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Stabilisation of a Rotating Beam Clamped on a Moving Inertia with Strong Dissipation Feedback

Andrea Mattioni[1] and Yongxin Wu[1] and Yann Le Gorrec[1] and Hans Zwart[2]

***Abstract*— In this paper we consider the stabilization problem of a beam clamped on a moving inertia actuated by an external torque and force. The beam is modelled as a distributed parameter port-Hamiltonian system (PDEs), while the inertia as a finite dimensional port-Hamiltonian system (ODEs). The control inputs correspond to a torque applied by a rotating motor and a force applied by a linear motor. In this paper we propose the use of a *strong dissipation* term in the control law, consisting of the time derivative of the restoring force at the clamping point. After a change of variables, the closed loop system shows dissipation at the boundaries of the PDEs. In this preliminary work we show that the closed loop operator is the generator of a contraction C_0-semigroup in a special weighted space, with norm equivalent to the standard one. Further, we prove the asymptotic stability of the closed loop system and we show the effectiveness of the proposed control law in comparison with a PD controller with the help of numerical simulations.**

I. INTRODUCTION

Stability of flexible manipulators has received considerable attention over the last 50 years. This is because of the multitude of possible applications ranging from spatial manipulators to microgippers. In these practical scenarios, it is usually not possible to neglect the actuators' inertias, therefore the model consists of the interconnection of a set of PDEs with a set of ODEs. The external inputs consist of the torque and/or the force applied by the actuators, therefore they acts on the set of ODEs. For these class of systems, in [9] Gibson shows that is not possible to obtain exponential stability with the use of classical feedbacks. It is for this reason that a different type of feedback, usually referred to as *strong dissipation* feedback, has been proposed to improve the closed loop performances. The *strong dissipation* feedback has been used in [14] to exponentially stabilize a wave equation with dynamic boundary conditions or, as in [5], for the exponential stabilization of an Euler Bernoulli beam with a tip mass. We consider here the stabilisation of a rotating beam clamped on a moving inertia, with control acting on the inertia, stemming from the modelling of flexible micro-manipulators. For that purpose we use the port-Hamiltonian (PH) framework. In the last decades, the Hamilton's principle has been extended to the modelling and control of open physical systems and of distributed parameter systems [18]. This formalism has been adapted for the definition of boundary controlled PH systems, where a simple matrix condition allows to define a well-posed (in the Hadamard sense) problem [13]. A complete exposition with some further extensions of these first results can be found in [20], [11]. Well-posedness and stabilization problems have been studied in case of static feedback [19], dynamic linear feedback [2] and dynamic non-linear feedback [17]. This class of systems encloses a wide class of mechanical systems like the wave equation with a tip mass [4] as well as models of rotating or translating flexible beams [1].

In this paper we study the stabilization of a plant composed by a flexible beam connected at one side to a moving inertia controlled through external torque and force, and free at the other side. This example is motivated by the application of a flexible micro-gripper in the approaching phase before the contact with DNA bundle [16], in case the actuators' inertias cannot be neglected. The proposed model can also be easily adapted to the contact scenario by putting a subset of boundary conditions equal to zero, as in [8]. The paper is organized as follows: in the next section we start by deriving the PH model of the rotating-translating beam, and next we formulate the closed loop equations with the proposed control law; in section III we present the main results of this paper, *i.e.* the contraction C_0-semigroup generation and the asymptotic stability of the closed loop system; in Section IV are given numerical simulations. Finally, some concluding remarks and comments on future works are given in Section V.

II. PRELIMINARIES

A. Modelling

We consider the system described in Fig. 1 composed of a rotating beam clamped on one side to a moving inertia (mass in translation along a slider plus rotating inertia) and free on the other side. We denote with I_h the moment of inertia of the rotating motor, and with m the mass of the translating inertia. The rotative motor's angle $\theta(t)$ and the inertia's position $s(t)$, as well as the external force $f(t)$ and torque $\tau(t)$ are real functions of time. $\xi \in [0,L]$ identifies the spatial coordinate of the beam. The deflection of the beam at a point ξ and at a time t, in the rotating frame, is identified with $w(\xi,t)$, while $\phi(\xi,t)$ represents the rotation

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[1]Andrea Mattioni, Yongxin Wu and Yann Le Gorrec are with AS2M department of the FEMTO-ST research lab University of Bourgogne Franche-Compté, 24 rue Savary, 25000 Besançon, France. andrea.mattioni@femto-st.fr yongxin.wu@femto-st.fr yann.le.gorrec@ens2m.fr

[2]Hans Zwart is with the Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands, and with the Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. h.j.zwart@utwente.nl

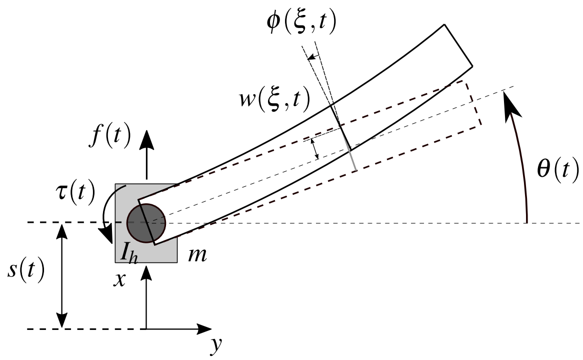

Fig. 1. Translating Rotating flexible Timoshenko's beam.

of a beam cross section relative to the angle $\theta(t)$. All the physical parameters are positive real.

