Bifurcation of symmetric domain walls for the Bénard-Rayleigh convection problem

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Abstract

We prove the existence of domain walls for the Bénard-Rayleigh convection problem. Our approach relies upon a spatial dynamics formulation of the hydrodynamic problem, a center manifold reduction, and a normal forms analysis of an eight-dimensional reduced system. Domain walls are constructed as heteroclinic solutions connecting suitably chosen periodic solutions of this reduced system.

Running head: Domain walls for the Bénard-Rayleigh convection problem

Keywords: Bénard-Rayleigh convection, rolls, domain walls, bifurcations

1 Introduction

The Bénard-Rayleigh convection is one of the most studied, both analytically and experimentally, and perhaps best understood, pattern-forming system. This hydrodynamic problem is concerned with the flow of a viscous fluid filling the region between two horizontal planes and heated from below. The difference of temperature between the two horizontal planes modifies the fluid density, tending to place the lighter fluid below the heavier one. Having an opposite effect, gravity induces, through the Archimedian force, an instability of the simple "conduction regime" leading to a "convective regime". While the fluid is at rest and the temperature depends linearly on the vertical coordinate in the conduction regime, various steady regular patterns, such as rolls, hexagons, or squares, are formed in the convective regime. The fluid viscosity prevents this instability up to a certain level, and there is a critical value of the temperature difference, below which nothing happens and above which a steady convective regime bifurcates. In dimensionless variables, this bifurcation occurs at a critical value of the Rayleigh number $\mathcal{R}_c$. The value $\mathcal{R}_c$, which depends on the chosen boundary conditions, has already been computed in the forties by

Pellew and Southwell [22]. Starting from the sixties, there has been extensive study of regular convective patterns and numerous mathematical existence results have been obtained. Without being exhaustive, we refer to the first works by Yudovich et al [27, 30, 31, 32], Rabinowitz [23], Görtler et al [7]; see also [16, 25], the monograph [17] for further references, and the recent work [2] on existence of quasipatterns.

The governing equations of the Bénard-Rayleigh convection consist of the Navier-Stokes system completed with an equation for energy conservation. We consider the Boussinesq approximation in which the dependency of the fluid density ρ on the temperature T is given by the relationship

$$\rho = \rho_0 \left(1 - \gamma(T - T_0)\right),$$

where γ is the (constant) volume expansion coefficient, T_0 and ρ_0 are the temperature and the density, respectively, at the lower plane. In Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$, where (x, y) are the horizontal coordinates and z is the vertical coordinate, after rescaling variables, the fluid occupies the domain $\mathbb{R}^2 \times (0, 1)$. Inside this domain, the particle velocity $\mathbf{V} = (V_x, V_y, V_z)$, the deviation of the temperature from the conduction profile θ, and the pressure p satisfy the system

$$\mathcal{R}^{-1/2}\Delta\mathbf{V} + \theta\mathbf{e}_z - \mathcal{P}^{-1}(\mathbf{V}\cdot\nabla)\mathbf{V} - \nabla p = 0, \tag{1.1}$$

$$\mathcal{R}^{-1/2}\Delta\theta + V_z - (\mathbf{V}\cdot\nabla)\theta = 0, \tag{1.2}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{1.3}$$

Here $\mathbf{e}_z = (0, 0, 1)$ is the unit vertical vector, and the dimensionless constants $\mathcal{R}$ and $\mathcal{P}$ are the Rayleigh and the Prandtl numbers, respectively, defined as

$$\mathcal{R} = \frac{\gamma g d^3(T_0 - T_1)}{\nu\kappa}, \quad \mathcal{P} = \frac{\nu}{\kappa}, \tag{1.4}$$

where ν is the kinematic viscosity, κ the thermal diffusivity, g the gravitational constant, d the distance between the planes, and T_1 the temperature at the upper plane. For notational simplicity, we set

$$\mu = \mathcal{R}^{1/2}.$$

This system is a steady version of the formulation derived in [17] in which $\mathbf{V}$ and θ are rescaled by $\mathcal{R}^{1/2}$ and $\mathcal{R}$, respectively. The equations (1.1)-(1.3) are completed by boundary conditions, and we consider here either the case of "rigid-rigid" boundary conditions:

