Experimental observations of topologically guided water waves within non-hexagonal structures

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We investigate symmetry-protected topological water waves within a strategically engineered square lattice system. Thus far, symmetry-protected topological modes in hexagonal systems have primarily been studied in electromagnetism and acoustics, i.e. dispersionless media. Herein, we show experimentally how crucial geometrical properties of square structures allow for topological transport that is ordinarily forbidden within conventional hexagonal structures. We perform numerical simulations that take into account the inherent dispersion within water waves and devise a topological insulator that supports symmetry-protected transport along the domain walls. Our measurements, viewed with a high-speed camera under stroboscopic illumination, unambiguously demonstrate the valley-locked transport of water waves within a non-hexagonal structure. Due to the tunability of the energy's directionality by geometry our results could be used for developing highly-efficient energy harvesters, filters and beam-splitters within dispersive media.

Considerable recent activity in wave phenomena is motivated through topological effects and focused on identifying situations where topological protection occurs that can enhance, or create, robust wave guidance along edges or interfaces. Remarkably, the core concepts that gave rise to topological insulators, originating within quantum mechanics[2] carry across, in part, to classical wave systems[3,4]. Topological insulators can be divided into two broad categories: those that preserve time-reversal symmetry (TRS), and those which break it. We concentrate upon the former due to the simplicity of their construction that solely requires passive elements. By leveraging the discrete valley degrees of freedom, arising from degenerate extrema in Fourier space, we are able to create robust symmetry-protected waveguides. These valley states are connected to the quantum valley-Hall effect and hence this research area has been named valleytronics[5,6].

Hexagonal structures are the prime candidates for valleytronic devices as they exhibit symmetry induced Dirac cones at the high-symmetry points of the Brillouin zone (BZ); when perturbed these Dirac points can be gapped, leading to well-defined KK' valleys distinguished, from each other, by their opposite chirality or pseudospin.

This pseudospin has been used in a wide variety of dispersionless wave settings to design valleytronic devices[7,8]. Here we extend the earlier research by examining a highly-dispersive physical system, i.e. water waves and move away from hexagonal structures. The topological protection afforded by these valley states is attributed to, both, the orthogonality of the pseudospins as well as the Fourier separation between the two valleys[9]. The vast majority of the valleytronics literature, inspired by graphene, opts to use hexagonal structures[10–19]. However a negative that emerges with these, especially when dealing with complex topological domains[9], is that certain propagation directions are restricted due to mismatches in chirality between incoming and outgoing modes. Notably, this has led to hexagonal structures being prohibited from partitioning energy in more than two-directions[13,14,16].

In this Letter, we demonstrate experimentally how a strategically designed square structure also allows for the emergence of valley-Hall edge states as well as allowing for the excitation of modes that are not ordinarily ignited within hexagonal valley-Hall structures. Additionally, the system chosen differs from the vast majority of the earlier literature[10–19] that has focussed on an idealised situation in which the dispersion of the host material has been avoided. This assumption restricts the applicability of the earlier studies to a small subset of, potentially useful, physical platforms that could host topological effects. Most notably, this assumption does not hold for water wave systems, which generally support highly dispersive surface waves[20]. The combination of topological physics applied to water waves is a relatively unexplored area[21,22]; those that have conducted experiments have either focused on 1D systems[21] or the hexagonal valley-Hall structure[22]. Potential applications of this budding area include controlling ocean wave energy[23], in a non-intrusive manner, for energy-harvesting or erosion mitigation[24].

