Dynamics: Intrinsic and Relational Presentation

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Abstract

Huygens (historical) procedure, which derived impulse from energy by imposing two basic physical requirements – relativity and conservation – is deepened and extended, in order to reveal all possible dynamics compatible with these two requirements. This extension is characterized by its intrinsic nature – viewpoint independent – not requiring any postulation of a specific motion parameter. It reveals, besides the conventional Newtonian and Einsteinian dynamical worlds, other ones, among which those recently proposed in the frameworks of Doubly Special Relativity and Finsler Geometry. Moreover, through an iterative procedure (inspired by what Leibniz called an architectonical approach), this formulation expresses Einsteinian dynamics in a relational way, with an infinity of possible parameters – including the velocity, the rapidity and the celerity, developed in the history of physics by use of analytical methods associated with the variational, group theoretical and geometrical formulations.

Keywords

Relativity principle, Conservation requirement, Huygens procedure, Analytical formulations, Leibnizian architectonical approach.

1. Introduction

As shown by Barbour [1], the relation between the conservation and relativity principles has been formulated by Huygens in his study of frontal elastic collisions (in 1+1 dimensions). His procedure, relative to the (parabolic) Newtonian world, using an additive motion parameter v, is rationally generalized, in order to characterize all dynamically admissible worlds (those which are compatible with the relativity and conservation requirements) and the associated points of view.

Historically, each one of the analytical methods (variational, group theoretical or geometrical) introduced a particular motion parameter (the velocity v, the rapidity w or the celerity u) providing a specific point of view on Einstein's dynamics [2-8].

The procedure of Huygens is recalled, and extended – according to Leibniz architectonical conceptualization, which looks for all dynamically admissible worlds and associated points of view. This method allows to cover the recently proposed dynamical frameworks [9-12] that generalize Einstein's dynamics. We show in particular that, when applied to Einstein's dynamics, this approach reveals an infinity of points of view – including those developed in the history of physics.

Remarks: Here, **t**he expressions "intrinsic dynamics" and "relational framework" mean respectively "dynamical structure independent of any point of view" and "framework relating together multiple points of view".

While an analytical formulation deals with a unique point of view [one motion parameter v related to energy E and impulse p by: $v = f(E) = g(p)$ or their inverse $E = h(v)$, $p = k(v)$], the "architectonical" approach deals simultaneously with an infinity of points of view [infinity of motion parameters: $v_\mu = f_\mu(E) = g_\mu(p)$ or their inverse $E = f^\mu (v_\mu)$ $p = g^\mu(v_\mu)$].

2. Elevation of Huygens procedure to the rank of a general principle

2.1 Generalization of Huygens procedure

J. Barbour explained through his equations (9.11)-(9.14) in Ref.[1], how Huygens, dealing with frontal elastic collisions, deduced impulse from energy by mobilizing the relativity requirement. He derived the expression of impulse $p = mv$ (initially called quantity of progression) from the so-called living force (or vis-viva) $L_f = mv^2$ (ancestor of kinetic energy: $T = ½ mv^2$). The main point is the use of the finite difference : $mv'^2 – mv^2$, associated with L_f in two reference frames (R_F and R_T): one fixed and one subject to a uniform additive translation ($v' = v + V$).

This procedure has been reexamined by Leibniz with his differential calculus, leading from the finite difference to the derivative: dT/dv.

The generalization of this procedure substitutes a general form: $E = f(v)$ to the particular one: $L_f = mv^2$ and an indeterminate composition law $v' = T(v, V) = v \, T \, V$ to the additive one: $v' = v + V$, such that: $T(v, 0) = v \, T \, 0 = v$.

The expression of impulse: $p = dE/dv$, associated with an additive composition law, is thus transformed into: $p = I(v) \, dE/dv$, associated with an indeterminate composition law, $I(v)$ being a function of v that will be determined at a later step.

