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DYNAMICS: ARCHITECTONICS IN (1+3) DIMENSIONS

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Abstract

In previous articles have been developed an architectonical approach that goes to the source of the different analytical methods derived in the history of physics. The attention being mainly focused on the formalization of the underlying concepts, this construction was restricted to $(1 + 1)$ dimensions. This methodology is extended here to $(1 + 3)$ dimensions, in order to be in conformity with practical applications.

Preliminary remark. This article expresses in $(1 + 3)$ dimensions what was expressed in $(1 + 1)$ dimensions in three recent articles [1-3]:

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Readers who want to better grasp the logic of the underlying methodology can refer to these articles.

1. Introduction

A first article [1] relative to dynamics had been devoted to the development of a unifying architectonical Leibnizian formulation which leads to the quantitative solutions usually derived by use of analytical principles (variational, geometrical, group theoretical …), each one revealing a particular point of view.

A second article [2] went further, showing the possibility of deriving, on an equal footing, the analytical structures that lead to these solutions, so that the different analytical principles become theorems (i.e., these are deduced simultaneously instead of being postulated separately as usually done).

In a third article [3], a different strategy was adopted, revealing a certain hierarchy in the passage from the architectonical approach to the analytical formulations.

In order to be in conformity with practical applications this fourth article develops the passage from $(1+1)$ to $(1+3)$ dimensions, following and prolonging what was announced and presented succinctly, without proof, in Section 9 of the third article [3].

In [1] and [2], dynamics is obtained, in $(1+1)$ dimensions, through a constraint C imposed on the second-order operator $O^2 - O = Id/dx$ being a generator of conserved entities. It allows to determine the two conserved entities (energy E and impulse p) required to get a well-posed physical problem ($C = O^2 E$ with $p = OE$).

Similarly to Eqs. (1) of Ref. [2], we start from the architectonical structure, (expressed here in natural units: $c = 1$):

$$C = E = O^2E = Id/dx\,[Id/dx]E = I^2d^2E\big/dx^2 + I[dI/dx]\,dE/dx$$

$$\text{with } p = OE = IdE/dx\,. \quad (1)$$

These equations are under-determinate [indeterminate as to the points of view $(O = Id/dx,\ I$ being an arbitrary function of $x)$ and determinate for the worlds: here Einstein's world $(C = E)$, where the constraint C coincides with the so-called "relativistic mass" $M = E$ (in natural units) or $M = E\big/c^2$ (otherwise, as shown in [2] when $c \neq 1$)].

Thanks to a "filtering" procedure, the indeterminate points of view, corresponding to the couple: $(I,\, x)$ of non-conserved entities are eliminated in favor of the conserved entities represented by the couple: $(E,\, p)$. As shown in Eqs. (2) and (3) of Ref. [2], this procedure (also expressed here in natural units, $c = 1$) led to the well-determinate (easily integrable) structure:

$$C = E = pdp/dE \Rightarrow E = (m^2 + p^2)^{1/2} \quad (2)$$

which is intrinsic (independent of any point of view, usually expressed by a specific parameter, accounting for motion: velocity, celerity or rapidity).

In the previous articles, we have been content with a $(1 + 1)$ dimensional framework in order to well-identify and underline the basic ideas, following thus the authors who developed new rational points of view [4-6], since the second half of the 20th century.

This restriction is also used in the pedagogical works of: (i) relativistic physics where one passes from Newton's absolute time and inertial mass $(dt = d\tau$ and $M = m)$ to Lorentz invariance $(dt^2 - dx^2\big/c^2 = d\tau^2$ and $M^2 - p^2\big/c^2 = m^2$ with $M = E\big/c^2$) and (ii) quantum physics, with the wave-particle duality through: $p = h/\lambda$ and $E = h/T$, where λ and T indicate, respectively, the wave-length and the period. All these cases correspond effectively to $(1 + 1)$ dimensional frameworks with respect to time and space as well as energy and impulse.

However, for the daily practice of physics, it is not the concepts that are targeted but the applications which usually require to consider four-dimensionality. Besides, one of the criticisms that kept coming back during my presentations of the transition from the analytical to the architectonical framework inspired by the conceptualization of Leibniz, was the limitation of this architectonical approach to $(1 + 1)$ dimensions while physical applications most often occur in $(1 + 3)$ dimensions. We propose here to overcome this critic by passing from $(1 + 1)$ to $(1 + 3)$ dimensions.