Formulation —The fluid within our domain has a constant depth of $h = 4$ cm and contains a periodic array of rigid, ver-

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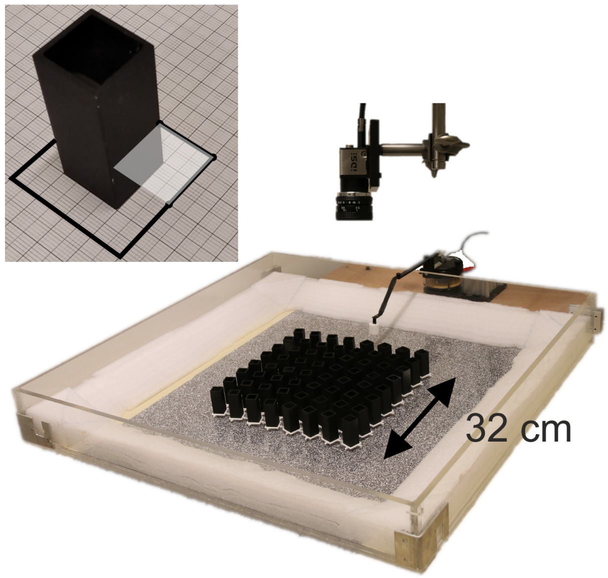

FIG. 1: Experimental setup: A crystal is assembled using square shaped aluminum tubes 7 cm in height arranged in a square array with different orientations using a plastic positioning frame at the bottom of the tank (80×80 cm^2 with 60-degree oblique edges made of soft polystyrene to mimic PMLs). A mechanical straight paddle holding a small plastic cylinder is used to generate water waves. The tank is continuously illuminated and images of water waves are recorded with a high speed camera placed on the top. A black and white random pattern is placed under the tank to provide water elevation measurement using an image cross-correlation algorithm. The experimental setup was inspired by the work of Moisy et al;[1].

tical and bottom mounted, square objects (2 cm of side length in a 4 cm square array) that perforate the free surface of the liquid (see the experimental set-up in Fig. 1). The planar coordinates are denoted by x_1, x_2 whilst the vertical upward direction has the coordinate x_3 ; the origin is prescribed to be at the mean free surface. Under the usual assumptions of linear water wave theory, where the fluid is assumed to be inviscid, incompressible with irrotational flow, there exists a velocity potential Φ such that

$$\Phi(\mathbf{x},t) = \Re e\left[\phi(x_1,x_2)\frac{\cosh(k(x_3+h))}{\cosh(kh)}\exp(-i\omega t)\right], \quad (1)$$

where ω denotes the angular frequency. The wavenumber, k the real positive solution of the dispersion relation

$$\left(gk + \frac{\sigma}{\rho}k^3\right)\tanh(kh) = \omega^2, \quad (2)$$

is used as a proxy for the frequency; in Eq. (2) $g = 9.81$ m s^{-2} is gravitational acceleration, $\sigma = 0.07$ N m^{-1} is the surface tension between air and water and $\rho = 10^3$ kg m^3 the water density. Then ϕ, the reduced potential, satisfies the Helmholtz equation,

$$\left(\nabla^2_{\mathbf{x}} + k^2\right)\phi(x_1,x_2) = 0, \quad (3)$$

where this equation holds at the mean free surface and the subscript $\mathbf{x}$ indicates differentiation with respect to $\mathbf{x}$. and no-flow boundary conditions on the vertical rigid cylinders: taking $\mathbf{n} = (n_1, n_2)$ as the unit outward normal to the square tubes' surface, $\partial\phi/\partial\mathbf{n} = 0$ on each of them.

When the problem is posed in terms of the reduced potential, ϕ, as the Helmholtz equation, with periodically arranged inclusions (the tubes), this directly maps across to the phononic crystal literature. Recognising the periodicity guides us to setting $\phi(x_1,x_2) = \phi_{\boldsymbol{\kappa}}(x_1,x_2)\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})$ with $\boldsymbol{\kappa}$ as the Bloch wavenumber and $\phi_{\boldsymbol{\kappa}}$ as the periodic piece of the Bloch solution. A key ingredient, that guides the experiments, is an understanding of the dispersion relation, relating k to the Bloch-wavenumber, $\boldsymbol{\kappa}$ spanning the BZ $\boldsymbol{\kappa} \in [0;\pi/L]\times[0;\pi/L]$ (see Fig 2(a)) for an infinite perfectly periodic square lattice system; we determine this relationship numerically. The geometry and band structures are shown in Fig. 2; for a square array, lattice constant L, the irreducible Brillouin zone (IBZ) is an eighth of the BZ. Despite this, we opt to plot around a quadrant of the BZ as this will incorporate the two distinct Dirac cones that are essential for our valley-Hall states. The desired quadrant has the following vertices: $X = (\pi/L, \pi/L)$, $N = (\pi/L, 0)$, $\Gamma = (0,0)$, $M = (0, \pi/L)$.