For the sake of compactness we shall not explicit the time and space dependency of the variables, unless it is not clear from the context. Assuming $\dot{\theta}w \approx 0$ and $|\theta|$ small enough, the kinetic and potential energy of the system depicted in Figure 1 write

$$\begin{aligned} H_k = & \ \tfrac{1}{2}\int_0^L \left[\rho\left(\xi\dot{\theta} + \frac{\partial w}{\partial t} + \dot{s}\right)^2 + I_\rho\left(\dot{\theta} + \frac{\partial \phi}{\partial t}\right)^2\right] d\xi \\ & + \tfrac{1}{2} I_h \dot{\theta}^2 + \tfrac{1}{2} m \dot{s}^2 \\ H_p = & \ \tfrac{1}{2}\int_0^L \left[K\left(\frac{\partial w}{\partial \xi} - \phi\right)^2 + EI\left(\frac{\partial \phi}{\partial \xi}\right)^2\right] d\xi . \end{aligned} \tag{1}$$

Using the Hamilton's principle as in [8], it is possible to obtain the following set of dynamical equations

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta} + \dot{s}\right)\right) = \frac{\partial}{\partial \xi}\left(K\left(\frac{\partial w}{\partial \xi} - \phi\right)\right) \\ \frac{\partial}{\partial t}\left(I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right)\right) = \frac{\partial}{\partial \xi}\left(EI\frac{\partial \phi}{\partial \xi}\right) + K\left(\frac{\partial w}{\partial \xi} - \phi\right) \\ I_h \ddot{\theta} = +EI\frac{\partial \phi(0)}{\partial \xi} + \tau \\ m\ddot{s} = K\left(\frac{\partial w}{\partial z}(0) - \phi(0)\right) + f \end{cases} \tag{2}$$

with boundary conditions

$$\begin{aligned} & w(0,t) = 0 && \phi(0,t) = 0 \\ & \tfrac{\partial w}{\partial \xi}(L,t) - \phi(L,t) = 0 && \tfrac{\partial \phi}{\partial \xi}(L,t) = 0. \end{aligned} \tag{3}$$

The energy variables related to the two PDEs in (2) are defined as

$$\begin{aligned} & \varepsilon_t = \tfrac{\partial w}{\partial \xi} - \phi && p_t = \rho\left(\tfrac{\partial w}{\partial t} + \xi\dot{\theta} + \dot{s}\right) \\ & \varepsilon_r = \tfrac{\partial \phi}{\partial \xi} && p_r = I_\rho\left(\tfrac{\partial \phi}{\partial t} + \dot{\theta}\right). \end{aligned} \tag{4}$$

Then, the two PDEs can be rewritten as a 1-D PH system of the form

$$\dot{z} = \mathscr{J} z = P_1 \frac{\partial}{\partial \xi}(\mathscr{H} z) + P_0(\mathscr{H} z) \tag{5}$$

where $z = [p_t \ p_r \ \varepsilon_t \ \varepsilon_r]^T \in Z = L^2([0,L],\mathbb{R}^4)$, and the matrices P_0, P_1, $\mathscr{H}$ are defined as

$$P_1 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \quad P_0 = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{bmatrix} \quad \mathscr{H} = \begin{bmatrix} \frac{1}{\rho}&0&0&0 \\ 0&\frac{1}{I_\rho}&0&0 \\ 0&0&K&0 \\ 0&0&0&EI \end{bmatrix}. \tag{6}$$

The state space Z is equipped with the weighted L^2 internal product $\langle z_1, z_2\rangle_Z = \langle z_1, \mathscr{H} z_2\rangle_{L^2}$, in order to express the energy related to the flexible part of the system by $H = \frac{1}{2}||z||_Z^2 = \frac{1}{2}\langle z, z\rangle_Z$. In order to define a well-posed (in the Hadamard sense) boundary control system, we use the flow and effort variables according to [13]

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathscr{H} z)(0,t) \\ (\mathscr{H} z)(L,t) \end{bmatrix}. \tag{7}$$