We shall proceed progressively by firstly extending the intrinsic Eq. (2) and then highlighting the existence of four basic (remarkable, singular and operational) points of view which will appear in a natural way, by use of mathematical generativity, associated with singular ratios and remarkable identities. These basic points of view turn out to be intimately related to those derivable from the infinitely multiple architectonical approach, given here in natural units, and derived in Eqs. (14) of Ref. [1]:

$$E = d_\mu{}^2 E \Big/ dv_\mu{}^2 = I_\mu{}^2 d^2 E \Big/ dv_\mu{}^2 + [\, I_\mu dI_\mu / dv_\mu \,] \, dE/dv_\mu$$

$$\text{with } p = d_\mu E \big/ dv_\mu = I_\mu dE \big/ dv_\mu . \qquad (3)$$

This infinitely multiple structure will be developed at the end of this article in $(1 + 3)$ dimensions, prolonging thus the results derived in [3] in $(1 + 1)$ dimensions.

2. Extension of the Intrinsic Structure and of its four basic points of view

The $(1 + 1)$ intrinsic structure of Einsteinian dynamics given in (2) can be expressed by:

$$E = (m^2 + |p|^2)^{1/2} \quad \text{or} \quad E^2 - |p|^2 = m^2. \tag{4}$$

The notation $|p|$ indicates the absolute value of p. In $(1 + 3)$ dimensions, the absolute value of p is replaced by the modulus of p_i with $i = 1, 2, 3$. One is led thus to:

$$E = (m^2 + |\mathbf{p}|^2)^{1/2} \quad \text{or} \quad E^2 - |\mathbf{p}|^2 = m^2 \tag{5}$$

with

$$|\mathbf{p}| = (\mathbf{p} \cdot \mathbf{p})^{1/2} = (p_i p_i)^{1/2}. \tag{6}$$

In Section 3 of Ref. [1], entitled: "Admissible Dynamics", we derived the general intrinsic form: $pdp/dE = \lambda E + \gamma p + \eta$ from which we deduced Einsteinian dynamics: $pdp/dE = E$, before deriving an infinitely multiple architectonical framework, obtained through Eqs. (15c) of Ref. [1], expressing simultaneously an infinity of points of view: $v_\mu = (1/m)\int dp/(1 + p^2/m^2c^2)^{(\mu-1)/2}$.

It appeared that among this infinity of points of view, four of them $(\mu = 1, 2, 3, 4)$ turn out to be basic, the others corresponding to more or less complicated expressions of these four basic ones, three of which are well-known, written as: $p = mu = m\sinh(w) = m[v/(1 - v^2)^{1/2}]$. The conventional notations: u, w and v (celerity, rapidity and velocity) - used when dealing with the various analytical (geometrical, group theoretical and variational) formulations - replace the ordered architectonical notations (μ = 1, 2 and 4): $u = v_1$, $w = v_2$ and $v = v_4$, derivable from the above infinitely multiple expression of v_μ. The extension to $(1 + 3)$ dimensions of the infinity of points of view will be derived at the end of this article.

In this Section, we shall be more practical keeping in touch with the analytical methods that follow the line of thought usually developed by physicists. Thus, we directly introduce four points of view, starting from (5) that we express by:

$$E^2 - m^2|\mathbf{u}|^2 = L^2 + m^2|\mathbf{v}|^2 = m^2, \tag{7}$$

where we have introduced L such that $L^2/m^2 = m^2/E^2$, and set:

$$u_i = p_i/m \quad \text{and} \quad v_i = p_i/E \Rightarrow |\mathbf{u}| = |\mathbf{p}|/m \quad \text{and} \quad |\ \mathbf{v}\ | = |\mathbf{p}|/E. \tag{8}$$

The velocity and the celerity vectors v_i and u_i are usually encountered in space-time physics. They are defined here in a dynamical way.

The form $L^2 + m^2|\mathbf{v}|^2 = m^2$ may appear as artificial since L, contrary to E, does not correspond to a conserved entity. However, in the light of the Lagrange-Hamilton variational formalism, which reflects the usual rationality of physics, it turns out that L, the Lagrangian, is fundamental since it allows deducing the different conserved entities (here impulse: $p_i = \partial L/\partial v_i$ and energy $E = v_i\, \partial L/\partial v_i - L$). These $(1 + 3)$ dimensional expressions, attached to the variational formulation, will be derived below, jointly with other points of view, associated with the geometrical and group theoretical formulations.

