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Farshid Hajir, Christian Maire, Ravi Ramakrishna

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Abstract. — For every prime number p, we show the existence of a solvable number field L ramified only at $\{p, \infty\}$ whose p-Hilbert Class field tower is infinite.

Keywords: *Hilbert class field tower, root discriminant*

For a number field L of degree n over $\mathbb{Q}$, the root discriminant is defined to be $D_{\mathrm{L}}^{1/n}$ where D_{L} is the absolute value of the discriminant of L. Given a finite set S of places of $\mathbb{Q}$, it is an old question as to whether there is an infinite sequence of number fields unramified outside S with bounded root discriminant. This question is related to the constants of Martinet [**10**] and Odlyzko's bounds [**12**]. Since the root discriminant is constant in unramified extensions, an approach to answering the previous question in the positive is to find a number field L (of finite degree) unramified outside S having an infinite class field tower. In the case of $\mathrm{K}/\mathbb{Q}$ quadratic, it is a classical result of Golod and Shafarevich that if $\mathrm{K}/\mathbb{Q}$ is ramified at at least 8 places, then K has an infinite 2-class field tower. On the other hand, if p is a prime, and $S = \{p, \infty\}$, this question becomes whether there exist number fields with p-power discriminant having an infinite unramified extension. Schmitals [**13**] and Schoof [**14**] produced a few isolated examples of this type. See also [**3**], [**9**], etc. For $p \in \{2, 3, 5\}$, Hoelscher [**4**] announced the existence of number fields unramified outside $\{p, \infty\}$ and having an infinite Hilbert class field tower; see remark 2.2. Here we prove:

Theorem. — *For every prime number p, there exists a solvable extension* $\mathrm{L}/\mathbb{Q}$*, ramified only at $\{p, \infty\}$, having an infinite Hilbert p-class field tower. Consequently, there exists an infinite nested sequence of number fields of p-power discriminant with bounded root discriminant.*

Our proof is based on the idea of "cutting" of wild towers introduced in [**2**]; in particular it does not involve the usual technique of genus theory. The strategy begins by choosing s such that $\mathbb{Q}(\zeta_{p^s})$ has large class group (always possible). We then let K be the Hilbert class field of $\mathbb{Q}(\zeta_{p^s})$. Clearly K_S, the maximal pro-p extension of K ramified only at S, the

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places of K above p, is infinite. For each positive integer k, we define a Galois extension $\mathrm{K}_S^{[k]}/\mathrm{K}$ contained inside K_S/K such that all decomposition groups of $\mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathrm{K})$ are finite and abelian. We then show that for all large enough k, $\mathrm{K}_S^{[k]}/\mathrm{K}$ is of infinite degree, and that there exists a finite Galois extension L/K contained in $\mathrm{K}_S^{[k]}$, with the property that the primes above p split completely from L to $\mathrm{K}_S^{[k]}$, and in particular are unramified, leading to the desired result.

We do not know whether for every prime p, there is a *totally real* number field of p-power discriminant having an infinite Hilbert class field tower. In [**14**, Corollary 4.4] it is shown that $\mathbb{Q}(\sqrt{39345017})$ (which is ramified only at the prime 39345017) has infinite Hilbert class field tower. In [**15**], Shanks studied primes of the form $p = a^2 + 3a + 9$ and the corresponding totally real cubic subfields $\mathrm{K} \subset \mathbb{Q}(\mu_p)$ and showed the minimal polynomials of K are $x^3 - ax^2 - (a+3)x - 1$. Taking $a = 17279$ so $p = 298615687$, one can compute that the 2-part of the class group of K has rank 6. It is not hard to see, using the Golod-Shafarevich criterion, that K has infinite 2-Hilbert class field tower. Thus some examples exist in the totally real case.

Also in [**5**] Joshi and McLeman, using ideas and data of [**15**], showed that $\mathbb{Q}(\zeta_p)$ has infinite Hilbert class field tower for sufficiently large primes p of the form $a^2 + 3a + 9$. Inasmuch as it specifies an explicit and easily constructed number field as base of the tower, this result is stronger than ours for the primes for which it applies.