Ignoring for the moment the boundary conditions in (3), we introduce the boundary input-output operators of the PH system (5) as

$$\begin{aligned} \mathscr{B}_1 z(t) &= W_{B1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = -\begin{bmatrix} \frac{1}{\rho} p_t(0,t) \\ \frac{1}{I_\rho} p_r(0,t) \end{bmatrix} = u_{z,1}(t) \\ \mathscr{B}_2 z(t) &= W_{B2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \begin{bmatrix} K\varepsilon_t(L,t) \\ EI\varepsilon_r(L,t) \end{bmatrix} = u_{z,2}(t) \\ \mathscr{C}_1 z(t) &= W_{C1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \begin{bmatrix} K\varepsilon_t(0,t) \\ EI\varepsilon_r(0,t) \end{bmatrix} = y_{z,1}(t) \\ \mathscr{C}_2 z(t) &= W_{C2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} p_t(L,t) \\ \frac{1}{I_\rho} p_r(L,t) \end{bmatrix} = y_{z,2}(t) \end{aligned} \tag{8}$$

where,

$$\begin{aligned} & W_{B1} = -\tfrac{\sqrt{2}}{2}\begin{bmatrix} 0&0&1&0&1&0&0&0 \\ 0&0&0&1&0&1&0&0 \end{bmatrix} && W_{B2} = \tfrac{\sqrt{2}}{2}\begin{bmatrix} -1&0&0&0&0&0&1&0 \\ 0&-1&0&0&0&0&0&1 \end{bmatrix} \\ & W_{C1} = \tfrac{\sqrt{2}}{2}\begin{bmatrix} 1&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&1 \end{bmatrix} && W_{C2} = \tfrac{\sqrt{2}}{2}\begin{bmatrix} 0&0&-1&0&1&0&0&0 \\ 0&0&0&-1&0&1&0&0 \end{bmatrix}. \end{aligned} \tag{9}$$

The input variables $u_{z,1}$ and $u_{z,2}$ correspond to the beam's boundary velocities at the clamped side and to the applied force and torque at the free side, respectively. The output variables $y_{z,1}$ and $y_{z,2}$ are the power conjugated[1] outputs of $u_{z,1}$ and $u_{z,2}$, respectively. The complete input and output operators are defined as the composition of the aforementioned operators

$$\begin{aligned} \mathscr{B} z &= \begin{bmatrix} \mathscr{B}_1 z \\ \mathscr{B}_2 z \end{bmatrix} = \begin{bmatrix} W_{B1} \\ W_{B2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = u_z \\ \mathscr{C} z &= \begin{bmatrix} \mathscr{C}_1 z \\ \mathscr{C}_2 z \end{bmatrix} = \begin{bmatrix} W_{C1} \\ W_{C2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = y_z. \end{aligned} \tag{10}$$

It is possible to prove that with the selected input and output variables (8), the PH system (5) is passive with storage function equal to the energy related to the flexible part

$$\dot{H} = u_z^T y_z = u_{z,1}^T y_{z,1} + u_{z,2}^T y_{z,2}. \tag{11}$$

We define the energy variables of the finite dimensional part of the system as $p_1 = m\dot{s}$, $p_2 = I_h\dot{\theta}$, $q_1 = s$ and $q_2 = \theta$, and consider u_p as the restoring forces of the infinite dimensional system and y_p the velocities of the finite dimensional part. Then the finite dimensional dynamics take the form

$$\begin{cases} \dot{p} = u_p + u \\ \dot{q} = M^{-1} p \\ y_p = M^{-1} p \end{cases} \tag{12}$$

where $p = [p_1 \ p_2]$, $q = [q_1 \ q_2]^T$, $M = \mathrm{diag}([m_1 \ m_2])$ with $m_1 = m$ and $m_2 = I_h$ and $u = [\tau \ f]^T$ corresponds to the vector

[1]A vector $u \in \mathbb{R}^n$ is *power conjugated* to a vector $y \in \mathbb{R}^n$ if the scalar product $\langle u, y\rangle_{\mathbb{R}^n}$ defines a power.

containing the external control inputs. The interconnection relations between the finite and the infinite dimensional part follows from the original boundary conditions (3) and the defined boundary variables (8)

$$u_{z,1} = -y_p \quad u_p = y_{z,1}, \tag{13}$$

where the velocity input of the PDEs equals the output of the ODEs, and the force input of the ODEs equals the output of the PDEs. Since the beam is free on the $\xi = L$ end, the remaining boundary input is set equal to zero $u_{z,2} = 0$.

B. Control design and closed loop system

The aim of the controller is to stabilize the system around a new slider's equilibrium position s^* and to reduce the vibrations of the flexible part. It is important to note that to respect the assumptions used to derive the model, the initial angle's condition should be close to zero $\theta(0) = 0$ and the control law must set to zero the equilibrium angle's position $\theta^* = 0$. Defining $q^* = [q_1^* \; q_2^*]^T = [s^* \; \theta^*]^T$, the proposed control law equals:

$$\begin{aligned} u = \; & -R_c M^{-1} p + (R_c M^{-1} K_p - I)\mathscr{C}_1 z - K(q - q^*) \\ & + K_p \tfrac{\partial}{\partial t}(\mathscr{C}_1 z), \end{aligned} \tag{14}$$

where the last term is known in the literature on stabilization of mixed PDEs-ODEs systems as *strong dissipation* feedback, and the matrices are defined such that $R_c = \text{diag}([r_1 \; r_2])$ and $K_p = \text{diag}([k_{p,1} \; k_{p,2}])$. In the literature on stabilization of flexible beams, the last term in (14) is known as "rate strain feedback" and can be computed calculating an approximated and filtrated version of the time derivative of the strain measurement as explained in [15]. Without loss of generality we consider the origin's stabilization problem (*i.e.* $q^* = 0$). With the extended state $x_f = \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ the closed loop system with the proposed control law becomes