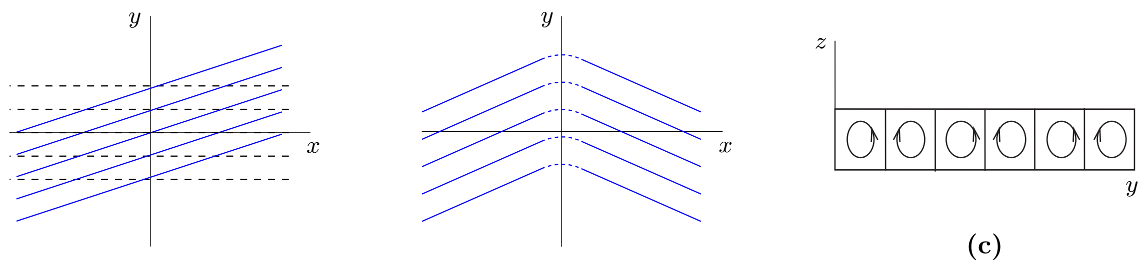
$$\mathbf{V}|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0, \tag{1.5}$$

or the case of "free-free" boundary conditions:

$$V_z|_{z=0,1} = \partial_z V_x|_{z=0,1} = \partial_z V_y|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0. \tag{1.6}$$

With these boundary conditions, the equations (1.1)-(1.3) are invariant under horizontal translations, reflections, and rotations, and the vertical reflection symmetry $z \mapsto 1 - z$. These symmetries play an important role in our analysis. We point out that the vertical symmetry only exists in these two cases where the boundary conditions are of the same type ("rigid-rigid" or "free-free"), the symmetry being lost in the case of "rigid-free" boundary conditions. We refer to [14, Vol. II] for a very complete discussion and bibliography on this problem, and in particular on the various geometries and boundary conditions.

At least locally, the most frequently observed patterns are convective rolls aligned along a certain direction (see Figure 1.1 (a) and (c)). However, such a pattern is only observed in a part of the apparatus, while the rolls take another direction in another part of the apparatus. The connection between the two regimes is quite sharp, occurring along a plane, and the two regimes of rolls make a definite angle between them (see Figure 1.1(b) and [11, 18, 4, 1] for experimental evidences not all on pure Bénard-Rayleigh convection). These line defects are referred to as domain walls or grain boundaries. In the present paper, we consider the case where two systems of rolls connect symmetrically with respect to a plane, even though such a perfectly symmetric pattern is not yet observed experimentally.



(c)

(a) (b)

Figure 1.1: In Cartesian coordinates (x, y, z), schematic plots of two-dimensional rolls (periodic in y and constant in x), rotated rolls, and domain walls. In the (x, y)-horizontal plane, **(a)** two-dimensional rolls (dashed lines) and rolls rotated by an angle α (solid lines); **(b)** symmetric domain walls constructed as heteroclinic connections between rolls rotated by opposite angles $\pm\alpha$. **(c)** In the vertical (y, z)-plane, streamlines of two-dimensional rolls (cross-section through the dashed lines in **(a)**).

The aim of this paper is to prove mathematically that such domain walls are indeed solutions of the steady Navier-Stokes-Boussinesq equations (1.1)-(1.3). Despite constant interest over the years, there is so far no existence result for these fluid dynamics equations. Many works gave tentative justifications of the existence of such patterns using formally derived amplitude equations (see [21, 20, 6] and the references therein). Beyond amplitude equations, the only mathematical results have been obtained for the Swift-Hohenberg equation, a toy model which exhibits many of the properties of the Bénard-Rayleigh convection problem [10, 26] (see also [19]). The domain walls constructed in [10] are symmetric, connecting rolls rotated by opposite angles $\pm\alpha$, for $\alpha \in (0, \pi/3)$. This result has been extended to arbitrary angles $\alpha \in (0, \pi/2)$ in [26]. We point out that there are no such results for domain walls which are not symmetric.