In order to get the above-mentioned four basic points of view, let us add to v_i and u_i two other points of view suggested by Eqs. (7) that naturally lead to the introduction of hyperbolic and elliptic parameters, thanks to the identities:

$$[\cosh(|\mathbf{w}|)]^2 - [\sinh(|\mathbf{w}|)]^2 = [\cos(|\mathbf{y}|)]^2 + [\sin(|\mathbf{y}|)]^2 = 1, \tag{9}$$

where w_i and y_i are introduced by analogy to u_i and v_i getting thus the unifying notation:

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = (x_i x_i)^{1/2} \qquad \text{with} \qquad x_i = \{u_i, v_i, w_i, y_i\}. \tag{10}$$

Some elementary calculations and formal manipulations lead to:

$$E = m(1 + |\mathbf{u}|^2)^{1/2} = m\big/(1 - |\mathbf{v}|^2)^{1/2} = m\cosh(|\mathbf{w}|) = m/\cos(|\mathbf{y}|), \tag{11}$$

$$p_i = M_u u_i = M_v v_i = M_w w_i = M_y y_i, \tag{12}$$

where we have set:

$$M_u = m, \qquad M_v = m[\,1\big/(1 - |\mathbf{v}|^2)^{1/2}\,],$$

$$M_w = m[\,\sinh(|\mathbf{w}|)/|\mathbf{w}|\,], \qquad M_y = m[\,\tan(|\mathbf{y}|)/|\mathbf{y}|\,]. \tag{13}$$

The elimination of the motion parameters $x_i = \{u_i, v_i, w_i, y_i\}$, in (11)-(13), allows recovering the intrinsic structure (5). In the next Sections, we shall go beyond these solutions and consider the extended architectonical method that leads to these solutions. In particular, we distinguish between finite and infinite extensions.

3. Finite Extension of the Architectonical Structure (finite number of points of view)

3.1. Demonstration of some previously unproved results

Section 9 of Ref. [3] was devoted to $(1 + 3)$ dimensions, with some particular results proposed succinctly without demonstrations. We shall firstly demonstrate these particular results before moving to the general case that unifies and includes the three points of view developed in the history of physics.

We asserted without any proof, in Section 9 of Ref. [3], that the $(1+1)$ dimensional couple of entities: $(E,\, p)$ verifying: $\{E = m(1+u^2)^{1/2}$, $p = D\, dE/du = mu$, with $D = (1+u^2)^{1/2}\}$ should transform into:

$$\{E = m(1+|\mathbf{u}|^2)^{1/2},\;\; p_i = \Delta\, \partial E/\partial u_i = mu_i$$

$$\text{with } \Delta = (1+|\mathbf{u}|^2)^{1/2}\}, \tag{14}$$

where the operator d/du and the function $D(u)$ become, in $(1+3)$ dimensions, $\partial/\partial u_i$ and $\Delta(|\mathbf{u}|)$. To prove the coherence of (14), we start from the first expression of energy given in (11), corresponding to:

$$E = m(1+|\mathbf{u}|^2)^{1/2} = m(1+u_i u_i)^{1/2}. \tag{15}$$

Noting that one has:

$$\partial E/\partial u_i = \partial|\mathbf{u}|/\partial u_i\; dE/d|\mathbf{u}| \qquad \text{with} \qquad \partial|\mathbf{u}|/\partial u_i = u_i/|\mathbf{u}| = e_i \tag{16}$$

one gets:

$$\partial E/\partial u_i = e_i\; dE/d|\mathbf{u}|. \tag{17}$$

A simple calculation shows that the first expression of (12) corresponding to: $p_i = M_u u_i = mu_i$ verifies:

$$p_i = \Delta\, \partial E/\partial u_i = \Delta e_i\; dE/d|\mathbf{u}| = m|\mathbf{u}|e_i = mu_i \tag{18}$$

if and only if:

$$\Delta = (1+|\mathbf{u}|^2)^{1/2} \tag{19}$$

as announced in (14).

3.2. Extension of the under-determinate architectonical structure

In order to extend the under-determinate second-order differential equation $C = E = Id/dx\,[Id/dx]E$ given explicitly in (1), we shall benefit from the above particular results where the indeterminate couple (I, x) attached to (1) becomes well-determined (D, u), in $(1+1)$ dimensions, and (Δ, u_i), in $(1+3)$ dimensions. Having shown above that the operator d/du and the function $D(u)$ become, in $(1+3)$ dimensions, $\partial/\partial u_i$ and $\Delta(|\mathbf{u}|)$, the same will hold for the operator d/dx and the indeterminate function $I(x)$.