1. The results we need

Let p be a prime number. Let $\mathrm{K}/\mathbb{Q}$ be a finite Galois extension. Assume $\mu_p \subset \mathrm{K}$ and moreover that K is totally imaginary when $p = 2$. For a prime $\mathfrak{p}$ of K dividing p denote by e (resp. f) the ramification index (resp. the residue degree) of $\mathfrak{p}$ in $\mathrm{K}/\mathbb{Q}$.

1.1. On the group G_S. —

Denote by S the set of places of K above p, and consider K_S the maximal pro-p extension of K unramified outside S; put $\mathrm{G}_S = \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$.

Theorem 1.1 below is well-known, see for example [**11**, Corollary 8.7.5 and Theorem 10.7.3].

Theorem 1.1. — *Let $\mathrm{K}/\mathbb{Q}$ be a totally imaginary Galois extension containing μ_p. Let $S = \{p, \infty\}$. If $p \nmid h_\mathrm{K}$, the class number of* K*, then*

$$\dim H^1(\mathrm{G}_S, \mathbb{F}_p) = g\left(\frac{ef}{2} + 1\right) \text{ and } \dim H^2(\mathrm{G}_S, \mathbb{F}_p) = g - 1$$

where (p) has the usual efg decomposition in $\mathrm{K}/\mathbb{Q}$.

1.2. The cutting towers strategy. —

1.2.1. The Golod-Shafarevich Theorem. — Let G be a finitely presented pro-p group. Consider a minimal presentation $1 \to \mathrm{R} \to \mathrm{F} \xrightarrow{\varphi} \mathrm{G}$ of G, where F is a free pro-p group on $d = d(\mathrm{G})$ generators $\sigma_1, \ldots, \sigma_d$ and r relations $\rho_1, \ldots, \rho_r$ with normal closure $\mathrm{R} = \langle \rho_1, \cdots, \rho_r \rangle^{\mathrm{Norm}}$. We note that $d = \dim H^1(\mathrm{G}, \mathbb{F}_p)$ and $r = \dim H^2(\mathrm{G}, \mathbb{F}_p)$. We recall the depth function ω on F. See [**8**, Appendice A.3] or [**7**] for more details. The augmentation ideal I of $\mathbb{F}_p[\![\mathrm{F}]\!]$ is, by definition, generated by the set of elements $\{g-1\}_{g \in \mathrm{F}}$. Then for $1 \neq g \in \mathrm{F}$, define $\omega(g) = \max\{k \geqslant 1 \mid g - 1 \in I^k\}$; we put $\omega(1) = \infty$. It is not difficult to see that $\omega([g,g']) \geqslant 2$ and that $\omega(g^{p^k}) \geqslant p^k$ for every $g, g' \in \mathrm{F}$ and $k \in \mathbb{Z}_{>0}$. Observe also that as the presentation φ is minimal, $\omega(\rho_i) \geqslant 2$ for all the relations ρ_i.

The Golod-Shafarevich polynomial associated to the presentation φ of G is $P_{\mathrm{G}}(t) = 1 - dt + \sum_i t^{\omega(\rho_i)}$.

***Theorem 1.2* (Golod-Shafarevich, Vinberg [16]).** — *If* G *is finite then* $P_{\mathrm{G}}(t) > 0$ *for all* $t \in]0,1[$.

Of course in generic situations, we have no information about the ρ_i's other than their being elements of F of depth at least 2. With that in mind, let us note that if $P_{\mathrm{G}}(t) \leqslant P(t)$ for all $t \in]0,1[$, for some polynomial $P(t)$ which takes on a non-negative value somewhere on the open unit interval, then G must be infinite. For example, for a pro-p group of generator rank d and relation rank r, since all r relations have depth at least 2, we have $P_{\mathrm{G}}(t) \leqslant 1 - dt + rt^2$ for all $t \in]0,1[$. Theorem 1.2 then yields the usual Golod-Shafarevich criterion: if G is a non-trivial finite p-group of generator rank d and relation rank r, then $r > d^2/4$.

