Infinite dimensional model of a double flexible-link manipulator: the port-Hamiltonian approach

Andrea Mattioni[☆a], Yongxin Wu[a], Yann Le Gorrec[a]

[a]*FEMTO-ST, Univ. Bourgogne Franche-Comté , CNRS, Besançon, France, 24 rue Savary, F-25000 Besançon, France*

Abstract

This paper proposes a modular and control oriented model of a double flexible-link manipulator stems from the modelling of a spatial flexible robot. The model consists of the power preserving interconnection between two infinite dimensional systems describing the beam's motion and deformation with a finite dimensional nonlinear system describing the dynamics of the actuated rotating joints. To derive the model, Timoshenko's assumptions are made for the flexible beams. Using Hamilton's principle, the dynamic equations of the system are derived and then written in the Port-Hamiltonian (PH) framework through a proper choice of the state variables. These so called energy variables allow to write the total energy as a quadratic form with respect to a state dependent energy matrix. The resulting model is shown to be a passive system, a convenient property for control design purposes.

Keywords: Flexible arms, Flexible Robotics, Port-Hamiltonian systems, Distributed parameter systems, Boundary control systems, Discretization, Finite dimensional approximation

☆CONTACT: Andrea Mattioni, Email: andrea.mattioni@femto-st.fr.
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1. Introduction

Starting from the early 80', the modelling of flexible robots has always been an important research topic, due to the need of high precision control for lightweight robots in industrial and spatial applications. The most common way of deriving the equations that describe the dynamics of a flexible manipulator is to use the Lagrangian principle [1]. Once flexibility is considered, the kinetic and potential energies depend on space dependent variables. To use the Lagrangian principle, the Lagrangian is approximated by the "Constant Mode Shapes Technique (CMST)", driving to a set of Ordinary Differential Equations (ODEs). In the literature, the same kind of procedure has been exploited for robots with flexible-joints [2], flexible-links flexible-joint [3], and flexible manipulators with also prismatic joints [4]. The Euler-Lagrangian modelling procedure, for general Flexible mechanism has been precisely detailed in the book of Junkins [5]. In this book, the Euler-Lagrangian procedure is used to derive infinite dimensional models of flexible mechanisms, including the double links manipulator with Euler-Bernoulli's assumptions. These equations have been written using the operator formalism in a compact form in [6] such to be able to design a control law and to analyse the asymptotic behaviour of solutions. A similar model has been used in [7] and [8], where respectively an adaptive and a special structure PD controller have been designed, and the closed loop behaviour analysed. In [9] are provided the required functional analysis tools for the study of well-posedness and stability of infinite dimensional systems, as well as some control strategies for flexible robots expressed in this framework. Among the literature, particular attention is devoted to the modelling and stabilization problem for single flexible link manipulators [10–12]. In this work we decided to use the PH framework to model the double flexible manipulator such to explicit the passivity properties of the systems, useful for control design purposes.

In the last decades an approach based on the extension of the Hamiltonian formulation to open distributed parameter systems has been developed for modelling and control. It has been initially introduced for finite dimensional nonlinear systems described by ODEs [13, 14], and then generalized to infinite dimensional systems described by partial differential equations (PDEs) in more recent years [15–17]. This provides a standardized framework for control design, especially suited for energy based control strategies both for finite dimensional [18, 19] and infinite dimensional systems [20, 21]. The PH modelling allows to express a system as the composition of different elements that exchange energy in a power preserving way. To this extent, the infinite dimensional model of a clamped-free flexible beam with Timoshenko's assumptions has been derived in [22]. In a similar manner, also the single flexible link manipulator model has been expressed as an infinite dimensional PH system [23]. A juncture element between the previous literature on flexible robot modelling and the PH framework could be found in [24]. Instead of the infinite dimensional boundary valued problem proposed in [23], the authors of [24] propose a finite dimensional model described by a set of nonlinear ODEs derived with Lagrangian equations from the discretized energy of the manipulator.

Figure 1: [Credits: www.nasa.org] Canadarm2 robotic arm attached to the International Space Station.

The aim of this paper is to use the PH formalism to derive a control oriented and physically meaningful model of the double flexible-links manipulator for spatial applications, as the Canadarm2 mounted on the International Space Station (ISS) showed in Figure (1). This robotic arm has seven actuating motors: three located in the first joint, one located in the second joint, and other three located in the end-effector. In this manuscript we only consider one motor in the first joint, such to assume that the motion of the overall arm remains in a plane, and we do not consider the three motors at the end effector since they do not affect the robot's dynamics. Throughout the rest of the paper, since the mass of the manipulator is negligible compared to the ISS, we make the assumption of neglecting the base dynamic, i.e. we consider the manipulator as connected to the ground framework. The PH framework allows to explicit the passivity property of the system that can be exploited for control law design. After the model derivation, a PH structure preserving discretization procedure (based on the mixed finite elements method [25]) is used to derive a finite dimensional version of the proposed infinite dimensional model.

2. Infinite dimensional modelling of the double flexible-link manipulator

In this paper, the double flexible-link manipulator is considered as depicted in Figure 2. The system is composed by two flexible links connected with actuated revolute joints, i.e. motors. The motor fixed to the ISS has only the shaft moving, while the other has both the stator and the shaft participating to the motion.

F_0 represents the reference frame connected to the stator of the ISS, F_i, $i = \{1, 2\}$ are the frameworks connected with the shaft respectively of the first and the second motor, and $\theta_i(t) \in \mathbb{R}$ represent their rotation with respect to the F_0 frame. With $z_1 \in [0, L_1]$ and $z_2 \in [0, L_2]$ we identify the spatial coordinates along the beams, belonging respectively to F_1 and F_2. The deflection of the two beams with respect to their own axis z_1 and z_2, has been denoted with

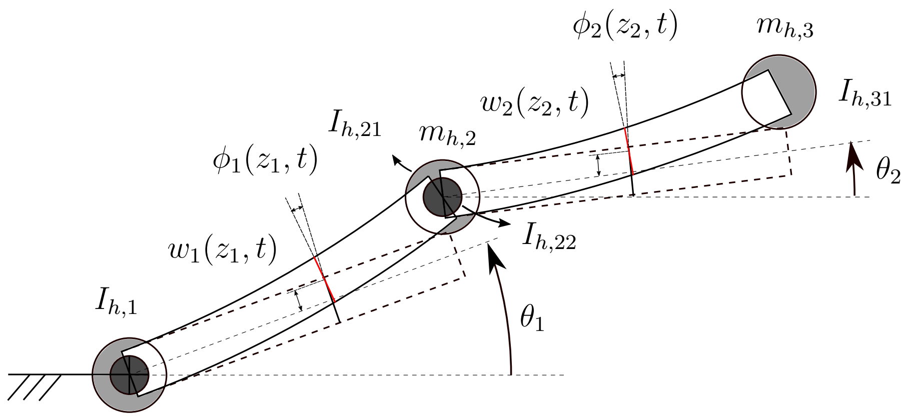


Figure 2: Flexible two links manipulator.

$w_1(t,z_1) \in L_2(0,L_1)$ and $w_2(t,z_2) \in L_2(0,L_2)$[1], while with $\phi_1(t,z_1) \in L_2(0,L_1)$ and $\phi_2(t,z_2) \in L_2(0,L_2)$ have been defined the relative (with respect to their own frame) rotation of the beam cross section. The beams are supposed to have a constant rectangular cross section width $L_{w,i}$, thickness $L_{t,i}$ and area $A_{s,i} = L_{w,1}L_{t,i}$, $i = \{1,2\}$. All the physical parameters of the system are positive real, and their meaning are given as follows: $I_{h,1}, I_{h,21}, I_{h,22}, I_{h,3}, m_{h,2}, m_{h,3}$ represent respectively the rotary inertia of the shaft of the first motor, of the stator of the second motor, of the shaft of the second motor and of the payload; $m_{h,2}, m_{h,3}$ represent respectively the mass of the second motor and of the tip payload at the end of the second link; $E_i, I_i \quad i = \{1,2\}$ are respectively the Young's modulus and the moment of inertia of the cross section. The inertia of the cross section of a beam with a rectangular section is defined as $I_i = \frac{L_{w,i}^3 L_{t,i}}{12}$; $\rho_1, \rho_2, I_{\rho 1}, I_{\rho 2}$ are respectively the mass per unit length and the mass moment of inertia of the cross section of both beams. The mass moment of inertia of the cross section is defined as $I_{\rho i} = \frac{I_i \rho_i}{A_{s,i}}$; K_1, K_2 are defined as $K_i = k_i G_i A_i \quad i = \{1,2\}$, where k_i is a constant depending on the shape of the cross section, G_i is the shear

[1]For the sake of simplicity, with w' has been denoted the spatial derivative of w, *i.e.* $w' = \frac{\partial w}{\partial z}$ and with $\dot{w}$ the time derivative of w *i.e.* $\dot{w} = \frac{\partial w}{\partial t}$.

modulus and A_i is the cross sectional area.

The model is derived through the Hamilton's principle. Since the system under study is a pure mechanical system, the energy will be composed by a kinetic and an elastic part. In this paper, the Timoshenko beam's assumptions are used to define the kinetic and potential energy related to the flexible links. The following assumptions are used for the model derivation for $i = \{1, 2\}$.