The relativity principle asserts that, for an additive parameter v, if $E = f(v)$ corresponds to a conserved entity in a fixed frame of reference, its counterpart: $f(v') = f(v + V) = E_T$ in another frame of reference must also correspond to a conserved entity. Since any linear combination of two conserved entities leads to a conserved entity, then the particular combination associated with a finite difference: $[f(v + V) – f(v)]/A(V)$ where $A(V) = \Sigma_{k\geq 1} a_k V^k$, verifying $A(0) = 0$, corresponds to a conserved entity. When $V\rightarrow 0$, one gets: df/dv (up to a multiplicative constant that may be identified to unity without loss of generality, in virtue of the conservation properties).

The extension of the above procedure from a well-determinate (here additive) composition law to an indeterminate one amounts to replace in $[f(v + V) – f(v)]/A(V)$, $f(v + V)$ by $f(v\, T\, V)$, getting thus: $[f(v\, T\, V) – f(v)]/A(V)$. Its infinitesimal counterpart ($V\rightarrow 0$) is: $I(v)\, df(v)/dv$. Since $E = f(v)$, one may write: $p = I\, dE/dv$.

[When the composition law $T(v, V)$ coincides with the additive one: $v' = v + V$, the indeterminate function I coincides with unity: $I = 1$ recovering thus: $p = dE/dv$.]

We shall introduce – instead of the indeterminate point of view expressed through the couple (I, v) – an infinity of indeterminate points of view expressed through the couples (I_μ, v_μ). The expression of impulse: $p = I\, dE/dv$ becomes infinitely multiple: $p = I_\mu\, dE/dv_\mu$, the indeterminate entities I_μ being functions of v_μ.

They will be expressed (in Section 4) in terms of E and/or p : $v_\mu = f_\mu(E) = g_\mu(p)$.

This extension from one to infinity of yet indeterminate points of view allows to establish numerous relations including finite and infinitesimal ratios, deductible from the infinitely multiple operators expressed by: $I_1\, d/dv_1 = I_2\, d/dv_2 = I_3\, d/dv_3 = \dots$ reflecting the relational character of the architectonical framework. This has no counterpart in the analytical frameworks where each one deals with a unique point of view, postulated from the start.

2.2 Elevation of Huygens' procedure to the rank of a principle

We are to transform Huygens particular procedure [which derives – thanks to the relativity requirement – impulse from a given expression of energy] into an autonomous principle apt to derive both energy and impulse.

Since the physical problem of frontal elastic collisions requires two and only two conserved entities (depending on motion) and since the successive applications to energy E of the relativity requirement lead to new conserved entities (dE/dv, d^2E/dv^2, d^3E/dv^3...), a constraint has to be imposed on the second application: d^2E/dv^2, in order to keep only two (independent) conserved entities. Such a constraint avoids the arbitrary number of conserved entities obtained through successive "derivations".

For Huygens dynamics, this constraint corresponds to: $d^2E/dv^2 = m$, with an extension to the infinity of points of view: $d_\mu^2E/dv_\mu^2 = m$

with $p = d_\mu E/dv_\mu$ and the limit conditions: $(E, p, v_\mu) = (E_0, 0, 0)$.

The compact notations: d_μ/dv_μ and d_μ^2/dv_μ^2 correspond to:

$d_\mu/dv_\mu = I_\mu d/dv_\mu$ and $d_\mu^2/dv_\mu^2 = I_\mu\, d/dv_\mu\, [I_\mu\, d/dv_\mu] = I_\mu^2 d^2/dv_\mu^2 + [I_\mu\, dI_\mu/dv_\mu]\, d/dv_\mu$

This autonomous procedure that we shall call “Huygens-Leibniz dynamical relativity principle” applies to any dynamical world compatible to the relativity and conservation principles.

For the “hyperbolic” Einsteinian world, instead of $d_\mu{}^2E/dv_\mu{}^2 = m$, the constraint corresponds to a linear relation with respect to energy:

$$d_\mu{}^2E/dv_\mu{}^2 = E/c^2$$

The possible points of view will be determined, rationally and relationally, in Section 4 [the resulting points of view will include those expressed in the analytical realm, through the velocity, the celerity and the rapidity parameters, attached respectively to the variational, geometrical and group theoretical formulations.]