Results —The unrotated cellular structure chosen, Fig. 2(a), contains, both, horizontal and vertical mirror symmetries along with four-fold rotational symmetry. Hence, in its entirety the structure has C_{4v} point group symmetries. Notably, it is the presence of these mirror symmetries that yield Dirac cones along the outer edges of the BZ, Fig. 2(b,c)[25–29]. Note, that rectangular structures (wallpaper group $P2mm$) which possess these mirror symmetries will also yield these non-symmetry repelled Dirac cones[26,27]. In contrast to hexagonal structures the position of these degeneracies can be tuned by varying the geometrical or material parameters of the system[28]. By rotating the internal square inclusion, both mirror symmetries are broken thereby yielding the band-gap shown for the dispersion curves in Fig. 2(d). The residual valleys, that demarcate the band-gap, are locally imbued with a nonzero valley-Chern number[30],

$$C_v = \frac{i}{2\pi}\int_S \nabla_{\boldsymbol{\kappa}} \times \phi^*_{\boldsymbol{\kappa}}(\mathbf{r})\cdot\nabla_{\mathbf{r}}\phi_{\boldsymbol{\kappa}}(\mathbf{r})d\boldsymbol{\kappa} = \frac{i}{2\pi}\oint_\gamma \phi^*_{\boldsymbol{\kappa}}(\mathbf{r})\nabla_{\mathbf{r}}\phi_{\boldsymbol{\kappa}}(\mathbf{r})\cdot d\mathbf{l} \quad (4)$$

where the path integrated around (γ) encircles a particular valley and the superscript $*$ denotes the complex conjugate. Despite the calculation (and name) of the C_v resembling that of its TRS breaking counterpart, namely the Chern number, there is an important difference: the former is not a quantized quantity whilst the latter is. The surface associated with γ is not on a closed manifold hence the Gauss–Bonnet theorem[31] does not hold. Despite this, the opposite pseudospin modes have a bijective relationship with $\mathrm{sgn}(C_v) = \pm 1$ which itself can be classed as a topological integer[32]. From this we can apply the bulk-boundary correspondence for certain edge terminations thereby guaranteeing the existence of valley-Hall edge states.

Motivated by this, we place a perturbed cellular structure, that contains a positively or negatively rotated inclusion, above its reflectional twin. This results in a pair of gapless edge modes that almost span the entirety of the band-gap,