Assumption 1. *The following hold throughout the remainder of the paper:*

1. $w_1\dot{\theta}_1 \approx 0$ *and* $w_2\dot{\theta}_2 \approx 0$.
2. *The* z_i *axes are always perpendicular to the beam's cross sections, and in particular they correspond to their principal axes of rotation.*
3. *The* z_i *axes are always perpendicular to the plane of the manipulator's motion.*

2.1. Hamilton's principle

Taking into account Assumption 1, the Kinetic energy of the system writes [5, 6]:

$$
\begin{aligned}
E_k = \frac{1}{2}\int_0^{L_1} & \left[\rho_1 \left(z_1\dot{\theta}_1 + \dot{w}_1\right)^2 + I_{\rho 1}\left(\dot{\theta}_1 + \dot{\phi}_1\right)^2\right] dz_1 + \frac{1}{2} I_{h,1}\dot{\theta}_1^2 \\
& + \frac{1}{2} I_{h,21}\left(\dot{\theta}_1 + \dot{\phi}(L_1)\right)^2 + \frac{1}{2}\left(m_{h,2} + m_2 + m_{h,3}\right)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)^2 \\
& + \frac{1}{2}\int_0^{L_2}\left[\rho_2\left(z_2\dot{\theta}_2 + \dot{w}_2\right)^2 + I_{\rho_2}\left(\dot{\theta}_2 + \dot{\phi}_2\right)^2\right] dz_2 + \frac{1}{2} I_{h,22}\dot{\theta}_2^{\ 2} \\
\frac{1}{2} m_{h,3} & \left(L_2\dot{\theta}_2 + \dot{w}_2(L_2)\right)^2 + m_{h,3}\cos\left(\theta_2 - \theta_1\right)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1,t)\right)\left(L_2\dot{\theta}_2\right. \\
& \left. + \dot{w}_2(L_2,t)\right) + \left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\cos\left(\theta_2 - \theta_1\right)\int_0^{L_2}\left[\rho_2\left(z_2\dot{\theta}_2 + \dot{w}_2\right)\right] dz_2
\end{aligned}
$$

where $m_2 = \displaystyle\int_0^{L_2} \rho_2 dz_2$.

The potential energy follows directly from the Timoshenko's assumption on the flexible beams,

$$E_p = \frac{1}{2}\int_0^{L_1} \left[K_1\left(\frac{\partial w_1}{\partial z_1} - \phi_1\right)^2 + EI_1\left(\frac{\partial \phi_1}{\partial z_1}\right)^2\right] dz_1 + \frac{1}{2}\int_0^{L_2} \left[K_2\left(\frac{\partial w_2}{\partial z_2} - \phi_2\right)^2 + EI_2\left(\frac{\partial \phi_2}{\partial z_2}\right)^2\right] dz_2.$$

Since we assume to model a manipulator for spatial applications, gravity is not taken into account. The Hamilton's principle states

$$\int_{t_1}^{t_2} (\delta\mathcal{L} + \delta W_{nc})\, dt = 0,$$

where $\mathcal{L} = E_k - E_p$ is the Lagrangian, $\delta\mathcal{L}$ is the variational derivative of the Lagrangian, δW_{nc} represents the virtual work of the non-conservative forces, while $t_1, t_2 \in \mathbb{R}^+,\ t_2 > t_1$ represents two successive instants of time. The non-conservative forces correspond in our case of study to the torques provided by the two motors $\tau_1, \tau_2 \in \mathbb{R}$ and the friction present in the mechanism. In the following we do the conservative assumption of not considering internal friction in the beam, but we assume that this friction operates only at the boundaries. As will be shown in Section 2.3, this leads to damped boundary dynamic equations. Hence, the virtual works' variational derivative writes

$$\begin{aligned}\delta W_{nc} = {} & \gamma_1(\dot\theta_1 + \tau_1)\delta\theta_1 + \gamma_2\dot w_1(L_1)\delta w_1(L_1) + \gamma_3\dot\phi_1(L_1)\delta\phi_1(L_1) \\ & + \gamma_4(\dot\theta_2 - \dot\theta_1 - \dot\phi_1(L_1) + \tau_2)\delta(\theta_2 - \theta_1 - \phi_1(L_1)) \\ & + \gamma_5\dot w_2(L_2)\delta w_2(L_2) + \gamma_6\dot\phi_2(L_2)\delta\phi_2(L_2)\end{aligned}$$

For the sake of brevity, it is not showed the tedious procedure involving integration by parts that allows the derivation of the following equations from Hamilton's principle.

2.1.1. ODEs associated with the motion of the actuated rotating joints

The ordinary differential equation governing the dynamic of the Shaft of the first motor θ_1 writes

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(I_{h,1}\dot{\theta}_1\right)+\frac{\partial}{\partial t}\int_0^{L_1}\left[\rho_1 z_1\left(z_1\dot{\theta}_1+\dot{w}_1\right)+I_{\rho_1}\left(\dot{\theta}_1+\dot{\phi}_1\right)\right]dz_1\\
&+\frac{d}{dt}\left(L_1\left(m_{h,2}+m_2+m_3\right)\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\right)+\frac{\partial}{\partial t}\left(I_{h,21}\left(\dot{\theta}_1+\dot{\phi}_1(L_1)\right)\right)\\
&+\frac{\partial}{\partial t}L_1\cos(\theta_2-\theta_1)\left(\int_0^{L_2}\left[\rho_2\left(z_2\dot{\theta}_2+\dot{w}_2\right)\right]dz_2+m_{h,3}\left(L_2\dot{\theta}_2+\dot{w}_2(L_2,t)\right)\right)=\\
&\qquad+\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\left(\int_0^{L_2}\left[\rho\left(z_2\dot{\theta}_2+\dot{w}_2\right)\right]dz_2\right.\\
&\qquad\left.+m_{h,3}\left(L_2\dot{\theta}_2+\dot{w}_2(L_2,t)\right)\right)+\tau_1-\tau_2+\gamma_4(\dot{\theta}_2-\dot{\theta}_1-\dot{\phi}_1(L_1)),
\end{aligned}\tag{1}$$

while the ordinary differential equation governing the dynamic of the Shaft of the second motor θ_2 is given as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(I_{h,21}\dot{\theta}_2\right)+\frac{\partial}{\partial t}\int_0^{L_2}\left[\rho_2 z_2\left(z_2\dot{\theta}_2+\dot{w}_2\right)+I_{\rho_2}\left(\dot{\theta}_2+\dot{\phi}_2\right)\right.\\
&+\left.\frac{\partial}{\partial t}\left(\rho_2 z_2\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\cos(\theta_2-\theta_1)\right)\right]dz_2+\frac{d}{dt}\left(I_{h,3}\left(L_1\dot{\theta}_1+\dot{\phi}_2(L_1,t)\right)\right)\\
&\quad\frac{d}{dt}\left(L_2 m_{h,3}\left(\left(L_2\dot{\theta}_2+\dot{w}_2(L_2,t)\right)+\left(L_1\dot{\theta}_1+\dot{w}_1(L_1,t)\right)\cos\left(\theta_1-\theta_2\right)\right)\right)\\
&\qquad=+\tau_2-\int_0^{L_2}\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\rho_2\left(z_2\dot{\theta}_2+\dot{w}_2\right)dz_2\\
&\qquad\qquad-\gamma_4(\dot{\theta}_2-\dot{\theta}_1-\dot{\phi}_1(L_1))
\end{aligned}\tag{2}$$

2.1.2. PDEs describing the first flexible beam

From Hamilton's principle, one can get the set of partial differential equations describing the absolute movement with respect to F_0 and the elastic deformations of the first flexible beam as follows:

$$\begin{cases}
\frac{\partial}{\partial t}\left(\rho_1\left(z_1\dot{\theta}_1+\dot{w}_1\right)\right)=\frac{\partial}{\partial z_1}\left(K_1\left(\frac{\partial w_1}{\partial z_1}-\phi_1\right)\right)\\
\\
\frac{\partial}{\partial t}\left(I_{\rho_1}\left(\dot{\theta}_1+\dot{\phi}_1\right)\right)=\frac{\partial}{\partial z_1}\left(EI_1\frac{\partial\phi_1}{\partial z_1}\right)+K_1\left(\frac{\partial w_1}{\partial z_1}-\phi_1\right).
\end{cases}\tag{3}$$

The above two PDEs describe respectively the translational and the rotational dynamic of every cross section of the first beam.