It appears that:

$$d_\mu/dv_\mu = I_\mu d/dv_\mu = (I_\mu dE/dv_\mu)d/dE = pd/dE = d_p/dE$$

This relation allows to eliminate the indeterminate couples: (I_μ, v_μ) – each corresponding to one point of view – in favor of the unique couple of conserved entities (E, p) which is independent of any point of view.

Thanks to this transformation, a sort of “**filtering procedure**”, the infinitely multiple structure: $d_\mu{}^2E/dv_\mu{}^2 = I_\mu{}^2d^2E/dv_\mu{}^2 + [I_\mu\, dI_\mu/dv_\mu]\, dE/dv_\mu = E/c^2$ reduces to: $pdp/dE = E/c^2$.

The integral form of this differential equation will actively contribute to the determination of the infinity of points of view associated with Einstein’s dynamical world, as shown in Section 4.

3. Admissible dynamics

The general dynamical structure compatible with the existence of two and only two (independent) conserved entities (E and p) needed to get a well-posed physical problem has to verify:

$$pdp/dE = F(E, p) = \lambda E + \gamma\, p + \eta$$

where λ, γ and η, are constant coefficients : any other combination violates the conservation requirement:

$$E_1 + E_2 = E_1' + E_2' \qquad p_1 + p_2 = p_1' + p_2'$$

Such a strong constraint, based on a criterion of conservation, which makes the initial indeterminacy [F(E, p)] sufficiently determined [$\lambda E + \gamma\, p + \eta$] is crucial for our purpose.

Remark: The passage from (1 + 1) to (3 + 1) dimensions can be obtained by replacing p and γ by three dimensional vectors: **p** and $\boldsymbol{\gamma}$ leading thus to:

$$\mathbf{p}\,.d\mathbf{p}/dE = \lambda E + \boldsymbol{\gamma}.\mathbf{p} + \eta$$

3.1. "Doubly Special Relativity"

This filtering procedure leads to a predictive framework and turns out to be general enough to encompass, in addition to the Newtonian and Einsteinian doubly particular dynamical worlds, that correspond respectively to: $(\lambda, \gamma, \eta) = (0, 0, m)$ and $(\lambda, \gamma, \eta) = (1/c^2, 0, 0)$, other recently developed ones, corresponding to the two particular cases: $(\lambda, \gamma, \eta) = (\lambda, 0, \eta)$ and $(\lambda, \gamma, \eta) = (\lambda, \gamma, 0)$.

The particular case: $(\lambda, \gamma, \eta) = (\lambda, 0, \eta)$, yielding:

$$p\, dp/dE = \lambda E + \eta \qquad (1)$$

turns out to correspond to the recently developed dynamical framework of: "Doubly Special Relativity". By using natural units ($c = 1$) and setting:

$$A = 1 - \lambda, \quad B = -2\eta \qquad (2)$$

the integration of (1) may be written in a particularly significant form:

$$E^2 - p^2 = f(E) = AE^2 + BE + C \qquad (3)$$

that generalizes Einstein's dynamics which corresponds to the particular case: $(A, B, C) = (0, 0, E_0^2)$.

Recalling that $f(E) = AE^2 + BE + C$, which verifies $f^{(3)}(E) = 0$, is compatible with the relativity and conservation requirements, the constant coefficients A, B and C will be determined in such a way that one adds to the usual initial condition: $p \to 0$, $E \to E_0 \leq E$ an upper limit condition: $p \to E_M$, $E \to E_M$ where the properties: $p \to \infty$, $E \to \infty$ that characterize Einstein's dynamics are assumed to be valid only locally when $E_M \to \infty$. Einstein's dynamics corresponds then to a local approach valid only for very small energies in comparison to some upper limit noted by E_M. In order to satisfy these additional requirements, one shows that the function f(E) should verify the two following constraints: $F(E_0) = E_0^2$ and $f(E_M) = 0$.