We can also define a depth function ω_{G} on G associated to the augmentation ideal I_G of $\mathbb{F}_p[\![\mathrm{G}]\!]$ by $\omega_{\mathrm{G}}(g) = \max\{k \geqslant 1 \mid g - 1 \in I_{\mathrm{G}}^k\}$, for $1 \neq g \in \mathrm{G}$; put $\omega_{\mathrm{G}}(1) = \infty$. Then:

Proposition 1.3. — *For every* $g \in \mathrm{G}$, *we have*

$$\omega_{\mathrm{G}}(g) = \max\{\omega(y) \mid \varphi(y) = g\}.$$

Proof. — See [**8**, Appendice 3, Theorem 3.5]. □

We now study quotients Γ of G such that $d(\mathrm{G}) = d(\Gamma)$. In this case, the initial minimal presentation of G induces a minimal presentation of Γ

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{R} & \longrightarrow & \mathrm{F} & \xrightarrow{\varphi} & \mathrm{G} & \longrightarrow & 1 \\ & & & & & \searrow & \downarrow & & \\ & & & & & & \Gamma & & \end{array}$$

Suppose that $\Gamma = \mathrm{G}/\langle x_1, \cdots, x_m\rangle^{\mathrm{Norm}}$. Lift the x_i's to $y_i \in \mathrm{F}$ such that $\omega_{\mathrm{G}}(x_i) = \omega(y_i)$ for each i. Hence, $\Gamma = \mathrm{F}/\mathrm{R}'$, where $\mathrm{R}' = \mathrm{R}\langle y_1, \cdots, y_m\rangle^{\mathrm{Norm}}$. In particular, if $\mathrm{R} = \langle \rho_1, \cdots, \rho_r\rangle^{\mathrm{Norm}}$, then $\mathrm{R}' = \langle \rho_1, \cdots, \rho_r, y_1, \cdots, y_m\rangle^{\mathrm{Norm}}$. In this situation, we say that we have 'cut' the group G by the elements $y_1, \ldots, y_m$. Even if we have no additional information about the ρ_i's, the estimate $P_\Gamma(t) \leqslant 1 - dt + rt^2 + \sum_i t^{\omega(y_i)}$ is valid on the open unit interval.

1.2.2. Cutting of G_S. — Fixing a prime p and a number field K, we let S be the set of primes of K above p. Recall that G_S is the Galois group over K of the maximal p-extension of K unramified outside S. We want to consider some special quotients Γ of G_S of the type that were introduced in [**2**]. In [**2**] tame ramification was allowed, and then a quotient was taken. Here G_S is wildly ramified and the quotient we take will have abelian decomposition groups with wild but finite image, and hence finite image of inertia. This quotient of course corresponds to a sub-extension so we will use the term 'cut' to apply both to Galois groups and the corresponding tower of fields.

Each place $v \in S$ corresponds to (a conjugacy class of) a decomposition group and hence to some extension $\mathrm{K}_v/\mathbb{Q}_p$ of degree ef (in fact these fields are isomorphic as $\mathrm{K}/\mathbb{Q}$ is Galois). Then, as $\mu_p \subset \mathrm{K}_v$, the $\mathbb{F}_p$-vector space $\mathrm{K}_v^\times/(\mathrm{K}_v^\times)^p$ has dimension $ef + 2$, and local class field theory implies the Galois group of the maximal pro-p extension of K_v is generated by $ef+2$ elements. Thus the decomposition subgroup G_v of v in K_S/K is generated by at most $ef+2$ elements $z_{i,v}$. Consider now the commutators $[z_{i,v}, z_{j,v}]$ of all these elements; they clearly yield at most $\binom{ef+2}{2}$ distinct elements of G_S. Now we cut G_S by the closed normal subgroup

$$\mathrm{R}_0 = \langle [z_{i,v}, z_{j,v}] \mid 1 \leqslant i,j \leqslant ef+2; v \in S \rangle^{\mathrm{Norm}}$$

these elements generate, and denote by Γ_0 the corresponding quotient. As $\omega_{\mathrm{G}_S}([z_{i,v}, z_{j,v}]) \geqslant 2$, we have the following estimate for the Golod-Shafarevich polynomial of Γ_0:

$$P_{\Gamma_0}(t) \leqslant 1 - dt + rt^2 + g\binom{ef+2}{2}t^2, \qquad \forall t \in]0,1[.$$

