which are based on empirical considerations in direct connection to physical measurements [2]. Others have recourse to rational considerations based on well-identified mathematical frameworks, particularly, the calculus of variations, corresponding to the usual rationality of physics, and more recently modern geometry and group theory, considered by some [3-8] as serious candidates for higher forms of rationality.

In another work, we shall derive, as for the above procedure with its three points of view that appear on an equal footing (G = E, G = F and F = E), a different procedure that reveals a certain hierarchy, allowing to deduce, from the architectonical approach, a scalar version of the geometrical formulation, which in turn will lead naturally to two other points of view, formally identical to those based on the calculus of variations and group theory (usually postulated).

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