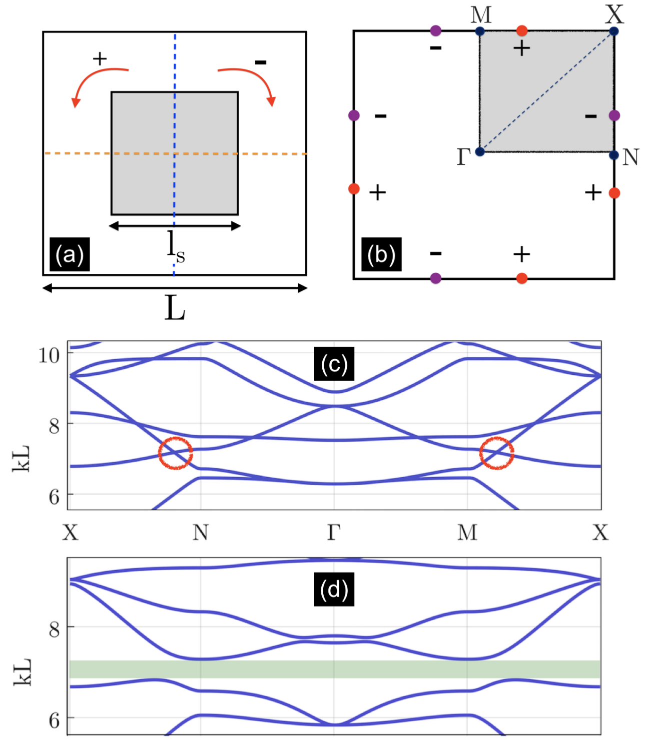

FIG. 2: Geometry, band structure and topological features: (a) Periodic cell (physical space) of the square lattice with sidelength L showing a square inclusion of sidelength l_s inside it. Mirror-symmetry breaking rotation (arrows) and lines (dashed) also shown. (b) In reciprocal space, the points ΓNX denote the extrema of the irreducible Brillouin zone that we extend to ΓNXM to show the two topologically inequivalent regions; the two distinct $\mathrm{sgn}(C_v)$ values are indicated by $\pm$ signs around the perimeter of the BZ and these are associated with the $+$ perturbation in panel (a) (the $-$ perturbation would result in opposite $\mathrm{sgn}(C_v)$'s, see[28]). The $\mathrm{sgn}(C_v)$ positions resemble those in[26,27]. (c) Band diagram for the configuration in (a), with two circles marking the position of the strategically engineered Dirac cones(d) Band diagram when the inclusion is rotated through an angle of 20°. A band-gap highlighted in green emerges from the symmetry breaking perturbation at Dirac points.

see Fig. 3. Here we use "gapless" to refer to the crossing of the concave and convex (opposite parity) modes. This distinguishes valley-Hall systems, that are topological, with those that are not and have coupled edge states; for example, the armchair termination within hexagonal structures produce gapped edge states that are, in turn, less robust[33].

The gapless nature of the states, and in turn the applicability of the Gauss–Bonnet theorem, is contingent upon the termination chosen containing projections of valleys with identical $\mathrm{sgn}(C_v)$. Unique to this specific square structure, the different parity eigenmodes belong to the *same* interface (see Fig. 3), rather than different interfaces. This result arises due to the mirror-symmetry relationship between the media either side of the interface in Fig. 3. This also implies that a right-propagating mode along one of the interfaces is a left-propagating mode on the other however, importantly, both states have opposite parity and hence remain orthogonal. This phenomena does not occur for hexagonal structures where the different parity eigenmodes belong to different interfaces. This relationship between the two interfaces allows for propagation, within our square structure, that is ordinarily forbidden within graphene-like structures. Coupling between modes, that are hosted along different interfaces, is crucial for energy navigation around sharp corners[17] and within complex topological domains[9,13,14,16]. Further explanation for this phenomenon can be found in[26,27].

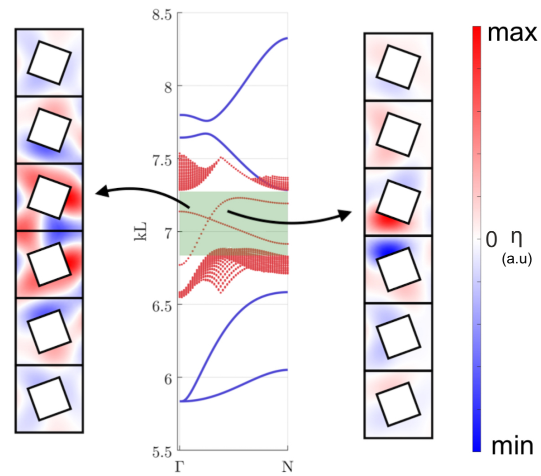

FIG. 3: Valley-Hall edge states: Band diagram for a ribbon with the upper/lower inclusions rotated clockwise/anti-clockwise. The real parts of the even and odd eigenmodes within the band-gap are shown (in red) as are several close-by ribbon modes (also in red). The blue curves are from Fig. 2(d), i.e. bulk modes along ΓN. Numerically, using finite elements, we take a long ribbon of N inclusions, apply Dirichlet boundary conditions top and bottom and extract the modes decaying away from the interface. The colormap for eigenmodes represents the normalized water elevation.