The whole deformation has been referred to the $z_1 = 0$ part of the beam *i.e.* $\delta w_1(0) = \delta\phi_1(0) = 0$. The resulting boundary deformation $\delta w_1(L_1) \neq 0$, $\delta\phi_1(L_1) \neq 0$ drives, through the Hamilton's principle, to two ordinary differential equations describing respectively the translational dynamic of the $z_1 = L_1$ part of the beam, and the rotational dynamic of the stator of the second motor. Hence, the boundary conditions write as:

$$\begin{aligned} K_1\left(\frac{\partial w_1}{\partial z_1}(L_1) - \phi(L_1)\right) + \frac{\partial}{\partial t}\int_0^{L_2}\left[\cos(\theta_2-\theta_1)\rho_2\left(z_2\dot{\theta}_2 + \dot{w}_2\right)\right]dz_2 \\ + \frac{d}{dt}\left(m_{h,3}\left(\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1,t)\right) + \cos\left(\theta_2-\theta_1\right)\left(L_2\dot{\theta}_2 + \dot{w}_2(L_2,t)\right)\right)\right) \\ + \frac{\partial}{\partial t}\left((m_{h,2}+m_2)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\right) + \gamma_2\dot{w}_1(L_1) = 0, \end{aligned} \quad (4)$$

$$EI_1\frac{\partial\phi_1}{\partial z_1}(L_1) + \tau_2 + \frac{\partial}{\partial t}I_{h,21}\left(\dot{\theta}_1 + \dot{\phi}_1(L_1,t)\right) + \gamma_3\dot{\phi}_1 = 0 \; \delta w_1(0) = 0,\; \delta\phi_1(0) = 0. \quad (5)$$

2.1.3. PDEs describing the second flexible beam

The set of PDEs describing the absolute movement with respect to frame F_2 and the elastic deformations of the second flexible beam writes as follows:

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho_2\left(z_2\dot{\theta}_1 + \dot{w}_1\right) + \rho_2\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\cos\left(\theta_2-\theta_1\right)\right) = \frac{\partial}{\partial z_2}\left(K_2\left(\frac{\partial w_2}{\partial z_2} - \phi_2\right)\right) \\ \frac{\partial}{\partial t}\left(I_{\rho_2}\left(\dot{\theta}_2 + \dot{\phi}_1\right)\right) = \frac{\partial}{\partial z_2}\left(EI_2\frac{\partial\phi_2}{\partial z_2}\right) + K_2\left(\frac{\partial w_2}{\partial z_2} - \phi_2\right) \end{cases} \quad (6)$$

with boundary conditions

$$\begin{aligned} K_2\left(\frac{\partial w_2}{\partial z_2}(L_2,t) - \phi(L_2,t)\right) + \frac{d}{dt}\left(m_{h,3}\left(\left(L_2\dot{\theta}_2 + \dot{w}_2(L_2,t)\right)\right.\right. \\ \left.\left. + \cos\left(\theta_2-\theta_1\right)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1,t)\right)\right)\right) + \gamma_5\dot{w}_2(L_2) = 0 \end{aligned} \quad (7)$$

$$EI_2\frac{\partial\phi_2}{\partial z_2}(L_2) + \frac{\partial}{\partial t}I_{h,31}\left(\dot{\theta}_2 + \dot{\phi}_2(L_2,t)\right) + \gamma_6\dot{\phi}_2(L_2) = 0,\; \delta w_1(0) = 0,\; \delta\phi_1(0) = 0. \quad (8)$$

2.2. PH formulation of the flexible beams

The application of Hamilton's principle returns two sets of PDEs describing the cross section translational and rotational dynamic of both beams, together with six ODEs describing the evolution of the boundary conditions. The system can be naturally split into three main parts: the first flexible beam (first set of PDEs), the second flexible beam (second set of PDEs) and concentrated inertia dynamics (set of ODEs). The energy variables of the infinite dimensional systems (3) (6), are defined as

$$
\begin{array}{ll}
\varepsilon_{1,t} = \frac{\partial w_1}{\partial z_1} - \phi_1, & \varepsilon_{2,t} = \frac{\partial w_2}{\partial z_2} - \phi_2, \\
\varepsilon_{1,r} = \frac{\partial \phi_1}{\partial z}, & \varepsilon_{2,r} = \frac{\partial \phi_2}{\partial z_2}, \\
p_{1,t} = \rho_1 \left(\frac{\partial w_1}{\partial t} + z_1 \dot{\theta}_1 \right), & \begin{aligned} p_{2,t} = \quad & \rho_2 \Big(\left(\tfrac{\partial w_2}{\partial t} + z_2 \dot{\theta}_2 \right) + \\ & \left(L_1 \dot{\theta}_1 + \dot{w}_1(L_1) \right) \cos(\theta_2 - \theta_1) \Big), \end{aligned} \\
p_{1,r} = I_\rho \left(\frac{\partial \phi}{\partial t} + \dot{\theta}_1 \right) & p_{2,r} = I_{\rho_2} \left(\frac{\partial \phi}{\partial t} + \dot{\theta}_2 \right)
\end{array}
\tag{9}
$$

where, $\varepsilon_{1,t}, \varepsilon_{1,r}, p_{1,t}, p_{1,r} \in \mathcal{L}_2(0, L_1)$ and $\varepsilon_{2,t}, \varepsilon_{2,r}, p_{1,t}, p_{2,r} \in \mathcal{L}_2(0, L_2)$. The state vectors of the two beam systems are defined as $x_1 = [p_{1,t}\ p_{1,r}\ \varepsilon_{1,t}\ \varepsilon_{1,r}]^T$, with the state space $X_1 = \mathcal{L}_2([0, L_1], \mathbb{R}^4)$ and $x_2 = [p_{2,t}\ p_{2,r}\ \varepsilon_{2,t}\ \varepsilon_{2,r}]^T$, with the state space $X_2 = \mathcal{L}_2([0, L_2], \mathbb{R}^4)$. The inner product in both state spaces is defined as the natural inner product in the $\mathcal{L}_2$ space with a slight modification: $\langle x_{ai}, x_{bi} \rangle_{Xi} = \langle x_{ai}, \mathcal{H}_i x_{bi} \rangle_{\mathcal{L}_2}$. The modification is such that half the square norm of $x_i \in X_i$ defined through the previously defined inner product, corresponds to the energy of the system. Hence, the Hamiltonian of both infinite dimensional systems writes,

$$
H_i = \frac{1}{2} ||x_i||^2 = \frac{1}{2} \langle x_i, x_i \rangle_{Xi} = \frac{1}{2} \int_0^{L_i} x_i^T \mathcal{H}_i x_i \, dz_i \quad \text{with} \quad i = \{1, 2\}. \tag{10}
$$

with $\mathcal{H}_i = \operatorname{diag}\left[\frac{1}{\rho_i}, \frac{1}{I_{\rho_i}}, K_i, EI_i \right]$, and the PDEs (3) (6) can be written using the same PH form:

$$
\begin{aligned}
\dot{x}_i &= \tfrac{\partial}{\partial z_i} P_1(\mathcal{H}_i x_i) + P_0(\mathcal{H}_i x_i), \\
\mathcal{B}_i(\mathcal{H}_i x_i) &= u_{if}, \quad i = \{1, 2\}
\end{aligned}
\tag{11}
$$

where $u_{if} \in \mathbb{R}^4 \quad i = \{1, 2\}$ represent the control functions, and matrices P_0 and P_1 are defined as:

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each system has four state variables, thus eight boundary variables results for each of them. According to [16], these boundary variables can be defined as a linear combination of the restriction of the co-energy variables computed at the boundaries according to a proper choice of the unitary matrix U:

$$\begin{bmatrix} f_{\partial i} \\ e_{\partial i} \end{bmatrix} = U \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} \mathcal{H}_i x(0) \\ \mathcal{H}_i x(L_i) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\rho_i} p_{i,t}(0) \\ -\frac{1}{I_{\rho,i}} p_{i,r}(0) \\ \frac{1}{\rho_i} p_{i,t}(L_i) \\ \frac{1}{I_{\rho,i}} p_r(L_i) \\ K_i \varepsilon_{i,t}(0) \\ EI_i \varepsilon_{i,r}(0) \\ K_i \varepsilon_{i,t}(L_i) \\ EI_i \varepsilon_{i,r}(L_i) \end{bmatrix}, \quad i = \{1, 2\}.$$

Four of these boundary variables will be imposed to define the boundary conditions of the set of PDEs through the boundary control input. The remaining four boundary variables form the conjugated outputs (i.e. the input-output product results in a power). The natural choice is selecting the velocities as inputs at both sides for both beams. Once the initial positions are known, fixing the velocities at both sides of both beams is the same as fixing their positions. With abuse of terminology, a beam on which the angular and translational positions are imposed at both sides by external factors, is called a "clamped-clamped beam". Hence, for both beams, we define for $i = \{1, 2\}$ the input and the output variables as:

$$\mathcal{B}_i x_i = W_i \begin{bmatrix} f_{\partial i} \\ e_{\partial i} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\rho_i} p_{t,i}(0) \\ -\frac{1}{I_{\rho_i}} p_{r,i}(0) \\ \frac{1}{\rho_i} p_{t,i}(L_i) \\ \frac{1}{I_{\rho_i}} p_{r,i}(L_i) \end{bmatrix}, \quad y_{if} = \tilde{W}_1 \begin{bmatrix} f_{\partial i} \\ e_{\partial i} \end{bmatrix} = \begin{bmatrix} K_i \varepsilon_{t,i}(0) \\ EI_i \varepsilon_{r,i}(0) \\ K_i \varepsilon_{t,1}(L_i) \\ EI_i \varepsilon_{r,i}(L_i) \end{bmatrix}. \quad (12)$$