Indeed, the replacement of d/du and $D(u)$ by $\partial/\partial u_i$ and $\Delta(|\mathbf{u}|)$ and the account for the properties associated with the unit vector e_i, defined in (16):

$$e_i e_i = 1 \qquad \text{and} \qquad e_i de_i = 0 \tag{20}$$

transform: $C = E = Id/dx\,[Id/dx]E$ with $(I, x) = (D, u)$ into: $C = E = Dd/du\,[Dd/du]E$ whose $(1+3)$ counterpart corresponds to:

$$\begin{aligned} C = E &= [\Delta\,\partial/\partial u_i\,][\Delta\,\partial/\partial u_i\,]E = [\Delta e_i\; d/d|\mathbf{u}|\,][\Delta e_i\; d/d|\mathbf{u}|\,]E \\ &= [\Delta\, d/d|\mathbf{u}|\,][\Delta\, d/d|\mathbf{u}|\,]E. \end{aligned} \tag{21}$$

This extension associated with the celerity can be generalized to any arbitrary motion parameter, transforming thus the under-determinate structure (1) into:

$$\begin{aligned} C = E &= [I\,\partial/\partial x_i\,][I\,\partial/\partial x_i\,]E = [In_i\; d/d\,|\mathbf{x}|][In_i\; d/d\,|\mathbf{x}|]E \\ &= [I\; d/d\,|\mathbf{x}|][I\; d/d\,|\mathbf{x}|]E, \end{aligned} \tag{22}$$

where the indeterminate function I is now a function of the modulus of x_i, noted $|\mathbf{x}|$. As to n_i, it is a unit vector that verifies:

$$n_i = \partial|\mathbf{x}|/\partial x_i = x_i/|\mathbf{x}| \Rightarrow n_i n_i = 1 \quad \text{and} \quad n_i dn_i = 0. \tag{23}$$

While the entity Δ, given in (19), is well-determinate thanks to the decoupling procedure developed in [1-3], particularly in [3] that deals mainly with the geometrical approach, the entity I corresponds to an arbitrary function of $|\mathbf{x}|$, apt to receive numerous determinations, each constituting one point of view. One may refer to the Appendix to better grasp the extension in its full generality.

Last but not least, notice that the replacement of $C = E$ (Einstein's world) in (21) by $C = \lambda E + \gamma p + \eta$, allows dealing with various other possible worlds developed in Ref. [1]. Obviously, Einstein's dynamical world $C = E$ corresponds to the doubly particular case: $(\lambda, \gamma, \eta) = (1, 0, 0)$. Its extension to $(1 + 3)$ dimensions corresponds to: $C = \lambda E + \gamma \cdot \mathbf{p} + \eta$.

3.3 Systematical derivation of the three usual points of view (simplifying procedure)

The architectonical under-determinate second-order differential structure given in (1) (points of view dependent), is formally cumbersome and mathematically complicated to solve and to integrate. It is possible, however, to simplify it, as shown in [2], by introducing two new complementary entities F and G, having the same dimension as E. The three identifications: $G = E$, $G = F$ and $F = E$ led to three well-determinate points of view, as shown in Eqs. (7)-(15) of Ref. [2]. These turned out to be structurally identical to those postulated by the variational, the geometrical and the group theoretical formulations, developed progressively in the history of physics. We shall revisit this basic article, entitled: "Dynamics: From Analytical Principles to Architectonical Theorems", and extend it to $(1 + 3)$ dimensions, by reproducing the same strategy developed in Section 2 of Ref. [2], except that the entities E, F and G depend henceforth on the modulus of x_i, namely $|\mathbf{x}|$ instead of x or rather its absolute value, since E, F and G correspond, in $(1 + 1)$ dimensions, to even functions of x (see Appendix for details).

Thus, similarly to Eq. (4) of Ref. [2], which corresponds to: $C = O^2 E = O_2 F = O_1 G$, where O_2 and O_1 are simplifying second-order and first-order operators that apply to F and G, respectively, we shall extend this method to $(1 + 3)$ dimensions, leading to the following operators: O_{i2} and O_{i1}, as shown below, with $i = 1, 2$ and 3. In particular, $C = O^2 E = O[OE]$ transforms into: $C = O_i{}^2 E = O_i[O_i E]$ where the operator $O = Id/dx$ [I being a function of x, through its absolute value] transforms into: $O_i = I\,\partial/\partial x_i = [n_i\; Id/d\,|\mathbf{x}|]$, where I is henceforth a function of the modulus of x_i, namely $|\mathbf{x}|$.

We introduce thus two new entities associated with the above-mentioned operators so that the rather complicated structure transforms into a simpler one apt to be integrated by use of elementary methods of integration. On adapting the methodology developed in Ref. [2], to $(1 + 3)$ dimensions, one may write:

$$C = O_i{}^2 E = O_i O_i E = [I\,\partial/\partial x_i\,][I\,\partial/\partial x_i\,]E$$

$$= [n_i\; Id/d|\mathbf{x}|\,][n_i\; Id/d|\mathbf{x}|\,]E. \tag{24}$$

This expression extends (22) limited to Einstein's world $(C = E)$.