After having derived the set of the admissible solutions – not reproduced here – one observes that the Maguejo-Smolin dynamical model [9] and the Hinterleitner one [10], both belonging to the framework associated with "Doubly special relativity" expressed by:

$$[E^2 - p^2]/[1 - (E/E_M)]^2 = E_0^2/[1 - (E_0/E_M)]^2 \qquad (4)$$

and

$$[E^2 - p^2]/[1 - (E/E_M)^2] = E_0^2/[1 - (E_0/E_M)^2] \qquad (5)$$

belong to the above-mentioned set.

3.2 "Finsler geometry": $(\lambda, \gamma, \eta) = (\lambda, \gamma, 0)$

Similarly to the previous case, we show here that for: $(\lambda, \gamma, \eta) = (\lambda, \gamma, 0)$, one is led to:

$$p\, dp/dE = \lambda E + \gamma\, p \qquad (6)$$

This form turns out to be comparable to the recently developed dynamical framework corresponding to the research program relative to "Finsler Geometry".

After having set:

$$\alpha = [4\lambda + \gamma^2]^{1/2} \quad \text{and} \quad \alpha_\gamma^+ = (\alpha + \gamma)/2, \ \alpha_\gamma^- = (\alpha - \gamma)/2 \tag{7}$$

the integration of (6) yields:

$$[\alpha_\gamma^+ E - p]^{1+\gamma/\alpha} [\alpha_\gamma^- E + p]^{1-\gamma/\alpha} = A \qquad \text{(A: integration constant)} \tag{8}$$

If this case: $(\lambda, \gamma, \eta) = (\lambda, \gamma, 0)$, is submitted to the additional restriction $(\lambda, \gamma, 0) = (1, 0, 0)$, using natural units, and identifying the integration constant to squared mass, one is led to:

$$\alpha = 2 \quad \gamma = 0 \quad \text{and} \quad A = m^2 \tag{9}$$

The vanishing of γ renders the two different coefficients: α_γ^+ and α_γ^- indiscernible, for one gets: $\alpha_\gamma^+ = \alpha_0^+ = \alpha/2$ and $\alpha_\gamma^- = \alpha_0^- = \alpha/2$. Since $\alpha = 2$, they become equal to unity:

$$\alpha_\gamma^+ = \alpha_\gamma^- = 1 \tag{10}$$

recovering thus Einstein's dynamics:

$$[E - p]\,[E + p] = E^2 - p^2 = m^2$$

It is possible to cast Eq. (8) into a more symmetrical but equivalent form, provided one redefines p, by considering a linear combination: $P = ap + bE$ (compatible with the conservation requirement), passing thus from the couple of conserved entities (E, p) to the other couple (E, P). Indeed, on setting:

$$P = p - (\gamma/2)\, E \tag{11}$$

which corresponds to a particular linear combination (with $a = 1$ and $b = -\gamma/2$) Eq.(8) takes a simpler form:

$$[E - P]^{1+\gamma/\alpha} [E + P]^{1-\gamma/\alpha} = A \tag{12}$$

where one recognizes the dynamical structure developed by use of Finsler geometry [11, 12].

For the particular case : $\gamma = 0$, and $A = m^2$, one recovers again Einstein's dynamics.

4. Determination of an infinity of points of view (for Einstein's world)

Einstein's dynamical world corresponds to the doubly particular case: $(\lambda, \gamma, \eta) = (1/c^2, 0, 0)$, with:

$$p\,dp/dE = E/c^2 \qquad \text{or its integral form:} \qquad E = (E_0^2 + c^2p^2)^{1/2} \tag{13}$$

The usual expression: $E = mc^2 (1 + p^2/m^2c^2)^{1/2}$, obtained with: $E_0 = mc^2$, plays a major role for the determination of the points of view derived below.

4.1 Viewpoint dependent structure

We are to establish a self-organization procedure leading to a relational structure involving an infinity of points of view.