where $W_i \in \mathbb{R}^{4\times 8}$ and $\tilde{W}_i \in \mathbb{R}^{4\times 8}$ such that $\begin{bmatrix} W_i \\ \tilde{W}_i \end{bmatrix}$ is non singular with $W_i = [I_4\ \mathbf{0}_4]$, $\tilde{W}_i = [\mathbf{0}_4\ I_4]$. I_4 and $\mathbf{0}_4$ stand respectively for the 4×4 identity matrix and for the 4×4 null matrices. The homogeneous operators (with inputs set equal to zero) $\mathcal{J}_i = \frac{\partial}{\partial z_i} P_1 + P_0$ with domain defined as $D(\mathcal{J}_i) = \{\mathcal{H}_i x_i \in \mathcal{L}_2([0, L_i], \mathbb{R}^4) | \begin{bmatrix} f_{\partial i} \\ e_{\partial i} \end{bmatrix} \in \ker(W_i)\}$ and $i = \{1, 2\}$, generate contraction semi-groups [16]. Since the homogeneous operators generate contraction semi-groups, and the range of the boundary operators $\mathcal{B}_i$ is the whole respective input space U_i, the defined systems (11) are Boundary control systems on X_i, $i = \{1, 2\}$, with unique classical solutions [17]:

$$\begin{aligned} \dot{x}_i &= \mathcal{J}_i(\mathcal{H}_i x_i), \\ u_{if} &= \mathcal{B}_i(\mathcal{H}_i x_i) = W_i \begin{bmatrix} f_{\partial, L_i x_i} \\ e_{\partial, L_i x_i} \end{bmatrix}, \\ y_{if} &= \mathcal{C}_i(\mathcal{H}_i x_i) = \tilde{W}_i \begin{bmatrix} f_{\partial, L_i x_i} \\ e_{\partial, L_i x_i} \end{bmatrix}. \end{aligned} \quad (13)$$

Remark 2. *The two beams' dynamics boundary control systems* (13) *are passive systems with respect to the Hamiltonian storage functions* (10)*. In fact, from Theorem 7.1.5 of [17], it holds*

$$\frac{dH_i}{dt}(t) = [(\mathcal{H}_i x_i)^T P_1 \mathcal{H}_i x_i]_0^{L_i}, \quad (14)$$

Because of the input output selection (12)*, the above equation implies*

$$\frac{dH_i}{dt}(t) = u_{if}(t)^T y_{if}(t). \quad (15)$$

that is equivalent to

$$H_i(T) - H_i(0) = \int_0^T u_{if}(t)^T y_{if}(t) dt$$

2.3. PH formulation of the actuated rotating joints

In this section, the previously defined set of ODEs (1)-(2), (4)-(5), (7)-(8), is re-formulated through a change of variables. The new selected variables are called "energy variables", and with this choice of states, the energy related to the set of ODEs is written as a quadratic form. To this end, substitute the infinite dimension energy variables (9) computed at $z_1 = L_1$ in the payload equations (7)-(8)

$$\frac{d}{dt}\left(m_{h,3}\left[\left(L_2\dot{\theta}_2 + \dot{w}_2(L_2)\right) + \cos(\theta_2 - \theta_1)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\right]\right) = \\ - K_2\varepsilon_{2,t}(L_2) - \gamma_5\dot{w}_2(L_2), \quad (16)$$

$$\frac{d}{dt} I_{h,3}\left(\dot{\theta}_2 + \dot{\phi}_2(L_2)\right) = -EI\varepsilon_{2,r}(L_2) - \gamma_6\dot{\phi}_2(L_2) \quad (17)$$

After several developments, and using equation (16), the ODE describing the boundary translational dynamic of the first beam (4) writes

$$\frac{d}{dt}\left(m_I\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\right) = \\ (\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\int_0^{L_2}\left[\rho_2\left(z_2\dot{\theta}_2 + \dot{w}_2\right) + \rho_2\cos(\theta_2 - \theta_1)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\right] dz_2 \\ + m_{h,3}(\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\left[\left(L_2\dot{\theta}_2 + \dot{w}_2(L_2)\right) + \cos(\theta_2 - \theta_1)\left(L_1\dot{\theta}_1 + \dot{w}_1(L_1)\right)\right] \\ - K_1\varepsilon_{1,t}(L_1) + \cos(\theta_2 - \theta_1)K_2\varepsilon_{2,t}(0) - \gamma_2\dot{w}_1(L_1) + \gamma_5(\dot{w}_2(L_2)) \quad (18)$$

The (virtual) mass term $m_I : \mathbb{R}^2 \to \mathbb{R}^+$ depends on the angle configuration of the manipulator, and it is defined as follows:

$$m_I(q_1, q_2) = m_{h,2} + (m_2 + m_{h,3})\sin^2(\theta_2 - \theta_1) > 0.$$

Similarly, substitute the infinite dimensional energy variables (9) in the boundary rotational dynamic equation (5)

$$\frac{d}{dt} I_{h,21}\left(\dot{\theta}_1 + \dot{\phi}_1(L_1)\right) = -EI_1\varepsilon_{1,r}(L_1) - \dot{\phi}_1(L_1) - \tau_2.$$

Using (9), (11) and (18), (17), the dynamic equations of the two motors' shaft are derived from (1) and (2):

$$\frac{d}{dt}\left(I_{h,1}\dot{\theta}_1\right) = +EI_1\varepsilon_{1,r}+\tau_1+\int_0^{L_2}\rho_2\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\left(z_2\dot{\theta}_2+\dot{w}_2\right)dz_2$$
$$+m_{h,3}\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\left(L_2\dot{\theta}_2+\dot{w}_2(L_2)\right)-\gamma_1\dot{\theta}_1+\gamma_2\dot{w}_1(L_1)+\gamma_3\dot{\phi}_1(L_1)$$

$$\frac{d}{dt}\left(I_{h,21}\dot{\theta}_2\right) = +EI_2\varepsilon_{2,r}(0)+\tau_2-\int_0^{L_2}\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\rho_2\left(z_2\dot{\theta}_2+\dot{w}_2\right)dz_2$$
$$-m_{h,3}\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\sin(\theta_2-\theta_1)\left(L_2\dot{\theta}_2+\dot{w}_2(L_2)\right)-\gamma_4(\dot{\theta}_2-\dot{\theta}_1-\dot{\phi}_1(L_1))+\gamma_6\dot{\phi}_2(L_2) \tag{19}$$

To define the PH representation of the above boundary dynamics, the energy states of the set of boundary dynamic equations (16)-(19) are defined as follows:

$$\begin{array}{ll}
p_1 = I_{h,1}\dot{\theta}_1, & p_2 = m_I\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right), \\
p_3 = I_{h,21}\left(\dot{\theta}_1+\dot{\phi}_1(L_1)\right), & p_4 = I_{h,22}\dot{\theta}_2, \\
\quad p_5 = m_{h,3}\Big[\left(L_2\dot{\theta}_2+\dot{w}_2(L_2)\right) & \\
\quad +\cos(\theta_2-\theta_1)\left(L_1\dot{\theta}_1+\dot{w}_1(L_1)\right)\Big] & p_6 = I_{h,3}\left(\dot{\theta}_2+\dot{\phi}_2(L_2)\right) \\
q_1 = \theta_1, & q_2 = \theta_2.
\end{array} \tag{20}$$

The state of the ODEs set is defined as $x_r = [p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ q_1\ q_2]^T \in X_r$, with state space $X_r \subset \mathbb{R}^8$. The related Hamiltonian can be written as a quadratic form $H_r = \frac{1}{2}x_r^T L_r x_r$ with the energy matrix $L_r \in \mathbb{R}^{8\times 8}$ defined as

$$L_r = \mathrm{diag}\left[\frac{1}{I_{h,1}},\ \frac{1}{m_I(q_1,q_2)},\ \frac{1}{I_{h,21}},\ \frac{1}{I_{h,22}},\ \frac{1}{m_{h,3}},\ \frac{1}{I_{h,31}},\ 0,\ 0\right]$$

To write the PH formulation of the ODEs set, the boundary conditions terms insides the ODEs are considered as inputs, that are in turn related to the boundary outputs of the PDEs. Thus, the input of the finite dimensional system is split in three vectors depending on the provenance of these quantities. The first input vector collect the two torques applied in the first and second joints, the other two input vectors are used for the interconnection respectively with the first and second set of PDEs:

$$u_r = \begin{bmatrix} u_{r1} \\ u_{r2} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \; u_1 = \begin{bmatrix} u_{r3} \\ u_{r4} \\ u_{r5} \end{bmatrix} = \begin{bmatrix} EI_1\varepsilon_{1,r}(0) \\ K_1\varepsilon_{1,t}(L_1) \\ EI_1\varepsilon_{1,r}(L_1) \end{bmatrix}, \; u_2 = \begin{bmatrix} u_{r6} \\ u_{r7} \\ u_{r8} \\ u_{r9} \end{bmatrix} = \begin{bmatrix} K_2\varepsilon_{2,t}(0) \\ EI_2\varepsilon_{2,r}(0) \\ K_2\varepsilon_{2,t}(L_2) \\ EI_2\varepsilon_{2,r}(L_2) \end{bmatrix} \tag{21}$$

The set of nonlinear ODEs (16)-(19) can be written in the PH formulation as:

$$\begin{cases} \dot{x}_r = (J_r(x_r, x_b) - R_r)\frac{\partial H}{\partial x_r}(x_r) + g_r u_r + g_1 u_1 + g_2(x_r)u_2 \\ y_r = g_r^T \frac{\partial H}{\partial x_r}(x_r) \\ y_1 = g_1^T \frac{\partial H}{\partial x_r}(x_r) \\ y_2 = g_2(x_r)^T \frac{\partial H}{\partial x_r}(x_r) \end{cases}, \tag{22}$$

where the non-linear interconnection matrix $J_r : X_r \times X_b \to \mathbb{R}^{8\times 8}$, $J_r(x_r, x_b) = -J_r(x_r, x_b)^T$, the damping matrix $R_r = R_r^T \geq 0 \in \mathbb{R}^{8\times 6}$ and the input matrices $g_r \in \mathbb{R}^{8\times 2}$, $g_1 \in \mathbb{R}^{8\times 3}$, $g_2 : X_r \to \mathbb{R}^{8\times 2}$ are defined in Appendix A. The three resulting output conjugated spaces are defined as $y_r \in Y_r \subset \mathbb{R}^2$, $y_1 \in Y_1 \subset \mathbb{R}^3$, $y_2 \in Y_2 \subset \mathbb{R}^4$.