By introducing a new second-order operator O_{i2} associated with a new entity F such that $dF = IdE$ one is led to:

$$C = O_i{}^2 E = O_i O_i E = O_i[n_i\; d/d|\mathbf{x}|]\,F = O_{i2} F, \tag{25}$$

where one replaces, in $O_i O_i E$, the second operator $O_i = [n_i\ Id/d|\mathbf{x}|]$ by $[n_i\ d/d|\mathbf{x}|]$ and E by F. Since

$$p_i = O_i E = I\ \partial E/\partial x_i = n_i I\ dE/d|\mathbf{x}|, \tag{26}$$

on replacing IdE by dF, one gets:

$$p_i = \partial F/\partial x_i = n_i\ dF/d|\mathbf{x}|. \tag{27}$$

One also introduces a new first-order operator O_{i1} associated with a new entity G such that:

$$C = O_i^{\ 2} E = O_i O_i E = [\ n_i/|\mathbf{x}|\ ] O_i G = O_{i1} G, \tag{28}$$

where one replaces, in $O_i O_i E$, the first operator $O_i = [n_i\ Id/d|\mathbf{x}|]$ by $[\ n_i/|\mathbf{x}|\ ]$ and E by G.

One deduces from (25) and (28):

$$C = O_i^{\ 2} E = O_{i2} F = O_{i1} G \tag{29}$$

that explicitly correspond to:

$$C = [n_i I\ d/d|\mathbf{x}|][n_i I\ d/d|\mathbf{x}|]\,E = [n_i I\ d/d|\mathbf{x}|][n_i\ d/d|\mathbf{x}|]\,F$$
$$= [\ n_i/|\mathbf{x}|\ ][n_i I\ d/d|\mathbf{x}|]\,G. \tag{30}$$

On account of: $n_i n_i = 1$ and $n_i dn_i = 0$, one gets:

$$C = [I\ d/d|\mathbf{x}|][I\ d/d|\mathbf{x}|]\,E = [I\ d/d|\mathbf{x}|][\ d/d|\mathbf{x}|]\,F$$
$$= [\ 1/|\mathbf{x}|\ ][I\ d/d|\mathbf{x}|]\,G \tag{31}$$

or equivalently:

$$C = [I^2\ d^2 E/d|\mathbf{x}|^2\ ] + [I\ dI/d|\mathbf{x}|][\ dE/d|\mathbf{x}|]$$

$$= I\, d^2F\big/d|\mathbf{x}|^2 = [\, I/|\mathbf{x}|\,]\, dG/d|\mathbf{x}|. \tag{32}$$

This greatly simplifies the second-order differential equation which can be easily integrated by use of the same procedure applied in Section 2 of Ref. [2], leading to:

$$G = |\mathbf{x}| dF/d|\mathbf{x}| - F = |\mathbf{x}||\mathbf{p}| - F \qquad \text{with} \qquad |\mathbf{p}| = dF/d|\mathbf{x}|, \tag{33}$$

where we have accounted for (27).

Thanks to the perfect similarity between Eqs. (32)-(33) and Eqs. (5)-(6) of Ref. [2], except that x is replaced here by $|\mathbf{x}|$, we shall be able to deduce different results without reproducing the calculations already developed in Ref. [2]. Indeed, we have previously shown that by setting: $G = E$, $G = F$ and $F = E$, which are functions of $x = \{v, u, w\}$, respectively, the under-determinate structure becomes well-determinate in three different ways, yielding, respectively, the three points of view: $v = dE/dp$ (compatible with $p = dF/dv$ and $E = vdF/dv - F$), $p = mu$ (compatible with $pdu = udp$) and $p = dE/dw$.

By setting here $G = E$, $G = F$ and $F = E$, which are functions of $|\mathbf{x}| = \{|\mathbf{v}|, |\mathbf{u}|, |\mathbf{w}|\}$, respectively, these points of view transform, in a first step, into: $|\mathbf{v}| = dE/d|\mathbf{p}|$ (compatible with $|\mathbf{p}| = dF/d|\mathbf{v}|$ and $E = |\mathbf{v}|dF/d|\mathbf{v}| - F$), $|\mathbf{p}| = m|\mathbf{u}|$ (compatible with $|\mathbf{p}|d|\mathbf{u}| = |\mathbf{u}|d|\mathbf{p}|$) and $|\mathbf{p}| = dE/d|\mathbf{w}|$. In a second step, one is led to: $v_i = \partial E/\partial p_i$ (compatible with $p_i = \partial F/\partial v_i$ and $E = v_i\, \partial F/\partial v_i - F$), $p_i = mu_i$ (compatible with $p_i du_i = u_i dp_i$) and $p_i = \partial E/\partial w_i$.