For the existence proof, we extend to the Navier-Stokes-Boussinesq system (1.1)-(1.3) the spatial dynamics approach used in [10] for the Swift-Hohenberg equation. The starting point of the analysis is a formulation of the steady problem as an infinite-dimensional dynamical system, in which one of the horizontal variables is taken as evolutionary variable. This idea goes back to the work of Kirchgässner [15], and since then it has been extensively used to prove the existence of nonlinear waves and patterns in many concrete problems arising in applied sciences, and in particular in fluid mechanics (see for instance [8] and the references therein). This infinite-dimensional dynamical system is typically ill-posed, but of interest are its small bounded solutions. An efficient way of finding these solutions is with the help of center-manifold techniques which reduce the infinite-dimensional system to a locally equivalent finite-dimensional dynamical system. An important property of this reduced system is that it preserves the symmetries of the original problem. Then normal forms and dynamical systems methods can be employed to construct bounded solutions of this reduced system.

We construct the domain walls as solutions of the steady Navier-Stokes-Boussinesq equations (1.1)-(1.3) which are periodic in the horizontal coordinate y (see Figure 1.1(b)). In our spatial dynamics formulation, we take as evolutionary variable the horizontal coordinate x and the boundary conditions, including the periodicity in y, determine the choice of the associated phase space and domain of definition of operators. An infinite-dimensional dynamical system is obtained as in the case of the Navier-Stokes equations in [12]. The rolls which are periodic in y and independent of x are then equilibria of this infinite-dimensional dynamical system, and through horizontal rotations they provide a family of periodic solutions. Domain walls are found as *heteroclinic solutions* of this infinite-dimensional dynamical system *connecting two symmetric periodic solutions* in this family.

We expect domain walls to bifurcate in the convective regime, at the same critical value $\mathcal{R}_c$ of the Rayleigh number as the rolls. In the bifurcation problem, we take the Rayleigh number $\mathcal{R}$ as bifurcation parameter, fix the Prandtl number $\mathcal{P}$ and also fix the wavenumber k_y in y of the solutions. We choose $k_y = k_c \cos\alpha$, where k_c is the wavenumber of the two-dimensional rolls bifurcating at $\mathcal{R}_c$ in the classical convection problem and α is a rotation angle. Then k_y represents the wavenumber in y of these bifurcating rolls rotated by the angle α.

The nature of the bifurcation is determined by the purely imaginary spectrum of the operator obtained by linearizing the dynamical system at the state of rest. Here, this operator has purely point spectrum and the number of its purely imaginary eigenvalues depends on the rotation angle α. We restrict to the simplest situation in which $\alpha \in (0, \pi/3)$. Then the linear operator possesses two pairs of complex conjugated purely imaginary eigenvalues $\pm ik_c$, $\pm ik_x$, where $\pm ik_c$ are algebraically double and geometrically simple, and $\pm ik_x$ are algebraically quadruple and geometrically double. In addition, 0 is a simple eigenvalue due to an invariance of our spatial dynamics formulation. (See Figure 1.2 for a plot of these eigenvalues and their continuation for Rayleigh numbers $\mathcal{R}$ close to $\mathcal{R}_c$). Except for the 0 eigenvalue, the other purely imaginary eigenvalues are of the same type as those found for the Swift-Hohenberg equation in [10]. Upon increasing the angle α in the interval $(\pi/3, \pi/2)$, the number of purely imaginary eigenvalues increases, and there are infinitely many eigenvalues when $\alpha = \pi/2$. For the Swift-Hohenberg equation, this case has been considered in [26].

The next step of our analysis is a center manifold reduction. The dimension of the reduced system being equal to the sum of the algebraic multiplicities of the purely imaginary above,