Remark 3. *The nonlinear finite dimensional PH system defined by* (22) *is a passive system with respect to the Hamiltonian function H_r [26]. In particular, it holds:*

$$\begin{aligned} \frac{dH_r}{dt}(t) \quad &= -\frac{\partial H_r}{\partial x_r}^T R_r \frac{\partial H_r}{\partial x_r} + u_r(t)^T y_r(t) + u_1(t)^T y_1(t) + u_2(t)^T y_2(t) \\ &= -\frac{\gamma_1}{I_1^2} p_1^2 - \gamma_2 \left(\frac{p_2}{m_I} - \frac{L^2 p_1}{I_1}\right)^2 - \gamma_3 \left(\frac{p_3}{I_3} - \frac{p_1}{I_1}\right)^2 - \gamma_4 \left(\frac{p_4}{I_4} - \frac{p_3}{I_3}\right)^2 \\ &\quad -\gamma_5 (\frac{p_5}{m_{h,3}} - L_2 \frac{p_4}{I_{h,22}} - \cos(q_2 - q_1)\frac{p_2}{m_I})^2 - \gamma_6 \left(\frac{p_6}{I_{h,3}} - \frac{p_4}{I_{h,22}}\right)^2 \\ &\quad + u_r(t)^T y_r(t) + u_1(t)^T y_1(t) + u_2(t)^T y_2(t) \end{aligned} \tag{23}$$

that using $\gamma_i \geq 0 \quad i = 1, \ldots, 6$, implies

$$H_r(T) - H_r(0) \leq \int_0^T u_r(t)^T y_r(t) + u_1(t)^T y_1(t) + u_2(t)^T y_2(t) dt$$

2.4. Interconnection relations definition and the global PH model of the double flexible-link manipulator

According to the states (9)-(20), and the input output definitions (12)-(21), the interconnection relations between the boundary control systems and the nonlinear set of ODEs are defined as

$$\begin{aligned} u_{1f} &= -G_1 y_1, \quad u_{2f} = -y_2, \\ u_1 &= G_1 y_{1f}, \qquad u_2 = y_{2f}, \end{aligned} \tag{24}$$

where $G_1 = [0_{3\times 1} \ \ I_{3\times 3}]^T$. The remaining input corresponds to the torques applied by the two motors $u_r = [\tau_1 \ \tau_2]^T$.

The global interconnected system can be represented with the use of an extended operator. The global state space is defined as $W = X_1 \times X_2 \times X_r$, with inner product $\langle w_a, w_b \rangle_W = \langle x_{a1}, x_{b1} \rangle_{X_1} + \langle x_{a2}, x_{b2} \rangle_{X_2} + x_{ar}^T L_r x_{br}$. According to the interconnection relations (24), the global system results in a collocated system defined as

$$\begin{aligned} \dot{w} &= \mathcal{J}_w \begin{bmatrix} \mathcal{H}_1 x_1 \\ \mathcal{H}_2 x_2 \\ \frac{\partial H_r}{\partial x_r} \end{bmatrix} + g_w u_r, \quad \mathcal{J}_w = \begin{bmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ g_1 G_1 \mathcal{C}_1 & g_2 \mathcal{C}_2 & (J_r - R_r) \end{bmatrix}, \\ y_w &= g_w^T \begin{bmatrix} \mathcal{H}_1 x_1 \\ \mathcal{H}_2 x_2 \\ \frac{\partial H_r}{\partial x_r} \end{bmatrix}, \qquad\qquad g_w = \begin{bmatrix} 0 \\ 0 \\ g_r \end{bmatrix} \end{aligned} \tag{25}$$

with domain $\mathcal{D}(\mathcal{J}_w) = \{ w \in W | \mathcal{H}_1 x_1 \in D(\mathcal{J}_1), \mathcal{H}_2 x_2 \in D(\mathcal{J}_2), \begin{bmatrix} \mathcal{H}_1 x_1 \\ \mathcal{H}_2 x_2 \\ \frac{\partial H_r}{\partial x_r}(x_r) \end{bmatrix} \in ker(\mathcal{B}_w) \}$, and

$$\mathcal{B}_w = \begin{bmatrix} \mathcal{B}_1 & 0 & G_1 g_1^T \\ 0 & \mathcal{B}_2 & g_2^T \end{bmatrix}. \tag{26}$$

Thanks to the energy preserving interconnections (24), the total energy of the manipulator $H = E_k + E_p$ can be rewritten as the sum of the energies of the three parts in which the system has been divided, and corresponds to the square norm defined with the inner product in the global state space,

$$H = \frac{1}{2}||w||^2 = \frac{1}{2}\langle w, w\rangle_W = +\frac{1}{2}\int_0^{L_1} x_1^T \mathcal{H}_1 x_1 dz_1 + \frac{1}{2}\int_0^{L_2} x_2^T \mathcal{H}_2 x_2 dz_2 + \frac{1}{2} x_r^T L_r x_r.$$

Remark 4. *Since the global system* (25) *is obtained through the power preserving interconnection of three passive systems, it is itself a passive system with respect to the sum of the the three storage functions, i.e. the total energy H. In fact, using relations* (15) *and* (23) *together with* (24)*, it holds*

$$\frac{dH}{dt}(t) \leq u_r(t)^T y_w(t).$$

That is equivalent to

$$H(T) - H(0) \leq \int_0^T u_r(t)^T y_w(t) dt$$

3. Structure preserving discretization of the double flexible-link manipulator

In the following, we exploit the discretization procedure introduced by Golo [25] to get the finite dimensional Dirac structure of the Timoshenko beam model. Then, the explicit system representing the finite dimensional version of both clamped -clamped beams is presented. Then, the two obtained time invariant models of the two beams are interconnected to the non-linear set of ODEs through their boundary conditions, to obtain the global model exploitable for simulation purposes.

3.1. Structure discretization of the Timoshenko beam

To use the discretization procedure introduced by Golo [25], we explicit the Dirac structure [15] underlying the previously defined Timoshenko model for both beams. To do so, the effort and the flow spaces are defined as $\mathcal{F} = \mathcal{E} = \mathcal{L}^2(0, L, \mathbb{R}^4) \oplus \mathbb{R}^4$, such to define the variable space $\mathcal{B} = \mathcal{E} \otimes \mathcal{F}$ with the associated bilinear form

$$\begin{aligned} \langle b_a, b_b\rangle = \quad & \textstyle\int_0^{L_i} \left(f_1^T e_2 + f_2^T e_1\right) dz_i + f_{\partial a}^T(L) e_{\partial b}(L) - f_{\partial a}^T(0) e_{\partial b}(0) \\ & + f_{\partial b}^T(L) e_{\partial a}(L) - f_{\partial b}^T(0) e_{\partial a}(0), \end{aligned}$$

where $b_a = [f_a \ f_{\partial a} \ e_a \ e_{\partial a}]^T, b_b = [f_b \ f_{\partial b} \ e_b \ e_{\partial b}]^T \in \mathcal{B}$. The Dirac structure of the Timoshenko beam equation is defined as

$$\mathcal{D} = \left\{ b = \begin{bmatrix} f_b \\ f_{\partial b} \\ e_b \\ e_{\partial b} \end{bmatrix} \in \mathcal{B} \;\middle|\; \begin{array}{c} e \text{ is absolutely continuous and } \frac{\partial e}{\partial z} \in \mathcal{L}^2(0, L, \mathbb{R}^4), \\ f = P_1 \frac{\partial}{\partial z} e + P_0 e, \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} U \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} e(a) \\ e(b) \end{bmatrix}. \end{array} \right\} \tag{27}$$

with $U^T \Sigma U = \Sigma$ and $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ [16]. The flux f and the effort e are approximated in an interval $z \in [a, b],\ a < b,\ a, b \in [0, L]$ as following:

$$f(t, z) = f^{ab}(t) w^{ab}(z) \quad e = e^a(t) w^a(z) + e^b w^b(z), \tag{28}$$

where the base function $w^{ab}(z)$ should satisfy $\int_a^b w^{ab}(z) = 1$ and the base functions $w^a(z),\ w^b(z)$ should satisfy $w^a(a) = 1,\ w^a(b) = 0,\ w^b(a) = 0$ and $w^b(b) = 1$. Depending on weather it is considered the first or the second beam, the points a and b must fulfil $a - b = \frac{L_i}{n_i}$. Hence, the base functions can be chosen as:

$$w^a(z) = -\tfrac{z}{b-a} + \tfrac{b}{b-a}, \quad w^b(z) = +\tfrac{z}{b-a} - \tfrac{a}{b-a}, \quad w^{ab}(z) = \tfrac{1}{b-a}. \tag{29}$$