Here $d = \dim H^1(\mathrm{G}_S, \mathbb{F}_p)$ and $r = \dim H^2(\mathrm{G}_S, \mathbb{F}_p)$. We note that the kernel R_0 of the surjection $\mathrm{G}_S \twoheadrightarrow \Gamma_0$ fixes the maximal sub-extension $\mathrm{K}_S^{loc-ab}/\mathrm{K}$ of K_S/K with abelian decomposition groups everywhere. Observe that $\mathrm{K}_S^{loc-ab}/\mathrm{K}$ contains the compositum of all $\mathbb{Z}_p$-extensions of K.

Next, we cut G_S a little bit further as follows. For each integer $k \geqslant 1$, define

$$\mathrm{R}_k = \mathrm{R}_0\langle z_{i,v}^{p^k} \mid 1 \leqslant i \leqslant ef+2, v \in S\rangle^{\mathrm{Norm}}.$$

Let Γ_k be the corresponding quotient of G_S and denote the fixed field of R_k by $\mathrm{K}_S^{[k]}$, so that $\Gamma_k = \mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathrm{K})$. Since $\omega_\Gamma(z_{i,v}^{p^k}) \geqslant p^k$ for all $z_{i,v}$, we observe that for $k \geqslant 1$,

$$P_{\Gamma_k}(t) \leqslant P_{\Gamma_0}(t) + g(ef+2)t^{p^k} \qquad \forall t \in]0,1[.$$

Suppose that there exists some $t_0 \in]0,1[$ such that $P_{\Gamma_0}(t_0) < 0$. Then, evidently for all sufficiently large k, $P_{\Gamma_0}(t_0) < 0 \implies P_{\Gamma_k}(t_0) < 0 \implies \mathrm{K}_S^{[k]}/\mathrm{K}$ is infinite.

We now show there exists a finite Galois extension L/K such that the infinite extension $\mathrm{K}_S^{[k]}/\mathrm{L}$ is unramified everywhere. We need a lemma.

Lemma 1.4. — *With the notation as above, fix an integer $k \geqslant 1$, set $\mathrm{K}_{(0)} = \mathrm{K}$ and for $i \geqslant 1$, define $\mathrm{K}_{(i+1)}$ to be the compositum of all $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathrm{K}_{(i)}$ contained in $\mathrm{K}_S^{[k]}$. Set $N_n = \mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathrm{K}_{(n)})$. Then $\bigcap_{n=0}^{\infty} N_n = \{1\}$.*

Proof. — It suffices to show $\bigcup_{n=0}^{\infty} \mathrm{K}_{(n)} = \mathrm{K}_S^{[k]}$. Let $\alpha \in \mathrm{K}_S^{[k]}$. The Galois closure $\mathrm{M}(\alpha)$ over K of $\mathrm{K}(\alpha)$ has Galois group a finite p-group. The solvability of finite p-groups implies that $\alpha \in \mathrm{K}_{(r)}$ for some r. ☐

Proposition 1.5. — *With the above notation, suppose $\mathrm{K}/\mathbb{Q}$ is Galois and $\mathrm{K}_S^{[k]}/\mathrm{K}$ is infinite. Then there exists a finite subextension L/K of $\mathrm{K}_S^{[k]}/\mathrm{K}$ which is Galois over $\mathbb{Q}$, has an infinite Hilbert p-class field tower, and has the property that all primes above p split completely from L to $\mathrm{K}_S^{[k]}$.*

Proof. — Since $\mathrm{K}/\mathbb{Q}$ is Galois and S is $\mathrm{Gal}(\mathrm{K}/\mathbb{Q})$-invariant, the fields K_S, $\mathrm{K}_S^{[k]}$ and $\mathrm{K}_{(n)}$ are all Galois over $\mathbb{Q}$.