In order to justify the passage from the expressions in terms of the modulus $|\mathbf{x}|$ to those in terms of the vectors x_i, let us note that since one has $n_i n_i = 1$, then $G = |\mathbf{x}|\, dF/d|\mathbf{x}| - F$ may also be written as:

$$G = |\mathbf{x}| n_i n_i \, dF/d|\mathbf{x}| - F = x_i \, \partial F/\partial x_i - F = x_i p_i - F$$

$$\text{with } p_i = \partial F/\partial x_i, \qquad (34)$$

where we have accounted for (27).

When $C = E$ (Einstein's dynamics), and in natural units $(c = l)$, the $(1 + 1)$ dimensional expressions of energy and impulse, given in Eqs. (9), (10) and (15) of Ref. [2], take the following $(1 + 3)$ dimensional forms:

$$E = m(1 + |\mathbf{u}|^2)^{1/2} = m/[(1 - |\mathbf{v}|^2)^{1/2}] = m[\cosh|\mathbf{w}|] \qquad (35a)$$

and

$$p_i = mu_i = m[v_i/[(1 - |\mathbf{v}|^2)^{1/2}] = m[\sinh(|\mathbf{w}|)/|\mathbf{w}|]w_i. \qquad (35b)$$

In Eqs. (11)-(14) of Ref. [2] we benefited from the definition of impulse: $p = \Gamma \, dE/du$, to deduce the expression: $\Gamma = (1 + u^2)^{1/2}$, which coincides with the Lorentz factor. We obtained thus: $p = mu$ and $E = m\Gamma$ with $\Gamma^2 - u^2 = 1$ [in natural units $(c = 1)$]. This allowed expressing the geometrical point of view (attached to u) in a compact way as follows: $\mathbf{p} = m\mathbf{u}$ with $\mathbf{u} \cdot \mathbf{u} = 1$, provided one sets: $\mathbf{u} = (\Gamma, u)$ and $\mathbf{p} = (E, p)$, with a Minkowskian signature $\eta = (1, -1)$, applied to the scalar product $\mathbf{u} \cdot \mathbf{u} = 1$.

This compact notation: $\mathbf{p} = m\mathbf{u}$ with $\mathbf{u} \cdot \mathbf{u} = 1$, remains formally valid in $(1 + 3)$ dimensions. The only difference appears in the expressions of the Minkowskian signature and the Lorentz factor which become equal to: $\eta = (1, -1, -1, -1)$ and $\Gamma = (1 + u_i u_i^2)^{1/2}$. As to the vectors: $\mathbf{u} = (\Gamma, u)$ and $\mathbf{p} = (E, p)$, they should be replaced by: $\mathbf{u} = (\Gamma, u_i)$ an $\mathbf{p} = (E, p_i)$ with $i = 1, 2, 3$.

4. Infinite Extension of the Architectonical Structure (infinity of points of view)

In Section 4 of Ref. [1], had been presented a systematic iterative method that allows deriving an infinity of points of view on Einstein's dynamics, in $(1+1)$ dimensions.

We shall show how one may extend this methodology to $(1+3)$ dimensions. To this end, one replaces Eqs. (14) of Ref. [1] which corresponds, in natural units $(c = 1)$ to:

$$E = d_\mu{}^2 E / dv_\mu{}^2 = I_\mu{}^2 d^2 E / dv_\mu{}^2 + [\, I_\mu dI_\mu / dv_\mu \,] dE/dv_\mu$$

$$\text{with } p = d_\mu E / dv_\mu = I_\mu dE / dv_\mu \qquad (36)$$

by the following structure

$$E = \partial_\mu{}^2 E / \partial v_{\mu i}{}^2 = I_\mu{}^2 \partial^2 E / \partial v_{\mu i}{}^2 + [\, I_\mu \partial I_\mu / \partial v_{\mu i} \,] \partial E / \partial v_{\mu i}$$

$$\text{with } p_i = \partial_\mu E / \partial v_{\mu i} = I_\mu \partial E / \partial v_{\mu i} \,, \qquad (37)$$

where I_μ is now a function of the modulus of $v_{\mu i}$ noted by $|\mathbf{v}_\mu| = (v_{\mu i} v_{\mu i})^{1/2}$.