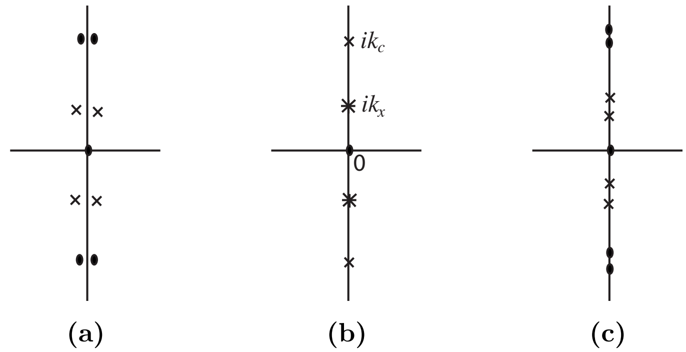


Figure 1.2: Spectrum of the linearized operator $\mathcal{L}_\mu$ lying on or near the imaginary axis, for a wave number $k_y = k_c \cos\alpha$ with $\alpha \in (0, \pi/3)$: **(a)** for $\mathcal{R} < \mathcal{R}_c$, **(b)** for $\mathcal{R} = \mathcal{R}_c$, **(c)** for $\mathcal{R} > \mathcal{R}_c$. Eigenvalues are either simple, double or quadruple denoted by a dot, a simple cross or a double cross, respectively.

we obtain here a reduced system of dimension 13. Due to the absence of the eigenvalue 0, the dimension of this reduced system was equal to 12 for the Swift-Hohenberg equation [10]. However, this additional dimension is easily eliminated, and then in the cases of "rigid-rigid" and "free-free" boundary conditions we use the reflection in the vertical coordinate to further eliminate 4 dimensions. This additional reduction of the dimension of the system has not been done in [10], but it is very helpful here, our reduced equations being much more complicated. The resulting system is 8-dimensional and the question of existence of domain walls consists now in the construction of a heteroclinic connection for this system.

In contrast to the Swift-Hohenberg equation, where the leading order terms of the reduced system have been computed explicitly, here the Navier-Stokes-Boussinesq equations are far too complicated to compute all these terms. We therefore need to extend the normal forms analysis of the particular reduced system found in [10] to a normal forms analysis for general 8-dimensional vector fields. On the other hand, the dimension of the reduced vector field being 8, it is too difficult to use the same methods for finding a general normal form, to any order, as usually done for lower dimensional vector fields (as for instance for four-dimensional vector fields in [8]). Instead, we restrict our computation of the normal form to cubic order, and using a standard normal form characterization, and the symmetries of the reduced system, we directly identify all possible resonant monomials, those which appear in the normal form. By this method it is not possible to obtain a normal form to any order, but a cubic normal form is enough for our purposes, and often in problems of this type.

The remaining part of the existence proof is based on the arguments from [10]. An appropriate change of variables allows us to identify a leading order system, determined by the cubic order terms of the normal form, for which the existence of a heteroclinic connection has been proved in [28]. Based on a variational method [24], this existence result requires that the quotient g of two coefficients in the cubic normal form is larger than 1. In [10] this quotient was equal to 2 and it was easily computed. Here, g depends on the angle α and on the Prandtl number $\mathcal{P}$ through complicated formulas (see (B.12)). We prove analytically that its value in the limit angle $\alpha = 0$ is also equal to 2, and for arbitrary angles and Prandtl numbers, we determine its numerical values using the package Maple. It turns out that indeed the condition $g > 1$ holds for all angles $\alpha \in (0, \pi/3)$ and all positive Prandtl numbers $\mathcal{P}$, for both "rigid-rigid" and "free-free" boundary conditions. The final step consists in showing that this heteroclinic connection found for the leading order system persists for the full system. We extend the persistence result in [10] from the case $g = 2$ to values $g \in (1, 4 + \sqrt{13})$, which implies the existence of domain walls for any Prandtl numbers $\mathcal{P}$ and any angles $\alpha \in (0, \alpha_*(\mathcal{P}))$, for some positive $\alpha_*(\mathcal{P}) \leqslant \pi/3$. A Maple computation allows us to identify the angles α and the Prandtl numbers $\mathcal{P}$ for which this property holds (see Figures 6.1 and 8.1). We point out that the persistence of the heteroclinic connection for $g \geqslant 4 + \sqrt{13}$ remains an open problem. We summarize our main result in the next theorem.