By taking the approximation of the flux and effort variables (28) in an interval $z \in [a, b]$ and the differential equations defined in Dirac structure (27), one can get the finite dimensional approximation of (27) in the interval $z \in [a, b]$ as follow:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \frac{b-a}{2} & -1 & \frac{b-a}{2} \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{F_{ab}} \underbrace{\begin{bmatrix} f_{pt}^{ab} \\ f_{pr}^{ab} \\ f_{\varepsilon t}^{ab} \\ f_{\varepsilon r}^{ab} \\ f_{Ba}^{t} \\ f_{Ba}^{r} \\ f_{Bb}^{t} \\ f_{Bb}^{r} \end{bmatrix}}_{f_{ab}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{b-a}{2} & 1 & -\frac{b-a}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}}_{E_{ab}} \underbrace{\begin{bmatrix} e_{pt}^{ab} \\ e_{pr}^{ab} \\ e_{\varepsilon t}^{ab} \\ e_{\varepsilon r}^{ab} \\ e_{Ba}^{t} \\ e_{Ba}^{r} \\ e_{Bb}^{t} \\ e_{Bb}^{r} \end{bmatrix}}_{e_{ab}} = 0, \tag{30}$$

where, $f^{ab} = [f_{pt}^{ab}\ f_{pr}^{ab}\ f_{\varepsilon t}^{ab}\ f_{\varepsilon r}^{ab}]^T$ and $e^{ab} = [e_{pt}^{ab}\ e_{pr}^{ab}\ e_{\varepsilon t}^{ab}\ e_{\varepsilon r}^{ab}]^T$ are flux vector and the effort vector of the discretized element respectively. $[f_{Ba}^t\ f_{Ba}^r\ f_{Bb}^t\ f_{Bb}^r]^T$ and $[e_{Ba}^t\ e_{Ba}^r\ e_{Bb}^t\ e_{Bb}^r]^T$ are the boundary variables of the discretized element which can be used to select the input output variables of the explicit system with different boundary considerations. From the definition of Dirac structure in [26], it is easy to prove that the above equation defines a Dirac structure with $E_{ab}^T F_{ab} + F_{ab}^T E_{ab} = 0$ and $\text{rank}[E, F] = 8$. The discretized Hamiltonian on the interval $z \in [a, b]$ writes:

$$\mathcal{H}_{ab} = \frac{1}{2}\frac{p_t^{ab}(t)^2}{\rho_{ab}} + \frac{1}{2}\frac{p_r^{ab}(t)^2}{I_{\rho ab}} + \frac{1}{2}K_{ab}\varepsilon_t^{ab}(t)^2 + \frac{1}{2}EI_{ab}\varepsilon_r^{ab}(t)^2. \tag{31}$$

with the approximated variables $p_t(t,z) = p_t^{ab}(t)w^{ab}(z)$, $p_r(t,z) = p_r^{ab}(t)w^{ab}(z)$, $\varepsilon_t(t,z) = \varepsilon_t^{ab}(t)w^{ab}(z)$, $\varepsilon_r(t,z) = \varepsilon_r^{ab}(t)w^{ab}(z)$. Assuming that the parameter values are constant along the whole interval, one can compute the approximated parameter values in the interval:

$$\begin{aligned} \rho_{ab} = \rho(b-a), \quad & I_{\rho ab} = I_\rho(b-a), \\ K_{ab} = \tfrac{K}{b-a}, \quad & EI_{ab} = \tfrac{EI}{b-a}. \end{aligned} \tag{32}$$

To achieve the explicit dynamic representations, we define the following relations

$$\begin{bmatrix} -\dot{p_t}^{ab} \\ -\dot{p_r}^{ab} \\ \dot{\varepsilon_t}^{ab} \\ \dot{\varepsilon_r}^{ab} \end{bmatrix} = \begin{bmatrix} f_{pt}^{ab} \\ f_{pr}^{ab} \\ f_{\varepsilon t}^{ab} \\ f_{\varepsilon r}^{ab} \end{bmatrix} \quad \begin{bmatrix} -\frac{\partial \mathcal{H}_{ab}}{\partial pt} \\ -\frac{\partial \mathcal{H}_{ab}}{\partial pr} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon t} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon r} \end{bmatrix} = \begin{bmatrix} e_{pt}^{ab} \\ e_{pr}^{ab} \\ e_{\varepsilon t}^{ab} \\ e_{\varepsilon r}^{ab} \end{bmatrix}.$$

The total model is derived through the power preserving interconnection of all elements in which the beam has been divided. To express the power preserving interconnection between two successive elements, it is convenient to split the input and the output of the $i-th$ element in two parts:

$$u_{i,2}^{ab} = \begin{bmatrix} u_{i,2}^{ab,1} \\ u_{i,2}^{ab,2} \end{bmatrix} = \begin{bmatrix} -\frac{p_t(a)}{\rho} \\ -\frac{p_r(a)}{I_\rho} \\ K\varepsilon_t(b) \\ EI\varepsilon_r(b) \end{bmatrix}, \quad y_{i,2}^{ab} = \begin{bmatrix} y_{i,2}^{ab,1} \\ y_{i,2}^{ab,2} \end{bmatrix} = \begin{bmatrix} K\varepsilon_t(a) \\ EI\varepsilon_r(a) \\ +\frac{p_t(b)}{\rho} \\ +\frac{p_r(b)}{I_\rho} \end{bmatrix}. \tag{33}$$

Then the explicit dynamic system with the above choice of input and output variables is given as:

$$\begin{bmatrix} -\dot{p}^{ab}_{t,2} \\ -\dot{p}^{ab}_{r,2} \\ \dot{\varepsilon}^{ab}_{t,2} \\ \dot{\varepsilon}^{ab}_{r,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & -(b-a) & 2 \\ -2 & (b-a) & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\partial \mathcal{H}_{ab}}{\partial pt,2} \\ -\frac{\partial \mathcal{H}_{ab}}{\partial pr,2} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon t,2} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon r,2} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} u^{ab}$$

$$y^{ab} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\partial \mathcal{H}_{ab}}{\partial pt,2} \\ -\frac{\partial \mathcal{H}_{ab}}{\partial pr,2} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon t,2} \\ \frac{\partial \mathcal{H}_{ab}}{\partial \varepsilon r,2} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} u^{ab} \tag{34}$$

Interconnecting all the elements in which the beam has been divided, we obtain a discretized beam model that takes as input in the boundaries the same input taken by a single element (33): rotational and traslational velocity at the $z_i = 0$ side, torque and force at the $z_1 = L_i, i = 1, 2$ side. The interconnection laws write

$$\begin{cases} u^{ab,1}_{i+1,1} = -y^{ab,2}_{i,2}, \\ u^{ab,2}_{i,2} = y^{ab,1}_{i+1,2}. \end{cases} \tag{35}$$

The model for the clamped-clamped beam should have as input translational and rotational velocity at both side of the beam. As a consequence, it is necessary to define a new element that takes velocities as input only speeds and forces as outputs, starting from the Dirac structure defined in (30). This element, that is referred to as elastic element, will be then connected at the free side to change the causality and convert it into a clamped side. The elastic element is derived setting in (30) $f_{pt}^{ab} = f_{pr}^{ab} = 0$ and $e_{pt}^{ab} = e_{pr}^{ab} = 0$:

$$\begin{array}{c} \begin{bmatrix} \dot{\varepsilon}_t^{ab} \\ \dot{\varepsilon}_r^{ab} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b-a}{2} & 1 & -\frac{b-a}{2} \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{p_t(a)}{\rho} \\ -\frac{p_r(a)}{I_\rho} \\ \frac{p_t(b)}{\rho} \\ \frac{p_r(b)}{I_\rho} \end{bmatrix} \\ \begin{bmatrix} K\varepsilon_t(a) \\ EI\varepsilon_r(a) \\ K\varepsilon_t(b) \\ EI\varepsilon_r(b) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{b-a}{2} & 1 \\ 1 & 0 \\ -\frac{b-a}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}_a b}{\partial \varepsilon_t} \\ \frac{\partial \mathcal{H}_a b}{\partial \varepsilon_r} \end{bmatrix} \end{array} \tag{36}$$

with the Hamiltonian in the following form:

$$H_{ab,e} = \frac{1}{2} K_{ab} (\varepsilon_t^{ab})^2 + \frac{1}{2} EI_{ab} (\varepsilon_r^{ab})^2$$

Connect the elastic element using the interconnection law (35) at the end of a chain composed of only normal elements (34), such to obtain a clamped-clamped beam. From now on all the quantities will be expressed with the subscript $(\cdot)_i$, implying that $i = \{1, 2\}$. After the interconnection, the resulting pair of input-output results into

$$u_{B,i} = \begin{bmatrix} -\frac{p_t(0)}{\rho} \\ -\frac{p_r(0)}{I_\rho} \\ \frac{p_t(L)}{\rho} \\ \frac{p_r(L)}{I_\rho} \end{bmatrix}, \quad y_{B,i} = \begin{bmatrix} K\varepsilon_t(0) \\ EI\varepsilon_r(0) \\ K\varepsilon_t(L) \\ EI\varepsilon_r(L) \end{bmatrix}.$$