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \zeta}(\mathscr{H} z) + P_0(\mathscr{H} z) \\ \dot{x}_f = (J - R)Qv + g_1 R_c M^{-1} K_p \mathscr{C}_1 z + g_1 K_p \frac{d}{dt}(\mathscr{C}_1 z) \end{cases} \tag{15}$$

where,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \; R = \begin{bmatrix} 0 & 0 \\ 0 & R_c \end{bmatrix} \; Q = \begin{bmatrix} K & 0 \\ 0 & M^{-1} \end{bmatrix} \; g_1 = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{16}$$

The total energy of the system, taking also into account the contribution of the "virtual" spring corresponding to the control action proportional to q, is

$$E = H_k + H_p + \frac{1}{2} q^T K q = \frac{1}{2}\langle z, z\rangle_Z + x_f^T Q x_f. \tag{17}$$

In order to analyse the closed loop system, we perform the change of variable $\eta = p - K_p \mathscr{C}_1(\mathscr{H} z)$, such to rewrite the system as

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \zeta}(\mathscr{H} z) + P_0(\mathscr{H} z) \\ \dot{v} = (J - R)Qv + g_2 M^{-1} K_p \mathscr{C}_1 z \end{cases} \tag{18}$$

where, $g_2 = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} q \\ \eta \end{bmatrix} \in \mathbb{R}^4$. This system can be written as a linear operator equation of the form

$$\dot{x} = \mathscr{A} x = \begin{bmatrix} \mathscr{J} & 0 \\ g_2 M^{-1} K_p \mathscr{C}_1 & (J - R)Q \end{bmatrix} x \tag{19}$$

where $x = \left[\begin{smallmatrix} z \\ v \end{smallmatrix}\right] \in L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ and domain defined as

$$\begin{aligned} D(\mathscr{A}) = \; & \{x \in X \mid z \in H^1([0,L], \mathbb{R}^4), \mathscr{B}_2 z = 0, \\ & \mathscr{B}_1 z = -M^{-1}(\eta + K_p \mathscr{C}_1 z)\}. \end{aligned} \tag{20}$$

The closed loop system highlights that the *strong dissipation* feedback can be considered as a dissipation action on the boundaries of the spatial domain.

III. SEMIGROUP GENERATION AND ASYMPTOTIC STABILITY

The closed-loop system (19)-(20) appears as a non power-preserving interconnection between two PH systems. The non power-preserving nature of the interconnection makes the dissipativity of $\mathscr{A}$ in the space $L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ equipped with the classical inner product difficult to show. Since the aim of this work is to show the stability of the system, we prove the contraction C_0-generation in $L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ equipped with a weighted inner product.

Theorem 3.1: There exists a weighted $L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ space such that the operator (19) with domain defined by (20) generates a contraction C_0-semigroup in this space, provided $r_i^2 > m_i k_i \quad i = \{1, 2\}$.

Proof: Consider $IP(L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4, \mathbb{R})$ the set of inner products from $L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ to $\mathbb{R}$. Define the set of inner products

$$\begin{aligned} \Gamma = \; & \{\gamma \in IP(L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4, \mathbb{R}) \mid \gamma(x_1, x_2, \Lambda) = \\ & \langle x_1, x_2\rangle_\Gamma, \langle x_1, x_2\rangle_\Gamma = \langle z_1, \mathscr{H} z_2\rangle_{L^2} + v_1^T M_v v_2\} \end{aligned} \tag{21}$$

where,

$$M_v = \begin{bmatrix} \Lambda R_c K_p^{-1} & K_p^{-1} \\ K_p^{-1} & 2\Lambda^{-1} R_c^{-1} K_p^{-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{22}$$

and, $\Lambda = diag([\alpha_1 \; \alpha_2])$ and $\alpha_i > 0$ for $i = \{1, 2\}$. The inner products in Γ are parametrized according to the α_i parameters. Using the Shur complements, it is easy to see that C and $A - B^T C^{-1} B$ are strictly positive definite matrices, from which it follows the positive definitiveness of M_v and that the inner product (21) is well-defined. Using the Lumer-Phillips' Theorem (see Theorem 6.1.7 of [11]), we have to show that $\gamma(x, \mathscr{A} x, \Lambda) \leq 0$ and that $Ran(\lambda I - \mathscr{A}) = L^2([0,L], \mathbb{R}^4) \times \mathbb{R}^4$ for some $\lambda > 0$. We start by the dissipativity of the operator $\mathscr{A}$, taking into account that $\mathscr{C}_1 z = y_{z,1}$,