Applied to Einstein's dynamics the under-determinate system of differential equations shown in 2-2 corresponds to

$$E/c^2 = d_\mu{}^2E/dv_\mu{}^2 = I_\mu{}^2d^2E/dv_\mu{}^2 + [I_\mu\, dI_\mu/dv_\mu]\, dE/dv_\mu \quad \text{with } p = d_\mu E/dv_\mu = I_\mu dE/dv_\mu \qquad (14)$$

combining $p = d_\mu E/dv_\mu$, with: $E/c^2 = d_\mu{}^2E/dv_\mu{}^2$ leads to: $E/c^2 = d_\mu p/dv_\mu = I_\mu\, dp/dv_\mu$, or

$$(1/m)\, dp/dv_\mu = (E/mc^2) / I_\mu = (1 + p^2/m^2c^2)^{1/2} / I_\mu \qquad (15, a)$$

where we have used : $E/mc^2 = (1 + p^2/m^2c^2)^{1/2}$, derived from $pdp/dE = E/c^2$, corresponding to the **filtering procedure** (derived above).

Setting $I_d = (1 + p^2/m^2c^2)^{1/2}$ (the index d standing for decoupling) in Eqs. (15, a) brings into action a **decoupling procedure**:

$$E = mc^2I_d \quad \text{and} \quad dp = mdv_d$$

This decoupled solution: $I_d = E/mc^2 = Y$, combined with the one corresponding to the additive point of view ($I_\mu = I_a = 1 = Y^0$, a for additive), suggests looking for solutions of the form: Y^n (a multiple scale law, corresponding to a geometric progression). This suggestion, resulting from the decoupled and additive particular solutions, is strengthened by the property, resulting from the filtering procedure:

$Y = I_d = (1 + p^2/m^2c^2)^{1/2} \geq 1$, for any p that verifies: $\ldots \geq Y^2 \geq Y^1 \geq Y^0 \geq Y^{-1} \geq Y^{-2} \geq \ldots$

Thus, the combination of these results derived from the above considerations turns out to be in full harmony with Leibniz's conceptualization relative to his "theoretical microscope", apt to reveal a world (here Einstein's one) under an infinity of well-ordered points of view corresponding to: $\ldots \geq I_{d-1} \geq I_d \geq I_{d+1} \geq I_{d+2} \geq I_{d+3} \geq \ldots$ *Such an ordering may be specified (or quantified) thanks to the global harmony that one may deduce from* $I_d = Y \geq 1$*, that leads to the unlimited number of inequalities:*

$$\ldots \geq I_d{}^2 \geq I_d{}^1 \geq I_d{}^0 \geq I_d{}^{-1} \geq I_d{}^{-2} \geq \ldots$$

valid for any value of the impulse p and/or the motion parameters v_μ.

An infinity of well-determined points of view is so obtained by introducing: $I_\mu = Y^{2-\mu}$ into (15, a), getting thus: $U_\mu = Y^{\mu-1}$ or more explicitly:

$$U_\mu = (1/m)\, dp/dv_\mu = (1 + p^2/m^2c^2)^{(\mu-1)/2} = Y^{\mu-1} \qquad (15, b)$$

from which results an infinity of integral expressions:

$$v_\mu = (1/m) \int dp/ Y^{\mu-1} = (1/m) \int dp/(1 + p^2/m^2c^2)^{(\mu-1)/2} = g_\mu(p) \qquad (15, c)$$

Each value of μ corresponds to a specific point of view. On account of the state of rest: $p = 0$, $v_\mu = 0 \quad \forall\mu$, one is led to an infinity of well-determined motion parameters v_μ, expressed in terms of impulse p.

It is possible to derive them in terms of energy E, using the above derived expressions: $Y = E/mc^2 = (1 + p^2/m^2c^2)^{1/2}$:

$$v_\mu = (1/m) \int Y^{1-\mu} (dp/dE)dE = (\pm 1/mc) \int [Y^{2-\mu} /(Y^2 - 1)^{1/2}]dE = f_\mu(E) \qquad (15, d)$$

Among the infinity of points of view, three of them correspond to those developed in the history of physics, associated with the concepts of velocity v, celerity u and rapidity w (usually introduced with the variational, geometrical and group theoretical formulations).