The obtained system maintains the PH form. The system has $2n_i$ momentum states and $2(n_i + 1)$ displacements states. Consequently, the kinetic energy is divided between $2n_i$ elements, while the potential elastic between $2(n_i + 1)$ elements $H_i = \sum_{k=1}^{n} H_{ab,k} + H_{ab,e}$. The clamped-clamped discretized Timoshenko beam can be written in the following compact notation

$$\begin{aligned} \dot{x}_i &= J_i \tfrac{\partial H_i}{\partial x_i}(x_i) + B_i u_{B,i} \\ y_{B,i} &= B_i^T \tfrac{\partial H_i}{\partial x_i} \end{aligned}$$

with the state variable $x_i = [x_{\varepsilon t,i} \; x_{pt,i} \; x_{\varepsilon r,i} \; x_{pr,i}]^T = [\varepsilon_{t,i} \; -p_{t,i} \; \varepsilon_{r,i} \; -p_{r,i}]^T$. In this case, it can also be noticed that the complete Hamiltonian can be expressed as a quadratic form of the discretized variables, with respect to the energy matrix

$$H_i = \frac{1}{2} x_i^T L_i x_i, \qquad \text{where} \qquad L_i = \mathrm{diag}\left[K_i, M_{\rho,i}, EI_i, M_{I\rho,i}\right] \in \mathbb{R}^{(2n_i+1)\times(2n_i+1)}$$

with $K_i = \mathrm{diag}\left[K_i^{ab}, \dots K_i^{ab}\right] \in \mathbb{R}^{(n_i+1)\times(n_i+1)}$, $M_{\rho,i} = \mathrm{diag}\left[\frac{1}{\rho_i^{ab}}, \dots \frac{1}{\rho_i^{ab}}\right] \in \mathbb{R}^{n_i \times n_i}$, $EI_i = \mathrm{diag}\left[EI_i^{ab}, \dots EI_i^{ab}\right] \in \mathbb{R}^{(n_i+1)\times(n_i+1)}$, $M_{I\rho,i} = \mathrm{diag}\left[\frac{1}{I\rho_i^{ab}}, \dots \frac{1}{I\rho_i^{ab}}\right] \in \mathbb{R}^{n_i \times n_i}$. Since the derivative of the Hamiltonian with respect to the the state x_i writes $\frac{\partial H_i}{\partial x_i}(x_i) = L_i x_i$, the approximate system can be rewritten as a linear time invariant system:

$$\begin{aligned} \dot{x}_i &= J_i L_1 x_i + B_i u_{B,i} \\ y_{B,i} &= B_i^T L_i x_i, \end{aligned} \tag{37}$$

Where J_i and B_i are defined in Appendix B.

3.2. Finite PHS of the double flexible-links manipulator

Take the two beams' equations (37) and the nonlinear equations of the boundary conditions (22), and using of the power preserving interconnections (24), define the total system:

$$\begin{bmatrix} \dot{x}_r \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} J_r(x_2) & g_1 G_1^T B_1^T & g_2(x_r) G_2^T B_2^T \\ -B_1 G_1 g_1^T & J_1 & 0 \\ -B_2^T G_2 g_2(x_r) & 0 & J_2 \end{bmatrix} - \begin{bmatrix} R_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H_r}{\partial x_r}(x_r) \\ L_1 x_1 \\ L_2 x_2 \end{bmatrix} + \begin{bmatrix} g_r \\ 0 \\ 0 \end{bmatrix} u_r$$

where we define the extended state $x_{tot} = [x_r \; x_1 \; x_2]^T \in X_{tot}$, with state space $X_{tot} = \mathbb{R}^{4n_1+4n_2+8}$. The resulting system can be written in the PH form

$$\begin{cases} \dot{x}_{tot} = \left(J_{tot}(x) - R_{tot}\right) \frac{\partial H_{tot}}{\partial x_{tot}}(x_{tot}) + g_{tot} u \\ y_{tot} = g_{tot}^T \frac{\partial H_{tot}}{\partial x_{tot}}(x_{tot}) \end{cases} . \tag{38}$$

with the Hamiltonian of the total system

$$H_{tot} = \frac{1}{2} x_r^T L_r(x_r) x_r + \frac{1}{2} x_1^T L_1 x_1 + \frac{1}{2} x_2^T L_2 x_2.$$

The total dissipative matrix is defined as $R_{tot} = \mathrm{diag}\left[c_1, 0, 0, c_2, 0, 0, 0, 0\right] \in \mathbb{R}^{(4n_1+4n_2+8)\times(4n_1+4n_2+8)}$. The inputs of the system (38) are the torques of the first and the second joint motors respectively. The input matrix of g_{tot} can be expressed as:

$$g_{tot} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^{4n_1+4n_2+8\times 2}.$$

A self-contained algorithm (Algorithm 3.2) is proposed to list the needed passages to obtain the model.

Algorithm 1 Finite dimensional model's derivation Algorithm

Input: PH PDEs-ODEs model (25)

Output: PH finite dimensional discretized model (38)

1: Define the base functions (29) and discretize the original Timoshenko beam's Dirac structure (27) into the finite dimensional one (30).

2: Define the discretized Hamiltonian on a single element interval (31), and the discretized parameters (32).

3: Use the power preserving interconnection law (35) to connect all the elements in which the beam has been divided.

4: Define the pure elastic element (36) and connect it at the end of the elements' chain to change the tip causality and obtain a discretized clamped-clamped beam (37).

5: Interconnect two discretized beam equations (37) to the finite dimensional part of the original model (22) through the energy preserving laws (24), such to obtain the PH PDEs-ODEs finite dimensional model (38).

System (38) has been obtained through a structure preserving mixed finite element discretization procedure, instead of the CMST used in the modelling procedure of the existing works [1, 24, 27]. The CMST allows to derive light models in terms of number of states, but the modes choice highly depends on the operating scenario of the manipulator, i.e. the selected modes can change drastically in case of impact/contact scenario. The proposed discretization procedure avoids this inconvenience, and if the manipulator's dynamics meet the conditions of Assumption 1, to obtain a finest simulation it is only necessary to augment the number of discretizing elements n_1, n_2. Further, the proposed model can be used for Control Design purposes using techniques developed for PH systems such as Control by Interconnection or IDA-PBC (Interconnection and Damping Assignment - Passivity Based Control) [18, 19].

4. Numerical simulations

In this section, the simulation results are used to illustrate the free response of the obtained finite dimensional model with a sinusoidal and a square inputs, to verify the qualitative behaviour of the proposed model. The manipulator

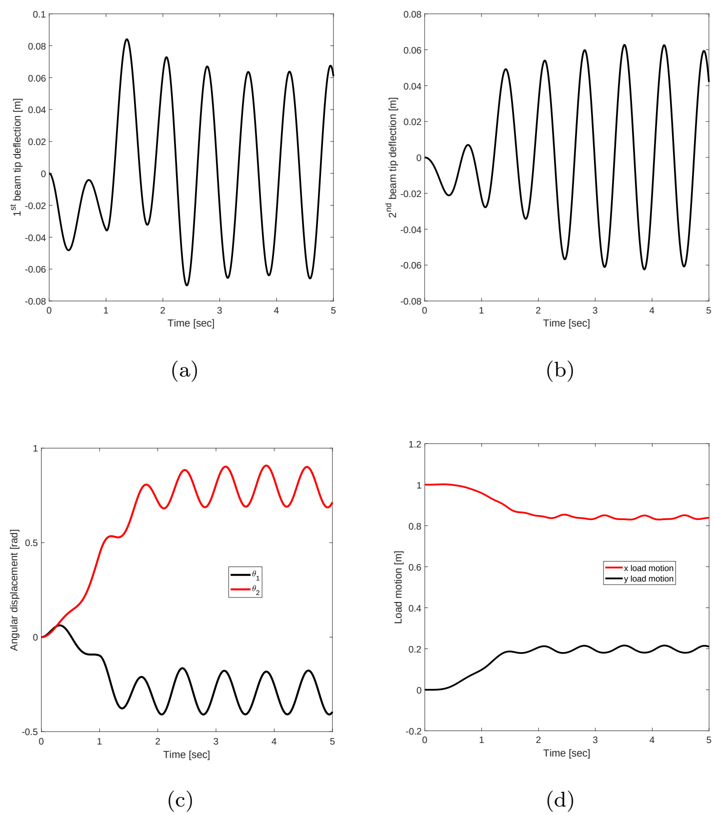
Table 1: Model parameters.