Let D be a (finite!) decomposition group above $v|p$ in $\mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathrm{K})$. Suppose now that $\mathrm{N}_n \cap D$ is nontrivial for all n. As these intersections are finite and decreasing in n, if they are all nontrivial, they stabilize at a finite nontrivial group, in which case $\bigcap_{n=0}^{\infty} N_n$ is nontrivial, contradicting Lemma 1.4. Thus there exists an m that $N_m \cap D = \{1\}$.

Since $\mathrm{K}_{(m)}/\mathbb{Q}$ is Galois, N_m intersects trivially with all $\mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathbb{Q})$-conjugates of D. We can then take $\mathrm{L} = \mathrm{K}_{(m)}$ and all decomposition groups above p in $\mathrm{Gal}(\mathrm{K}_S^{[k]}/\mathrm{L})$ are trivial so primes above p split completely from L to $\mathrm{K}_S^{[k]}$. ☐

2. Proof

In Proposition 2.1 below we give a general criterion for a number field L to exist satisfying the conclusion of the Theorem. We then prove the Theorem by giving a fairly explicit conditions under which the criterion of Proposition 2.1 holds.

Proposition 2.1. — *Let* $\mathrm{K}/\mathbb{Q}$ *be finite Galois and totally complex with* $\mu_p \subset \mathrm{K}$. *Let* S *be the set of primes of* K *dividing* p. *Assume the cardinality of* S*, denoted* g_{K}*, is at least* 8. *Then there exists a finite extension* L *of* K *contained in* K_S *which is Galois over* $\mathbb{Q}$ *and has infinite Hilbert* p*-class field tower.*

Proof. — Let $\mathrm{H} = K_{\varnothing}$ be the "top" of the p-Hilbert class field tower of K, i.e. the maximal unramified p-extension of K. If H/K is infinite, we are done, so suppose $[\mathrm{H} : \mathrm{K}] < \infty$. Recall $\mathrm{H}_S = \mathrm{K}_S$. Note that H has class number prime to p so by Theorem 1.1, working over H,

$$\dim H^1(\mathrm{Gal}(\mathrm{H}_S/\mathrm{H}), \mathbb{F}_p) = g_{\mathrm{H}}\left(\frac{e_{\mathrm{H}} f_{\mathrm{H}}}{2} + 1\right) \text{ and } \dim H^2(\mathrm{Gal}(\mathrm{H}_S/\mathrm{H}), \mathbb{F}_p) = g_{\mathrm{H}} - 1.$$

As in §1.2.2, consider the quotient Γ_0 of $\mathrm{Gal}(\mathrm{H}_S/\mathrm{H})$ by the normal subgroup generated by the local commutators at each $v \in S$ (all commutators of generators of the decomposition group at v); one has $\binom{e_{\mathrm{H}} f_{\mathrm{H}}+2}{2}$ such commutators. We have

$$P_{\Gamma_0}(t) \leqslant 1 - \dim H^1(\Gamma_0, \mathbb{F}_p)t + \dim H^2(\Gamma_0, \mathbb{F}_p)t^2 \leqslant 1 - dt + rt^2 \qquad \forall t \in]0, 1[,$$

where $d := g_{\mathrm{H}}\left(\frac{e_{\mathrm{H}} f_{\mathrm{H}}}{2} + 1\right)$, and $r := g_{\mathrm{H}} - 1 + g_{\mathrm{H}}\frac{(e_{\mathrm{H}} f_{\mathrm{H}}+2)(e_{\mathrm{H}} f_{\mathrm{H}}+1)}{2}$. The first inequality we have seen simply comes from the fact that all relations have depth at least 2. For the second inequality, we note first that Γ_0 and $\mathrm{Gal}(\mathrm{H}_S/\mathrm{H})$ have the same generator rank, namely d; moreover, since Γ_0 is constructed using a presentation on d generators using at most r relations, we have $\dim H^2(\Gamma_0, \mathbb{F}_p) \leqslant r$.

Clearly $d/2r < 1$, and $P_{\Gamma_0}(d/2r) \leqslant 1 - \frac{d^2}{4r}$. If $P_{\Gamma_0}(d/2r) < 0$, then one has, as in §1.2.2, room to cut by some large p-power of the generators of the abelian decomposition group at $v|p$ and obtain an infinite extension of K whose decomposition groups at p are finite. Proposition 1.5 would then give the result.