They correspond respectively to: $\mu = 4$, $\mu = 1$ and $\mu = 2$. Some calculations and formal manipulations lead to the well-known expressions:

$$p = mv/(1 - v^2/c^2)^{1/2} \qquad E = mc^2/(1 - v^2/c^2)^{1/2} \qquad \text{for } v = v_4 \qquad (16, a)$$

$$p = mu \qquad E = mc^2(1 + u^2/c^2)^{1/2} \qquad \text{for } u = v_1 \qquad (16, b)$$

$$p = mc \sinh(w/c) \qquad E = mc^2 \cosh(w/c) \qquad \text{for } w = v_2 \qquad (16, c)$$

each couple of equations (p, E) expressing a specific point of view.

4.2 The relational character of the present approach

In order to underline the relational character of the present Leibnizian architectonical approach, let us note that Eq.(15, a) leads to the geometric progression of functions: $r = U_{\mu+1}/U_\mu$ where the ratio r corresponds to: $r = Y = E/mc^2 = (1 + p^2/m^2c^2)^{1/2}$. This clearly shows how the passage from one point of view to an adjacent one can be derived iteratively (ad infinitum).

More generally, one has: $U_\eta/U_\mu = r^{\eta-\mu}$ from which one deduces:

$$v_\mu = \int Y^{\eta-\mu}\, dv_\eta = f_\mu^{\eta}(v_\eta) \qquad (17)$$

This formula allows especially to derive the infinity of points of view in terms of anyone of the three points of view developed in the history of science, expressed by the velocity, the celerity or the rapidity.

Remarks: *The extended Huygens procedure, deriving impulse from energy, is based on a finite difference which is only a particular case of the most general admissible form: a linear combination (up to an additive constant). Another study explicitly exposes this extension. Moreover, instead of dealing only with the intrinsic structures developed above through (1)-(12), it is also possible to derive the corresponding points of view.*

We finally show, in another work, how to deduce the three formal axiomatic structures, corresponding to the variational, geometrical and group theoretical formulations, conventionally used to derive the three usual points of view given by (16, a)-(16, c), showing, especially how the Lagrangian (considered as magical by Penrose) arises from this architectonical formulation (thanks to a change of variable facilitating the integration of the second-order differential equation corresponding to the velocity) and how the space-time metrical structure emerges from one of the points of view, through the notion of duality. This last point has been succinctly explained in the third part of a synthetic paper [13], that deals with the connection between the present architectonical approach and the energy (or scalar) formulation, known as the "Principle of virtual power", particularly adapted to the study of electro-magneto-mechanical interactions in continuum mechanics with singular surfaces and interfaces [14,15].

Epistemology: *Let us emphasize that the present work owes much to formal and physical articles [1-6, 9-12], but also to epistemological and conceptual ones devoted to Leibniz philosophy of nature. While Leibniz investigations were not fully understood and appreciated by his contemporaries, his foundational thought was taken seriously in the 20th century by scholars such as A. N. Whitehead and K. Gödel who firmly adhere to what Hans Reichenbach wrote in his book ("The philosophy of space and time"):*

"*It is the more remarkable that Leibniz, this genuine philosopher, was able to understand the nature of scientific knowledge to such an extent that, two hundred years later, a new development of physics and an analysis of its philosophical foundations confirmed his views*".

The present Leibnizian architectonical formulation that accounts simultaneously for a variety of possible worlds and points of view differs radically from the usual "analytical" formulations, (which are, by construction, limited to a unique point of view a priori specified and postulated from the start). In the same way as the Lagrange "analytical" approach is also designated by: "Lagrange-Hamilton formulation", we propose to designate the Leibniz "architectonical" approach by: "Huygens-Leibniz formulation".

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