Parameters	Values
Lengths L_1 L_2	0.5 (m)
Widths $L_{w,1}$ $L_{w,2}$	0.1 (m)
Thickness $L_{t,1}$ $L_{t,2}$	0.1 (m)
Densities ρ_1 ρ_2	0.2 (kg/m)
Young's Modulus E_1 E_2	1.2×10^5 (N/m^2)
Rotor' First motor Inertia $I_{h,1}$	0.1 $(kg \cdot m^2)$
Stator' second motor Inertia $I_{h,21}$	0.0001 $(kg \cdot m^2)$
Rotor' second motor Inertia $I_{h,22}$	0.1 $(kg \cdot m^2)$
Second motor's Mass $m_{h,2}$	1 (kg)
Payload mass $m_{h,3}$	1 (kg)
Payload's inertia $I_{h,31}$	0.0001 $(kg \cdot m^2)$
Friction $\gamma_i \quad i = \{1, \ldots, 6\}$	0 $(N \cdot m \cdot s)$
Number of discretization elements n_1 n_2	2

parameters are selected as in [1] such to be able to compare the obtained simulations in case of Bang-Bang input. The used parameters are listed in Table 1. We refer to the parameters definitions at the beginning of Section 2 and the ones of Section 3, for the computation of the remaining parameters. The applied Bang-Bang input is defined as

$$\tau_k = \begin{cases} 0.2 & 0 \leq t < 1 \\ -0.2 & 1 \leq t < 2 \\ 0 & t \geq 4 \end{cases} \qquad k = \{1, 2\},$$

and in Figure 3 are shown his related simulation results. The sinusoidal inputs



(a) (b) (c) (d)

Figure 3: Bang-Bang input dynamic response: (a) First beam's tip deflection, (b) Second beam's tip deflection, (c) Motors rotor's anglular displacements, (d) Payload motion in cartesian coordinates.

are defined as

$$\tau_k = \begin{cases} 0.2\sin(\pi t) & 0 \leq t \leq 2 \\ 0 & t > 2 \end{cases} \qquad k = \{1, 2\},$$

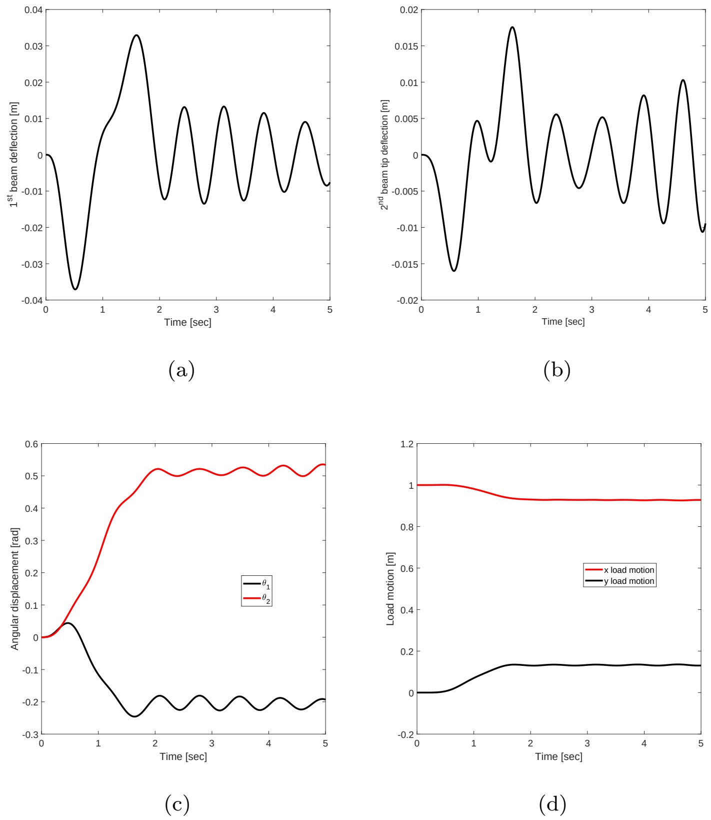


Figure 4: Sine input dynamic response: (a) First beam's tip deflection, (b) Second beam's tip deflection, (c) Motors rotor's anglular displacements, (d) Payload motion in cartesian coordinates.

and the dynamic simulation results are shown in Figure 4. It can be observed that the obtained results with Bang-Bang inputs, and in case of no friction, are in good agreement with those in [1]. A different deflection behaviour results from the different type of inputs application. It can be noticed that the maximal tip deflection in case of Bang-Bang input is approximately 0.08 (m) for the first beam and 0.06 (m) for the second beam , while in case of sinusoidal input are 0.03 (m) and 0.015 (m) for respectively the first and the second beam. The resulting vibration amplitudes after the input application $(t > 2)$ have for both beams a value of $0.06(m)$ in case of Bang-Bang inputs, while they are of $0.01(m)$

in case of sinusoidal inputs.

5. Conclusions

A boundary control problem describing the double flexible links manipulator in the port Hamiltonian (PH) form has been fully characterized and derived from general principles. The derived dynamic model consists of an hybrid set of PDEs-ODEs with actuation on the ODEs. It has been shown that the PDEs describing the motion of the flexible part can be put in the classical Timoshenko's Beam equations form interconnected with a nonlinear set of ODEs, if the state variables are chosen properly. The proposed system representation highlights its passivity properties, that can be exploited for control design. The system has been constructed in a modular way, making possible to add or remove parts dependently on the needed application or to change boundary conditions such to cope with contact or impact scenario. The obtained infinite dimensional model has been approximated to a finite dimensional one using a PH structure preserving discretization. The obtained finite dimensional nonlinear model could be used for simulation or control design purposes. Further, since the resulting finite dimensional system maintains the PH form, the well known Control by interconnection or IDA-PBC can be used to design the control law. The proposed model uses the Timoshenko's assumptions for the description of the flexible dynamics, instead of the commonly used Euler-Bernoulli assumptions. This leads to a finite dimensional system with a number of states bigger than the already existing models. Nevertheless the computational time remains acceptable: with the "ode23tb" integration scheme it took 3.473 *sec* to simulate 5 *sec* of trajectories' dynamics, in case of number of discretizing elements $n_1 = n_2 = 2$.

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Appendix A. Matrices definition: non-linear finite dimensional system

The interconnection and the dissipative matrices of the finite dimensional non-linear system describing the boundary dynamics are defined as:

$$J_r = \begin{bmatrix} 0 & \alpha & 0 & 0 & -1 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_r = \begin{bmatrix} \gamma_1 + L_1^2\gamma_2 + \gamma_3 & -L\gamma_2 & -\gamma_3 & 0 & 0 & 0 & 0 & 0 \\ -L\gamma_2 & \gamma_{21} + c_{21}^2\gamma_5 & 0 & +L_2c_{21}\gamma_5 & -c_{21}\gamma_5 & 0 & 0 & 0 \\ -\gamma_3 & 0 & \gamma_3 + \gamma_4 & -\gamma_4 & 0 & 0 & 0 & 0 \\ 0 & +L_2c_{21}\gamma_5 & -\gamma_4 & \gamma_4 + L_2^2\gamma_5 + \gamma_6 & -L_2\gamma_4 & -\gamma_6 & 0 & 0 \\ 0 & -c_{21}\gamma_5 & 0 & -L_2\gamma_5 & \gamma_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_6 & 0 & \gamma_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where $c_{21} = \cos(q_2 - q_1)$ and $\alpha : \mathbb{R}^2 \times \mathcal{L}_2(0, L_2) \to \mathbb{R}$ is given as:

$$\alpha(\theta_1, \theta_2, p_{2,t}) = \sin(\theta_2 - \theta_1)\left(\int_0^{L_2} p_{2,t}dz_2 + p_5\right). \quad \text{(A.1)}$$

The input matrices are defined as

$$g_r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ +\cos(q_2 - q_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Appendix B. Matrices definition: discretized beams

The matrices in the discretized model are defined, for $i = 1, 2$, as:

$$J_i = \begin{bmatrix} 0 & J_{v_i} & 0 & -S_i^T \\ -J_{v,i}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{v,i} \\ S_i & 0 & -J_{v,i}^T & 0 \end{bmatrix} B_i = \begin{bmatrix} B_i^1 & B_i^2 & B_i^3 & B_i^2 \\ 0 & 0 & 0 & 0 \\ 0 & B_i^1 & 0 & 0 \\ 0 & -B_i^4 & 0 & B_i^4 \end{bmatrix} \tag{B.1}$$

with $B_i^1 = \left[2, -2, \ldots, (-1)^{n_i-1}2, -1\right]^T \in \mathbb{R}^{n_i+1}$, $B_i^2 = \left[0, \ldots, 0, -\frac{b-a}{2}\right]^T \in \mathbb{R}^{n_i+1}$, $B_i^3 = [0, \ldots, 0, 1]^T \in \mathbb{R}^{n_i}$ and $B_1^4 = [1, 0, \ldots, 0]^T \in \mathbb{R}^{n_i}$. Since two displacement states have been added to the system, the J_1 sub-matrices become $J_{i,1} \in \mathbb{R}^{(n_1+1)\times n_1}$, $S_c \in \mathbb{R}^{n_1\times(n_1+1)}$ defined as

$$J_{v,i} = \begin{bmatrix} -2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ +4 & -2 & 0 & 0 & \cdots & 0 & 0 \\ -4 & +4 & -2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^{n_i-1}4 & (-1)^{n_i-2}4 & \cdots & +4 & -2 & 0 & 0 \\ (-1)^{n_i}4 & (-1)^{n_i-1}4 & \cdots & -4 & +4 & -2 & 0 \\ (-1)^{n_i+1}2 & (-1)^{n_i}2 & \cdots & -2 & +2 & -2 & -2 \end{bmatrix},$$

$$S_i = \begin{bmatrix} -(b-a) & 0 & \cdots & 0 & 0 \\ 0 & -(b-a) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(b-a) & 0 \\ (-1)^{n_i}(b-a) & (-1)^{n_i-1}(b-a) & \cdots, & -(b-a) & +(b-a) \end{bmatrix}$$