By use of the already developed arguments relative to the even character of energy, given in the Appendix, one is led to:

$$E = d_\mu{}^2 E / d|\mathbf{v}_\mu|^2 = I_\mu{}^2 d^2 E / d|\mathbf{v}_\mu|^2 + [\, I_\mu dI_\mu / d|\mathbf{v}_\mu| \,] dE/d|\mathbf{v}_\mu|$$

$$\text{with } |p| = d_\mu E / d|\mathbf{v}_\mu| = I_\mu dE / d|\mathbf{v}_\mu|. \qquad (38)$$

Notice the perfect symmetry between (36) and (38), where v_μ and p are replaced by their modulus $|\mathbf{v}_\mu|$ and $|\mathbf{p}|$. Thus, the infinity of well-determined points of view obtained by the integral expressions given in (15c) of Ref. [1], which correspond to:

$$v_\mu = (1/m)\int dp/(1 + p^2/m^2)^{(\mu-1)/2} \tag{39}$$

should be replaced by:

$$|\mathbf{v}_\mu| = (1/m)\int d|\mathbf{p}|/(1 + |\mathbf{p}|^2/m^2)^{(\mu-1)/2}. \tag{40}$$

Each value of μ corresponds to a specific point of view. On account of the state of rest that verifies: $|\mathbf{p}| = 0,\ |\mathbf{v}_\mu| = 0\ \ \forall\mu,$ one is led to an infinity of well-determinate motion parameters v_μ, expressed in terms of impulse p. Indeed, after some calculations and manipulations that we do not reproduce here, one gets, among the infinity of points of view, the following four basic points of view:

$$|\mathbf{p}| = m|\mathbf{u}| = m\sinh(|\mathbf{w}|) = m\tan(|\mathbf{y}|) = m([\,|\mathbf{v}|/(1 - |\mathbf{v}|^2)^{1/2}\,], \tag{41}$$

where we have replaced the set of basic parameters $\{v_{1i}, v_{2i}, v_{3i}$ and $v_{4i}\}$ by $\{u_i, w_i, v_i$ and $y_i\}$. As to the remaining infinite set of parameters, it corresponds to more or less complicated combinations of the four basic ones.

The passage from $|\mathbf{p}|$ to p_i and from $|\mathbf{v}_\mu|$ to $v_{\mu i}$ is obtained by use of:

$$n_i dv_{\mu i} = n_i\, dp_i/Y^{\mu-1} \Leftrightarrow d|\mathbf{v}_\mu| = d|\mathbf{p}|/Y^{\mu-1}, \quad Y = (1 + |\mathbf{p}|^2/m^2)^{1/2} \tag{42}$$

which renders possible the determination of the expressions of $v_{\mu i}$. Indeed, since one has: $p_i/|\mathbf{p}| = v_{\mu i}/|\mathbf{v}_\mu| = n_i$, Eqs. (41) may be written as:

$$p_i = m|\mathbf{u}|n_i = m[\,|\mathbf{v}|/(1 - |\mathbf{v}|^2)^{1/2}\,]n_i = m\sinh(|\mathbf{w}|)n_i = m\tan(|\mathbf{y}|)n_i. \tag{43}$$

As to the expressions of energy, they can be obtained in different ways, the simplest one corresponding to the substitution of (44) into the intrinsic form given in (5), yielding:

$$E = m(1 - |\mathbf{u}^2|)^{1/2} = m\big/(1 - |\mathbf{v}|^2)^{1/2} = m\cosh(|\mathbf{w}|) = m/\cos(|\mathbf{y}|). \quad (44)$$

Eqs. (43)-(44) are equivalent to the ones derived in (12)-(13), thanks to the property: $n_i = p_i/|\mathbf{p}| = v_{\mu i}\big/|\mathbf{v}_{\mu}|$ with $\{v_{1i}, v_{2i}, v_{3i}$ and $v_{4i}\} = \{u_i, w_i, v_i$ and $y_i\}$ as indicated above.

5. Conclusion

In Refs. [1-3], our derivation of dynamics from the architectonical framework generalized and unified, in different ways, the works performed by authors like Barbour, Landau, Sampanthar, Lévy-Leblond, Provost and Comte, recalled in Refs. [1-6] of our basic paper [1]. But this unification that transforms the different analytical principles into theorems, as shown in [2], remained confined into $(1+1)$ dimensions, insufficient for many physical applications. Only Section 9 of Ref. [3] was devoted to $(1+3)$ dimensions, where some results have been proposed succinctly without demonstrations.

Let us finally note that we could have developed the general case from the start before deducing the consequences as in any axiomatico-deductive approach but we preferred to keep a direct contact with the daily practice of the physicist whose approach is at best analytical, when it is not simply empirical and/or heuristic.