$$\begin{aligned} \gamma(x, \mathscr{A} x, \Lambda) = \; & \langle z, \mathscr{J} z\rangle_Z + v^T M_v g_2 M^{-1} K_p y_{z,1} \\ & + v^T M_v (J - R)Qv \\ = \; & + \tfrac{1}{2}\langle z, \mathscr{J} z\rangle_Z + \tfrac{1}{2}\langle \mathscr{J} z, z\rangle_Z \\ & + v^T M_v g_2 M^{-1} K_p y_{z,1} + v^T M_v (J - R)Qv \\ = \; & + \tfrac{1}{2}\tfrac{d}{dt}||z(t)||^2 - 2\eta^T \Lambda K_p^{-1} M^{-1} \eta \\ & - 2\eta^T \Lambda^{-1} R_c^{-1} K_p^{-1} K q + \eta^T K_p^{-1} M^{-1} \eta \\ & + \eta^T M^{-1} y_{z,1} - q^T K_p^{-1} R_c M^{-1} \eta \\ & - q^T K_p^{-1} K q + q^T \Lambda R_c K_p^{-1} M^{-1} \eta \\ & + q^T \Lambda R_c M^{-1} y_{z,1}. \end{aligned} \tag{23}$$

Then, use (11) together with the domain definition (20) to obtain,

$$\begin{array}{rl}\gamma(x,\mathscr{A}x,\Lambda)\leq & -y_{z,1}^TM^{-1}K_py_{z,1}-q^TK_p^{-1}Kq\\ & -\eta^T(2\Lambda^{-1}K_p^{-1}M^{-1}-K_p^{-1}M^{-1})\eta\\ & +\eta^T(R_cK_p^{-1}M^{-1}(\Lambda-I)\\ & -2\Lambda^{-1}R_c^{-1}K_p^{-1}K)q+q^T\Lambda R_cM^{-1}y_{z,1}\end{array} \tag{24}$$

The latter inequality can be rewritten in matrix form

$$\gamma(x,\mathscr{A}x,\Lambda)\leq-\begin{bmatrix}\eta^T & q^T & y_{z,1}^T\end{bmatrix}\begin{bmatrix}P_{11} & P_{12} & 0\\ P_{12}^T & P_{22} & P_{23}\\ 0 & P_{23}^T & P_{33}\end{bmatrix}\begin{bmatrix}\eta\\ q\\ y_{z,1}\end{bmatrix} \tag{25}$$

with,

$$\begin{array}{l}P_{11}=(2\Lambda^{-1}-I)K_p^{-1}M^{-1}\\ P_{12}=\Lambda^{-1}R_c^{-1}K_p^{-1}K+\frac{1}{2}R_cK_p^{-1}M^{-1}(I-\Lambda)\\ P_{22}=K_p^{-1}K\\ P_{23}=-\frac{1}{2}\Lambda R_cM^{-1}\\ P_{33}=M^{-1}K_p.\end{array} \tag{26}$$

Since all the matrices are diagonal, and considering $y_{z,1}=[y_1\ y_2]^T$, the last inequality can be split in all its different components as

$$\gamma(x,\mathscr{A}x,\Lambda)\leq-\sum_{i=1}^{2}\begin{bmatrix}\eta_i & q_i & y_i\end{bmatrix}P_i\begin{bmatrix}\eta_i\\ q_i\\ y_i\end{bmatrix}, \tag{27}$$

where

$$P_i=\begin{bmatrix}\frac{(2-\alpha_i)}{m_ik_{p,i}\alpha_i} & \frac{k_i}{\alpha_ir_ik_{p,i}}+\frac{r_i(1-\alpha_i)}{2m_ik_{p,i}} & 0\\ \frac{k_i}{\alpha_ir_ik_{p,i}}+\frac{r_i(1-\alpha_i)}{2m_ik_{p,i}} & \frac{k_i}{k_{p,i}} & -\frac{\alpha_ir_i}{2m_i}\\ 0 & -\frac{\alpha_ir_i}{2m_i} & \frac{k_{p,i}}{m_i}\end{bmatrix}. \tag{28}$$

We first note that all the terms on the diagonal of (28) are positive definite as soon as $\alpha_i<2$. Afterwards, to show that the matrix P_i is semi-positive definite, we have to check that every principal minor is non-negative. We start from the determinant of the complete matrix, obtaining the condition to have it equal to zero

$$\begin{array}{rl}\det|P_i| & =\frac{k_i(2-\alpha_i)}{m_i^2k_{p,i}\alpha_i}-\left(\frac{k_i}{\alpha_ir_ik_{p,i}}+\frac{r_i(1-\alpha_i)}{2m_ik_{p,i}}\right)^2\frac{k_{p,i}}{m_i}\\ & -\frac{\alpha_i(2-\alpha_i)r_i^2}{4m_i^4k_{p,i}}=0.\end{array} \tag{29}$$