Experiments —The propagation of water waves is imaged at the surface of the water tank of Fig. 1. A mechanical paddle holding a circular cylinder is shaken at a controlable frequency. Cylindrical waves originating from the monopolar source are observed numerically and experimentally in Fig. 4(a, e). The experimental setups for a topologically nontrivial interface, with two different lengths, are shown in Figs. 4(b, f); the upper/lower half has square inclusions rotated clockwise/ anti-clockwise in order to break the mirror symmetries and generate the valley-edge states required. Images were acquired by a high speed camera and post processed using a cross-correlation algorithm[1]; each image was discretized into 360 areas each composed of 16 pixels.

Full-wave numerical simulations, performed using COMSOL Multiphysics (a commercial finite element scheme), for

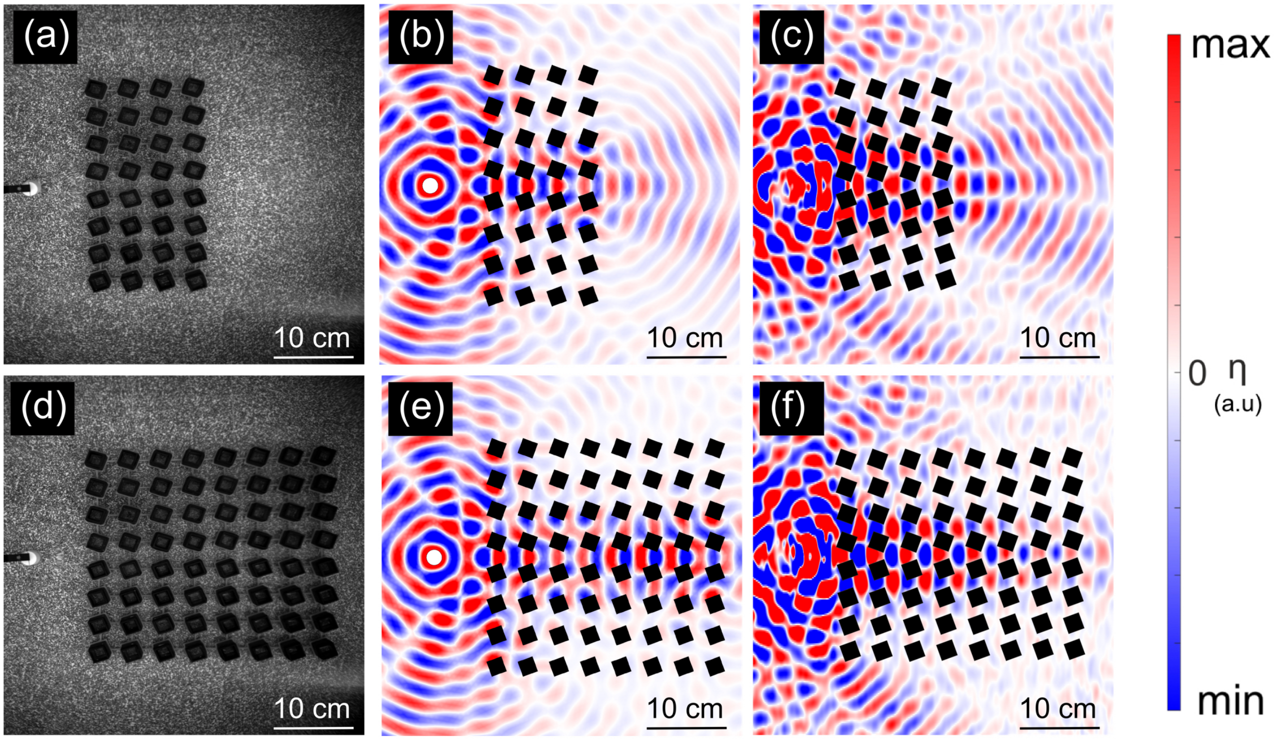

FIG. 4: Valley-Hall edge states: Experiments, and simulations. (a,d) Experimental set-up showing the top view of the water surface perforated by 4×8 and 8×8 square rigid inclusions respectively. (b,e) Corresponding numerical calculations; (c,f) Experimentally observed wavepatterns. These valley-Hall states are generated by a monopolar source operating at a frequency of 7.3 Hz and placed 6 cm from the domain wall edge and propagate from left to right between inclusions rotated by 20° clockwise and counter-clockwise.