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Appendix

Extension of even and odd functions of x to multidimensionality x_i

The aim of the first paragraph of this Appendix is to show that the expressions:

$$E = f(x) = f(-x) \text{ and } p = g(x) = -g(-x) \text{ with } x \in R$$

can be written as:

$$E = f(|x|) \qquad \text{and} \qquad p = h(|x|)x.$$

This paves the way to their extension to multidimensionality yielding:

$$E = f(|\mathbf{x}|) = f[(x_i x_i)^{1/2}] \quad \text{and} \quad p_i = h(|\mathbf{x}|)x_i = h[(x_i x_i)^{1/2}]x_i.$$

In the second paragraph, we show that $p = I(x)\,dE/dx$ extends to

$$p_i = I(|\mathbf{x}|)\,\partial E/\partial x_i = I(|\mathbf{x}|)\,(dE/d|x|)\,n_i \text{ with } n_i = x_i/|x|.$$

General properties of even and odd functions

Starting from regular even and odd functions in R:

$$E = f(x) = f(-x) = \sum A_{2n} x^{2n}$$

and

$$p = g(x) = -g(-x)\sum A_{2n+1} x^{2n+1},$$

we show that they can be written in a compact form as:

$$E = f(|x|) \text{ and } p = h(|x|)x,$$

where $|x|$ indicates the absolute value of x belonging to R. This writing results from:

$$E = \sum A_{2n} x^{2n} = \sum A_{2n} |x|^{2n}$$

and

$$g(x) = \sum A_{2n+1} x^{2n+1} = [\sum A_{2n+1} |x|^{2n}]x = h(|x|)x.$$

Thus, the even and odd properties: $f(x) = f(-x)$ and $g(x) = -g(-x)$ do not need to be specified anymore: they are included in the expressions of $E = f(|x|)$ and $p = h(|x|)x$.

This writing paves the way for the extension to multidimensionality. Indeed, the variable x, belonging to R, and its absolute value, noted by

$|x|$, transform into the vector x_i which belongs to R^n, (in dynamics $n = 3$), and its modulus, noted by $|\mathbf{x}| = (x_i x_i)^{1/2}$.

One gets thus:

$$E = f(|\mathbf{x}|) = f[(x_i x_i)^{1/2}] \text{ and } p_i = h(|\mathbf{x}|)x_i = h[(x_i x_i)^{1/2}]x_i.$$

Remark: On defining a unit vector by:

$$e_i = p_i/|\mathbf{p}|$$

and accounting for p_i expressed in terms of x_i, one gets:

$$e_i = [h(|\mathbf{x}|)x_i]/[h(|\mathbf{x}|)|x|] = x_i/|x| = n_i$$

getting thus:

$$n_i = p_i/|\mathbf{p}| = x_i/|\mathbf{x}|.$$

Extension of the expression of $p = I(x)\, dE/dx$

Let us show now that the extension of $p = I(x)\, dE/dx$ leads to the following result:

$$p_i = I(|\mathbf{x}|)\, \partial E/\partial x_i = I(|\mathbf{x}|)(\, dE/d|x|)\, n_i.$$

Since the derivative of an even function of x (here E) corresponds to an odd function of x, and since p is an odd function of x (as mentioned in the first paragraph), then the function $I(x)$ present in $p = I(x)\, dE/dx = I(x)E'(x)$, corresponds necessarily to an even function: $I(x) = I(-x)$. Indeed, the ratio between two odd functions [here $p(x)/E'(x) = I(x)$] corresponds to an even function. Thus, $I(x) = I(-x)$ can be written as $I(|x|)$ yielding thus:

$$p = I(|\mathbf{x}|)\, dE/dx.$$

On replacing the variable x belonging to R by the vector x_i belonging to R^3 the expression of E depends then on the three variables x_1, x_2 and x_3. One replaces thus dE/dx by $\partial E/\partial x_i$, getting:

$$p_i = I(|\mathbf{x}|)\, \partial E/\partial x_i.$$

Moreover, since E is an even function of x_i, it can be expressed in terms of the modulus $(|\mathbf{x}|)$ leading thus to:

$$\partial E/\partial x_i = (\, dE/d|\mathbf{x}|)\,[\partial|\mathbf{x}|/\partial x_i\,] = (\, dE/d|\mathbf{x}|)\, n_i,$$

where we have used the following property:

$$\partial|\mathbf{x}|/\partial x_i = x_i/|\mathbf{x}| = n_i.$$

One is led thus to:

$$p_i = I(|\mathbf{x}|)\, \partial E/\partial x_i = I(|\mathbf{x}|)(\, dE/d|\mathbf{x}|)\, n_i$$

which is none other than the above-mentioned result.