Theorem 1. *Consider the Navier-Stokes-Boussinesq system* (1.1)-(1.3) *with either "rigid-rigid" boundary conditions* (1.5) *or "free-free" boundary conditions* (1.6)*. Denote by $\mathcal{R}_c$ the critical Rayleigh number at which convective rolls with wavenumbers k_c bifurcate from the conduction state. Then for any Prandtl number $\mathcal{P}$, there exists a positive number $\alpha_*(\mathcal{P}) \leqslant \pi/3$ such that for angles $\alpha \in (0, \alpha_*(\mathcal{P}))$, a symmetric domain wall bifurcates for Rayleigh numbers $\mathcal{R} = \mathcal{R}_c + \epsilon$, with $\epsilon > 0$ sufficiently small. The domain wall connects two rotated rolls which are the rotations by opposite angles $\pm(\alpha + O(\epsilon))$ of a roll with wavenumber $k_c + O(\epsilon)$, continuously linked to the amplitude which is of order $O(\epsilon^{1/2})$.*

In our presentation we focus on the case of "rigid-rigid" boundary conditions. In Section 2 we briefly recall the classical convection problem and give a short proof of the existence of convective rolls. The spatial dynamics formulation is given in Section 3 and the bifurcation problem is analyzed in Section 4. The center manifold reduction is done in Section 5 and the normal forms analysis in Section 6. The existence of the heteroclinic connection is proved in Section 7. Finally, in Section 8, we discuss the differences which occur in the case of "free-free" boundary conditions, and briefly comment on the case of "rigid-free" boundary conditions. Some technical results, including the proof of the cubic normal form and the formula for g, are given in Appendices A and B.

Acknowledgments. M.H. was partially supported by the EUR EIPHI program (Contract No. ANR-17-EURE-0002). The authors thank the referee for the careful reading of the manuscript and the constructive comments and suggestions.

2 The classical Bénard-Rayleigh convection

In the classical approach, the steady system (1.1)-(1.3) is written in the form

$$\mathbf{L}_\mu \mathbf{u} + \mathbf{B}(\mathbf{u}, \mathbf{u}) = 0, \tag{2.1}$$

where $\mathbf{u} = (\mathbf{V}, \theta)$ lies in a suitable function space of divergence free velocity fields $\mathbf{V}$ and the pressure term in (1.1) is eliminated via a projection on the divergence free vector field (see, for instance, [8, Chapter 5]). Then $\mathbf{L}_\mu \mathbf{u}$ is the linear part and $\mathbf{B}(\mathbf{u}, \mathbf{u})$ is the nonlinear part, quadratic in $(\mathbf{V}, \theta)$, of the equations (1.1) and (1.2). The Prandtl number $\mathcal{P}$ which only appears in the quadratic part is kept fixed, and the square root μ of the Rayleigh number is taken as bifurcation parameter. We recall below some of the basic results which are used later in the paper.

2.1 Two-dimensional convection

The simple classical convection problem restricts to velocity fields $\mathbf{V} = (0, V_y, V_z)$ which are two-dimensional and functions which are independent of x and periodic in y. The corresponding function space for the system (2.1) is

$$\mathcal{H} = \{\mathbf{u} \in \{0\} \times (L^2_{per}(\Omega))^3 \; ; \; \nabla \cdot \mathbf{V} = 0, \; V_z = 0 \text{ on } z = 0, 1\},$$

where $\Omega = \mathbb{R} \times (0, 1)$ and the subscript per means that the functions are $2\pi/k$-periodic in y, for some fixed $k > 0$. The boundary conditions (1.5) are included in the domain $\mathcal{D}$ of the linear operator $\mathbf{L}_\mu$ by taking

$$\mathcal{D} = \{\mathbf{u} \in \{0\} \times (H^2_{per}(\Omega))^3 \; ; \; \nabla \cdot \mathbf{V} = 0, \; V_y = V_z = \theta = 0 \text{ on } z = 0, 1\}.$$

In this setting, the linear operator $\mathbf{L}_\mu$ is selfadjoint with compact resolvent and the quadratic operator $\mathbf{B}$ in (2.1) is symmetric and bounded from $\mathcal{D}$ to $\mathcal{H}$.