Hence, after some computations the former equation becomes

$$\begin{array}{rl}0= & 4m_i\alpha_ir_i^2k_i(2-\alpha_i)-4k_i^2m_i^2-\alpha_i^2(1-\alpha_i)^2r_i^4\\ & -4\alpha_ik_im_i(1-\alpha_i)r_i^2-\alpha_i^3(2-\alpha_i)r_i^4\\ = & -\alpha_i^2r_i^4+4k_im_i\alpha r_i^2-4k_i^2m_i^2.\end{array} \tag{30}$$

The solution of this equality writes

$$r_i^2=\frac{2m_i}{\alpha_i}k_i. \tag{31}$$

With some trivial computation, it is possible to show that also the three remaining principal minors have positive definite determinant as soon as $\alpha_i<2$. To sum up, for any choice of the parameters r_i and k_i $i=\{1,2\}$ such that $r_i^2>m_ik_i$, it is possible to define an inner product $\gamma(x_1,x_2,\Lambda)\in\Gamma$ whereby the operator $\mathscr{A}$ is dissipative.

The range condition consists in finding for a certain $\lambda>0$, a couple $(z,v)\in D(\mathscr{A})$ such that

$$\lambda\begin{bmatrix}z\\ v\end{bmatrix}-\mathscr{A}\begin{bmatrix}z\\ v\end{bmatrix}=\begin{bmatrix}f_z\\ f_v\end{bmatrix},\quad\forall\begin{bmatrix}f_z\\ f_v\end{bmatrix}\in L^2([0,L],\mathbb{R}^4)\times\mathbb{R}^4. \tag{32}$$

The range condition relies on the existence of the right inverse of the operator $(\mathscr{B}+K\mathscr{C})$ with $\mathscr{B}$ and $\mathscr{C}$ defined in (10) and K a singular matrix. The existence of this right inverse follows from the non-singularity of $\begin{bmatrix}W_B\\ W_C\end{bmatrix}$. ■

Now, we are interested in the C_0-semigroup generation of the operator (19)-(20) in the state space defined as $X=L^2([0,L],\mathbb{R}^4)\times\mathbb{R}^4$ equipped with the standard norm $||\cdot||=\sqrt{\langle z,z\rangle_{L^2}+v^Tv}$.

Corollary 1: The closed-loop operator (19)-(20) generates a C_0-semigroup in the state space X.

Proof: It is sufficient to show that the standard norm of X is equivalent to the one associated with (21), *i.e.* there exist $c,C\in\{r\in\mathbb{R}\mid r>0\}$ such that $c||\cdot||\leq||\cdot||_\Gamma\leq C||\cdot||$. It is easy to see that the previous inequalities are met if there exist $c_{\mathscr{H}},C_{\mathscr{H}},c_{Mv},C_{Mv}\in\{r\in\mathbb{R}\mid r>0\}$ such that,

$$c_{\mathscr{H}}I\leq\mathscr{H}\leq C_{\mathscr{H}}I\quad c_{Mv}I\leq M_v\leq C_{Mv}I. \tag{33}$$

The first inequalities follow directly from the definition of $\mathscr{H}$. The second inequalities follow from the strictly positive definiteness of M_v. ■

In the following Theorem we conclude about the asymptotic stability of the closed loop system (19)-(20) assuming the plant's approximate observability. In [12], it has been proven that a rotating Timoshenko beam is approximate observable in infinite time. It is consequently reasonable to assume approximate observability for the system under study, where we added a degree of freedom allowing the beam also to translate.

Theorem 3.2: Assume that the system obtained by the interconnection between (5) and (12) through the interconnection relations (13) is approximately observable. Then, the system (19) with domain (20) is asymptotically stable if $r_i^2\geq m_ik_i$, $i=\{1,2\}$.

Proof: Consider $LF(X,\mathbb{R}^+)$ the space of continuous functions, with continuous first derivative and locally positive definite from X to $\mathbb{R}^+$. Define the Lyapunov's functions set as

$$\Gamma_V=\{V\in LF(X,\mathbb{R}^+)\mid V(x,\Lambda)=\frac{1}{2}\langle z,\mathscr{H}z\rangle_{L^2}+\frac{1}{2}v^TM_vv\} \tag{34}$$

with M_v defined in (22). Because of the linearity of $\mathscr{A}$, the time derivative of a Lyapunov's function in Γ_V correspond to $\dot{V}(x,\Lambda)=\gamma(x,\mathscr{A}x,\Lambda)$ of (27), and is non-positive since $r_i^2\geq m_ik_i$, $i=\{1,2\}$. We substitute relation (31) in the time derivative of the Lyapunov's function to obtain

$$\begin{array}{rl}\dot{V}(x,\Lambda)\leq & -\frac{(2-\alpha_i)}{m_ik_{p,i}}\left(\frac{1}{\sqrt{\alpha}}\eta_i+\frac{\sqrt{\alpha}r_i}{2}q_i\right)^2\\ & -\frac{1}{m_i}\left(\sqrt{k_{p,i}}y_i-\frac{r_i\alpha_i}{2\sqrt{k_{p,i}q_i}}q_i\right)^2.\end{array} \tag{35}$$