It thus suffices to check that $4r < d^2$, or equivalently

$$16(g_{\mathrm{H}} - 1) + 8g_H(e_{\mathrm{H}} f_{\mathrm{H}} + 2)(e_{\mathrm{H}} f_{\mathrm{H}} + 1) \overset{?}{<} g_{\mathrm{H}}^2(e_{\mathrm{H}} f_{\mathrm{H}} + 2)^2.$$

Replacing the $16(g_{\mathrm{H}}-1)$ term on the left by $16g_{\mathrm{H}}$ and dividing by g_{H}, and setting $x = e_{\mathrm{H}}f_{\mathrm{H}}$, it now suffices to verify

$$16 + 8(x+2)(x+1) \overset{?}{<} g_{\mathrm{H}}(x+2)^2$$

for all $x \geqslant 1$. It is easy to see that this inequality holds as long as $g_{\mathrm{H}} \geqslant 8$. Since $g_{\mathrm{H}} \geqslant g_{\mathrm{K}}$, we have therefore checked that $P_{\Gamma_0}(d/2r) < 0$. By Proposition 1.5 we conclude that $\mathrm{K}_S^{[k]}/\mathrm{K}$ is infinite for all sufficiently large k and the field L with the desired properties exists. ☐

Proof of Theorem : Recall that the principal prime $\mathfrak{p} = (1-\zeta_{p^s})$ of $\mathbb{Q}(\zeta_{p^s})$ is the unique prime dividing p and by class field theory $\mathfrak{p}$ splits completely in the Hilbert class field H of $\mathbb{Q}(\zeta_{p^s})$. Thus if the class group has order at least 8, Proposition 2.1 applied to the solvable number field H gives the result.

In the proof of [**17**, Corollary 11.17], the class number of $\mathbb{Q}(\zeta_{p^s})$ is shown to be at least 10^9 for $\phi(p^s) = p^{s-1}(p-1) > 220$. Choosing $s \geqslant 9$ for *any* p completes the proof of the Theorem.

A slightly more detailed analysis using Table §3 of [**17**] shows the fields below suffice:

p	K	$h_{\mathrm{K}}(= g_{\mathrm{K}})$
$p > 23$	$\mathrm{K} = \mathbb{Q}(\zeta_p)$	$\geqslant 8$
$7 \leqslant p \leqslant 23$	$\mathrm{K} = \mathbb{Q}(\zeta_{p^2})$	$\geqslant 43$
$p = 5$	$\mathrm{K} = \mathbb{Q}(\zeta_{125})$	57708445601
$p = 3$	$\mathrm{K} = \mathbb{Q}(\zeta_{81})$	2593
$p = 2$	$\mathrm{K} = \mathbb{Q}(\zeta_{64})$	17

☐

Remark 2.2. — In [**4**] a proof of the Theorem for $p = 2, 3$ and 5 was announced. There are two cases there: Case I, where the Hilbert class field tower is infinite; and Case II, where ramification is allowed at one prime above p in the Hilbert class field H and a $\mathbb{Z}/p$-extension of H ramified at exactly this prime is used. Gras has given a criterion for such an extension to exist: see [**1**, Chapter V, Corollary 2.4.4]. Gras' criterion is not verified in [**4**]. Given the size of the number fields H, it seems very difficult to do so. We therefore we regard the results of [**4**] as incomplete. See [**6**] for a related description of the same error.

Our proof is partially modeled on the ideas of [**4**], namely considering the Hilbert class field of a cyclotomic field.

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Farshid Hajir, Christian Maire, Ravi Ramakrishna, Department of Mathematics, University of Massachussetts, Amherst, MA 01003, USA • FEMTO-ST Institute, Université Bourgogne Franche-Comté, CNRS, 15B avenue des Montboucons, 25000 Besançon, FRANCE • Department of Mathematics, Cornell University, Ithaca, NY 14853 USA • *E-mail :* hajir@math.umass.edu, christian.maire@univ-fcomte.fr, ravi@math.cornell.edu