As a consequence of the invariance of the equations (1.1)-(1.3) under horizontal translations and reflections, the system (2.1) is $O(2)$-equivariant: both its linear and quadratic parts commute with the one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a \in \mathbb{R}/2\pi\mathbb{Z}}$ and the discrete symmetry $\mathbf{S}_2$ defined through

$$\boldsymbol{\tau}_a \mathbf{u}(y, z) = \mathbf{u}(y + a/k, z), \quad \mathbf{S}_2 \mathbf{u}(y, z) = (0, -V_y, V_z, \theta)(-y, z),$$

for any $\mathbf{u} \in \mathcal{H}$, and satisfying

$$\boldsymbol{\tau}_a \mathbf{S}_2 = \mathbf{S}_2 \boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_0 = \boldsymbol{\tau}_{2\pi} = \mathbb{I}.$$

An additional equivariance, under the action of the symmetry $\mathbf{S}_3$ defined through

$$\mathbf{S}_3 \mathbf{u}(y, z) = (0, V_y, -V_z, -\theta)(y, 1 - z),$$

which commutes with $\boldsymbol{\tau}_a$ and $\mathbf{S}_2$, is obtained from the invariance of the equations (1.1)-(1.3) under the vertical reflection $z \mapsto 1 - z$.

Instabilities and bifurcations are determined by the kernel of $\mathbf{L}_\mu$. Elements in the kernel of $\mathbf{L}_\mu$ are found by looking for solutions of the form $e^{iky} \widehat{\mathbf{u}}_k(z)$ for the linear equation

$$\mathbf{L}_\mu \mathbf{u} = 0, \tag{2.2}$$

and the boundary conditions $V_y = V_z = \theta = 0$ on $z = 0, 1$. A direct computation (see also [3]) gives

$$e^{iky} \widehat{\mathbf{u}}_k(z) = e^{iky} \begin{pmatrix} 0 \\ \frac{i}{k} DV \\ V \\ \theta \end{pmatrix}, \tag{2.3}$$

where $D = d/dz$ denotes the derivative with respect to z, and the functions $V = V(z)$ and $\theta = \theta(z)$ are real-valued solutions of the boundary value problem

$$(D^2 - k^2)^2 V = \mu k^2 \theta, \quad V = DV = 0 \text{ in } z = 0, 1, \tag{2.4}$$

$$(D^2 - k^2)\theta = -\mu V, \quad \theta = 0 \text{ in } z = 0, 1. \tag{2.5}$$

Yudovich [30] showed that, for any fixed $k > 0$, there is a countable sequence of parameter values $\mu_0(k) < \mu_1(k) < \mu_2(k) < \dots$ for which the boundary value problem (2.4)-(2.5) has a unique, up to a multiplicative constant, nontrivial solution (V_j, θ_j), and that the function V_0 is positive for $\mu = \mu_0(k)$. The vertical reflection symmetry $z \mapsto 1 - z$ further implies that V_0 is symmetric with respect to $z = 1/2$. The functions $\mu_j(k)$ are analytic in k and in an analogous case Yudovich [29] showed that they tend to ∞ as k tends to 0 or ∞. Of particular interest for the classical bifurcation problem, and also in our context, is the global minimum of $\mu_0(k)$. Combining analytical arguments and numerical calculations, Pellew and Southwell [22] computed a unique global minimum $\mu_c = \mu_0(k_c)$, for some $k = k_c$, but a complete analytical proof of this property is not available, so far. Solving the boundary value problem (2.4)-(2.5) using the symbolic package Maple leads to the numerical values

$$k_c \approx 3.116, \quad \mu_c \approx 41.325, \quad \mu_0''(k_c) \approx 6.265, \tag{2.6}$$

which are consistent with the ones found in [22].

Going back to the kernel of $\mathbf{L}_\mu$, as expected by the general theory of $O(2)$-equivariant systems, for $\mu = \mu_0(k)$ and any k sufficiently close to the minimum k_c, the kernel of $\mathbf{L}_{\mu_0(k)}$ is two-dimensional and spanned by the vectors

$$\boldsymbol{\xi}_0 = e^{iky}\widehat{\mathbf{u}}_k(z), \quad \overline{\boldsymbol{\xi}_0} = e^{-iky}\overline{\widehat{\mathbf{u}}_k}(z), \tag{2.7}$$

satisfying

$$\boldsymbol{\tau}_a \boldsymbol{\xi}_0 = e^{ia}\boldsymbol{\xi}_0, \quad \mathbf{S}_2\boldsymbol{\xi}_0 = \overline{\boldsymbol{\xi}_0}, \quad \mathbf{S}_3\boldsymbol{\xi}_0 = -\boldsymbol{\xi}_0.$$