TABLE I

SIMULATION PARAMETERS

Name	Variable	Value
Beam's Length	L	$1\ m$
Beam's Width	L_w	$0.1\ m$
Beam's Thickness	L_t	$0.02\ m$
Density	ρ	$950\ \frac{kg}{m^3}$
Young's modulus	E	$8\times10^8\ \frac{N}{m^2}$
Bulk's modulus	K	$1.7\times10^9\ \frac{N}{m^2}$
Hub's inertia	I	$1\ kg\cdot m^2$
Slider's mass	m	$1\ kg$
Beam's discretizing elements	n_b	50

To find the largest invariant subset Ω of $\{x\in X|\dot V(x,\Lambda)=0\}$, we substitute the relations $\eta_i=-\frac{\alpha_i r_i}{2}q_i$ and $y_i=\frac{r_i\alpha_i}{2k_{p,i}}q_i$ for $i=\{1,2\}$ in the closed loop system (19), and after some computations we get

$$\begin{cases}\dot z=\mathscr{J}z\\ \dot\eta=0\\ \dot q=0\end{cases}\tag{36}$$

with $\mathscr{B}z=0$, $\mathscr{C}_2z=\tilde y_z(t)$ and the other part of the boundary output

$$\mathscr{C}_1z=\begin{bmatrix}\frac{r_1\alpha_1}{2k_{p,1}}q_1\\ \frac{r_2\alpha_2}{2k_{p,2}}q_2\end{bmatrix}.\tag{37}$$

From (36), one obtains that η,q are constants $\eta=\eta^*$, $q=q^*$. Hence, z should verify

$$\begin{matrix}\dot z=\mathscr{J}z\\ \mathscr{B}z=0\end{matrix}\qquad \mathscr{C}_1z=\begin{bmatrix}\frac{r_1\alpha_1}{2k_{p,1}}q_1^*\\ \frac{r_2\alpha_2}{2k_{p,2}}q_2^*\end{bmatrix}.\tag{38}$$

From the approximate observability's assumption it is possible to prove that the only solution of (38) is the zero solution. Hence, it follows that $\Omega=\{0\}$. From Theorem 5.1.1 in [3] follows that the resolvent of the differential operator $\mathscr{A}$ is compact, hence it follows from theorem 11.2.25 of [7] that the solutions are pre-compact in X. From La Salle's invariance principle, we conclude that the closed loop system converges asymptotically to the origin. ■

IV. NUMERICAL SIMULATIONS

To appreciate the achievable performances of the control law proposed in Section II-B, we show the numerical simulations of the closed loop system (19)-(20) compared with the one obtained with the application of a classical PD controller of the form

$$u=-R_cM^{-1}p-Kq.\tag{39}$$

The numerical simulations have been carried out with a finite element approximation of the infinite dimensional PH system. In particular, it has been used the finite element discretization for infinite dimensional PH systems presented in [10]. Simulations are made in the Simulink® environment using the "ode23t" time integration algorithm. The set of parameters used for the simulations are listed in Table I.

In line with Section II-B, the desired equilibrium position is $q^*=0$. Figures 2 and 3 show respectively the beam movement and deformation evolution in time with the control actions (39) and (14). The control parameters used in the simulations are defined as

$$R_c=\begin{bmatrix}15&0\\0&15\end{bmatrix}\quad K=\begin{bmatrix}10&0\\0&10\end{bmatrix}\quad K_p=\begin{bmatrix}2&0\\0&2\end{bmatrix}.\tag{40}$$

In the proposed simulations the system is initialized with the initial conditions $q_1(0)=1$, $q_2(0)=p_1(0)=p_2(0)=0$ and $z(\xi,0)=0$.

Figure 4 shows that the closed loop system with the PD control (39) has a decreasing energy function (17), while it is not the case for the control law with the *strong dissipation* feedback (14). This is the reason why it was not possible to choose the closed loop energy as Lyapunov's

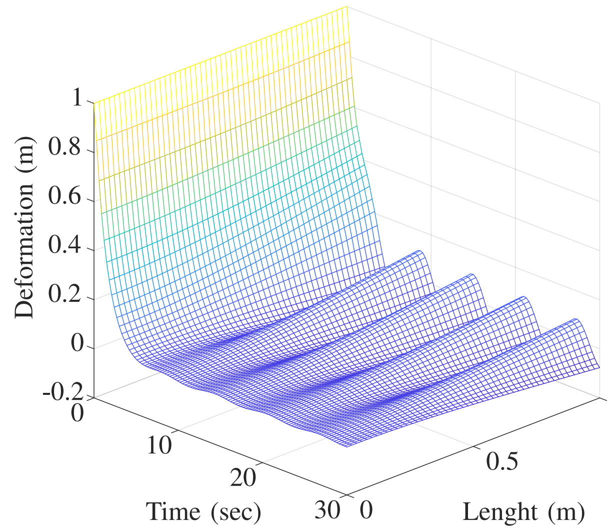

Fig. 2. Beam's deformation in time with PD control action.