tightly confined valley-Hall edge states, Figs. 4(c, d), show excellent agreement with the experiments, Figs. 4(g, h), despite our model not taking into account contact-line effects that occur between the water and the solid pillars, viscosity or nonlinearity. These square structure valley-Hall edge states have longer-wavelengths than their hexagonal counterparts and hence the distance between the pillars is subwavelength. A frequency modulated monopolar source is generated that ignites the even-parity valley-Hall edge state. The observed patterns are associated to the surface curvature where the coloured regions are indicative of the vertical elevation of the water level. The localisation of the topological edge state is clearly evident when comparing two interfaces of differing lengths, i.e. four or eight squares in Figs. 4(c, f). Notably, the valley-Hall state that propagates across four columns, Fig. 4(c), radiates almost isotropically upon exit. In the absence of any rods the energy would radiate isotropically away from the source[34]. The broadbandedness of this effect is demonstrated via the experimental results shown in[34]. The tight-confinement of these dispersive water waves, within a strategically designed square structure, is a highly nontrivial and unique observation.

We now strategically extend our earlier design, Figs. 4(b, f), to engineer four structured quadrants that results in a three-way topological energy-splitter, Fig. 5. We rotate the bottom-right and top-right inclusion sets anti-clockwise and clockwise, respectively, thereby creating four distinct domain walls upon which the valley-Hall states reside. The monopolar source triggers a wave, from the leftmost interface, into upward and downward modes along with continued rightward propagation. Incidentally, the most pronounced displacement pattern is along the two geometrically distinct horizontal interfaces. This continued rightward propagation is *forbidden* for hexagonal systems[9,13,14,16]. For coupling between the incident mode and the right-sided mode the chirality's must match and this does *not* happen for hexagonal structures. Contrastingly, this mismatch in chirality is overcome for the square structure as the right-sided interface is the reflectional partner of the left-sided interface. Hence, the incident mode need only to couple to itself in order to continue it's rightward propagation. This subtle relationship between the mirror-symmetry generated Dirac cones, and the subsequent, mirror-symmetry related interfaces allows for propagative behaviour not readily found within the valleytronics literature, Fig. 5. Note that Fig. 5 is a simulation as, unfortunately, the experiments suffered from a capillarity effect leading to enhanced dissipation that resulted in inconclusive experimental results.

Conclusion —We have experimentally shown the existence of topological valley-Hall transport for gravity-capillary water waves within a non-hexagonal structure. We have also

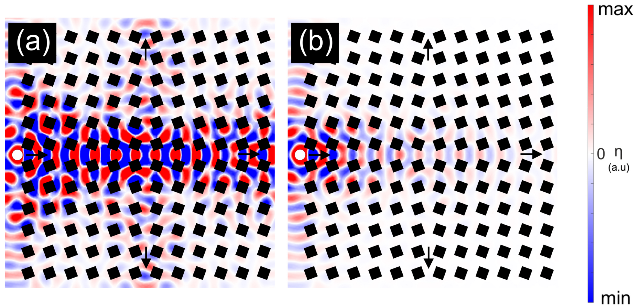

FIG. 5: Numerical illustration for a three-way splitter at 7.3 Hz: Four quadrants of alternately squares rotated clockwise and counter-clockwise. (a) has no losses whilst (b) incorporates attenuation introduced by considering a complex wave velocity with 2% imaginary part relative to the real part.

simulated a three-way topological multiplexer for the same highly-dispersive system. These demonstrations open up a new way for design in energy transport: the conventional symmetry constraints associated with hexagonal structures can be relaxed leading to richer designs of waveguides and multiplexers within highly-dispersive systems.

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