Since the operator has compact resolvent, this shows that 0 is an isolated double semi-simple eigenvalue of $\mathbf{L}_{\mu_0(k)}$. Furthermore, all other eigenvalues are negative, so that the selfadjoint operator $\mathbf{L}_{\mu_0(k)}$ is nonpositive with a two-dimensional kernel. This property is a key ingredient in the proof of existence of rolls, which bifurcate from the trivial solution at $\mu = \mu_0(k)$, for any fixed k sufficiently close to k_c, in a steady bifurcation with $O(2)$ symmetry.

2.2 Existence of rolls

We give below a short and simple proof of the existence of convective rolls. This type of proof was first made by Yudovich [31].

The $O(2)$ symmetry of the system (2.1) allows to restrict the existence proof to solutions $\mathbf{u}$ which are invariant under the action of $\mathbf{S}_2$, and then the one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}/2\pi\mathbb{Z}}$ gives the non-symmetric solutions (a "circle" of solutions). Using the Lyapunov-Schmidt method, symmetric rolls can be constructed as convergent series in $\mathcal{D}$, under the form

$$\mathbf{u} = \sum_{n\in\mathbb{N}} \delta^n \mathbf{u}_n, \quad \text{for} \quad \mu = \mu_0(k) + \sum_{n\in\mathbb{N}} \delta^n \mu_n, \tag{2.8}$$

and fixed k close enough to k_c. We insert these expansions into (2.1), and solve the resulting equations at orders δ, δ^2 and δ^3.

The equality at order δ shows that $\mathbf{u}_1$ belongs to the kernel of $\mathbf{L}_0 = \mathbf{L}_{\mu_0(k)}$, which by the restriction to symmetric solutions is one-dimensional, so that

$$\mathbf{u}_1 = \boldsymbol{\xi}_0 + \overline{\boldsymbol{\xi}_0}. \tag{2.9}$$

Next, by taking the L^2-scalar product of the equality found at order δ^2 with $\mathbf{u}_1$, we find

$$\mu_1 \langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle = -\langle \mathbf{B}(\mathbf{u}_1, \mathbf{u}_1), \mathbf{u}_1 \rangle,$$

where $\mathbf{L}_1 = \frac{d}{d\mu} \mathbf{L}_\mu \big|_{\mu=\mu_0(k)}$. A direct computation gives (dropping the index 0 in V_0 and θ_0)

$$\langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle = 2 \operatorname{Re} \langle \mathbf{L}_1 \boldsymbol{\xi}_0, \boldsymbol{\xi}_0 \rangle = \frac{2}{k^2\mu^2} \langle (D^2 - k^2)V, (D^2 - k^2)V \rangle + \frac{2}{\mu^2} (\|D\theta\|^2 + k^2 \|\theta\|^2) > 0, \tag{2.10}$$

and a remarkable property of the Navier-Stokes equations is that

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{u}), \mathbf{u} \rangle = 0, \tag{2.11}$$

for any real-valued $\mathbf{u} \in \mathcal{D}$. Consequently, $\mu_1 = 0$ and then $\mathbf{u}_2$ is a symmetric solution of

$$\mathbf{L}_0 \mathbf{u}_2 = -\mathbf{B}(\mathbf{u}_1, \mathbf{u}_1).$$

Without loss of generality, $\mathbf{u}_2$ may be chosen orthogonal to $\mathbf{u}_1$. Finally, the scalar product of the equality found at order δ^3 with $\mathbf{u}_1$, leads to

$$\mu_2 \langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle = -\langle 2\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_1 \rangle.$$