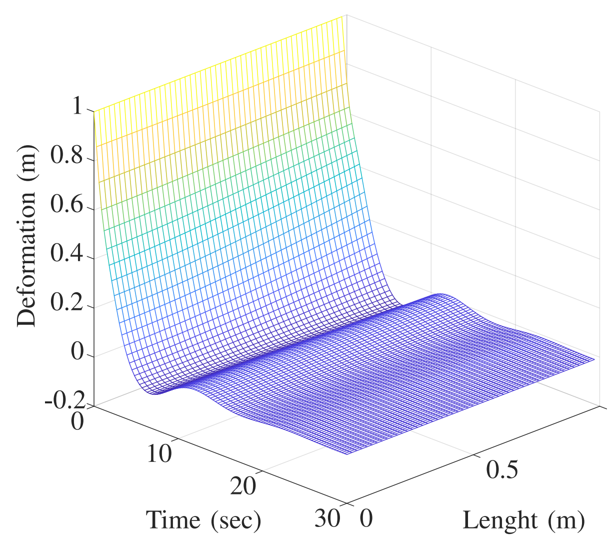

Fig. 3. Beam's deformations in time with *strong dissipation* feedback control action.

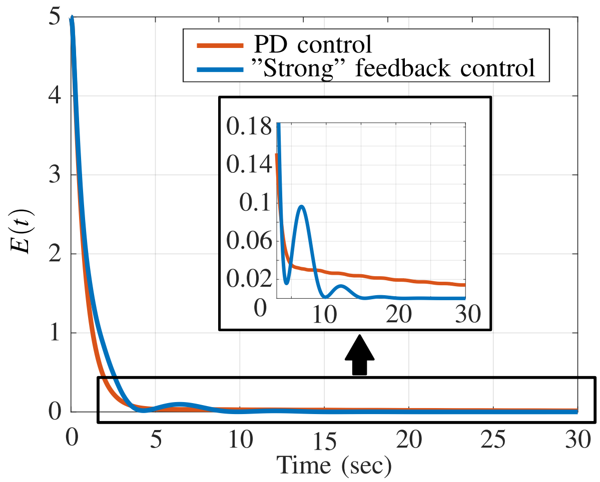

Fig. 4. Closed loop energy evolution in time.

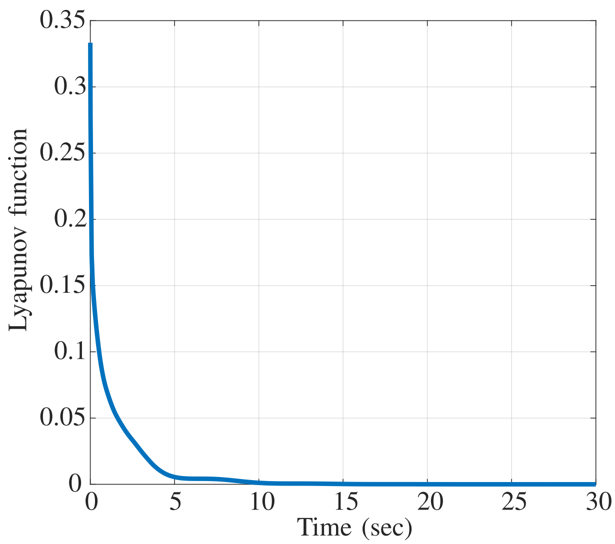

Fig. 5. Lyapunov function evolution in time.

function. Figure 5 shows instead that the chosen Lyapunov function $V(x,\Lambda) \in \Gamma_V$ (34) is decreasing in time when the control parameters are chosen such as to satisfy the sufficient conditions in Theorem 3.1. In this simulation scenario, the Λ matrix inside the inner product definition (21)-(22), results to be defined as

$$\Lambda = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{2mk_1}{r_1^2} & 0 \\ 0 & \frac{2Ik_2}{r_2^2} \end{bmatrix} = \begin{bmatrix} 0.0889 & 0 \\ 0 & 0.0889 \end{bmatrix}. \quad (41)$$

V. CONCLUSIONS

In this preliminary work, a PH model of a rotating and translating flexible beam has been derived and a control law making use of a *strong dissipation* term has been proposed. The contraction C_0-semigroup generation by the closed loop operator in a special weighted space has been proven making use of the Lumer-Phillips theorem. Further, the asymptotic stability of the closed loop system has been obtained. In order to validate the theoretical development, the numerical simulations of the closed loop system have been compared with the same system controlled by a PD controller. To conclude, our future research efforts will focus on the generalization of the proposed control law for a larger class of PH systems.

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