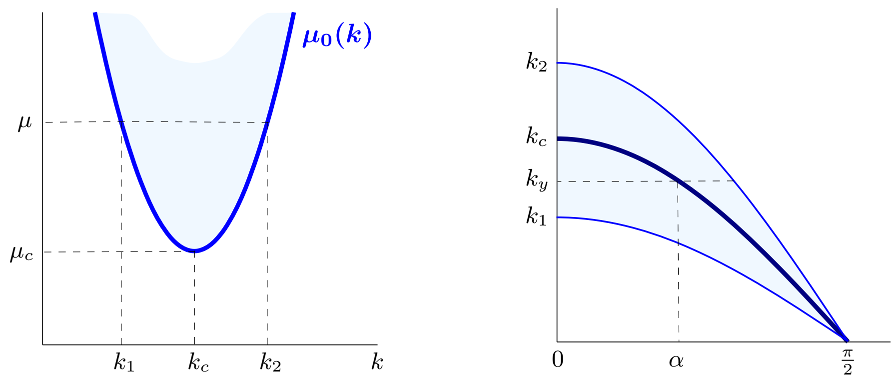
Writing the equality (2.11) for $\mathbf{u} = \mathbf{u}_1 + t\mathbf{u}_2$ and taking the term linear in t we find

$$\langle 2\mathbf{B}(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_1 \rangle + \langle \mathbf{B}(\mathbf{u}_1, \mathbf{u}_1), \mathbf{u}_2 \rangle = 0,$$

hence

$$\mu_2 = \frac{\langle \mathbf{B}(\mathbf{u}_1, \mathbf{u}_1), \mathbf{u}_2 \rangle}{\langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle} = -\frac{\langle \mathbf{L}_0 \mathbf{u}_2, \mathbf{u}_2 \rangle}{\langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle}. \tag{2.12}$$

The sign of μ_2 determines the type of the bifurcation. We have $\langle \mathbf{L}_1 \mathbf{u}_1, \mathbf{u}_1 \rangle > 0$ by (2.10), and $\langle \mathbf{L}_0 \mathbf{u}_2, \mathbf{u}_2 \rangle < 0$, since $\mathbf{L}_0$ is a nonpositive selfadjoint operator and $\mathbf{u}_2$ is orthogonal to its kernel. Consequently, $\mu_2 > 0$, implying that rolls bifurcate supercritically, for $\mu > \mu_0(k)$ (see Figure 2.1(a)). Summarizing, for any fixed k close enough to k_c, for any $\mu > \mu_0(k)$, sufficiently close to $\mu_0(k)$, there exists a "circle" of rolls $\boldsymbol{\tau}_a(\mathbf{u}_{k,\mu})$, $a \in \mathbb{R}/2\pi\mathbb{Z}$, in which $\mathbf{u}_{k,\mu}$ and $\boldsymbol{\tau}_\pi(\mathbf{u}_{k,\mu})$ are invariant under the action of $\mathbf{S}_2$ and exchanged by the action of $\mathbf{S}_3$. These two solutions correspond to values δ in the expansion (2.8) with opposite signs, and we choose $\delta > 0$ for $\mathbf{u}_{k,\mu}$. For the convection problem, we obtain a periodic pattern with adjacent cells, with vertical separations, having half the period (see Figure 1.1(c)).



(a) **(b)**

Figure 2.1: **(a)** Graph of $\mu_0(k)$. Two-dimensional rolls bifurcate into the shaded region situated above the curve $\mu_0(k)$. For $\mu > \mu_c$ sufficiently close to μ_c, two-dimensional rolls exist for wavenumbers $k \in (k_1, k_2)$ with $\mu = \mu_0(k_1) = \mu_0(k_2)$. **(b)** Plot of the wavenumbers $k_y = k \cos\alpha$ in y of the rolls rotated by angles $\alpha \in (0, \pi/2)$, for $k = k_1, k_c, k_2$. For $\mu > \mu_c$ sufficiently close to μ_c, rotated rolls exist in the shaded region. In the bifurcation analysis we fix $k_y = k_c \cos\alpha$, for some $\alpha \in (0, \pi/3)$.

3 Spatial dynamics

The starting point of our analysis is a formulation of the steady system (1.1)-(1.3) as a dynamical system in which the evolutionary variable is the horizontal spatial coordinate x.

Set $\mathbf{V} = (V_x, V_\perp)$, where $V_\perp = (V_y, V_z)$, and consider the new variables

$$\mathbf{W} = \mu^{-1}\partial_x \mathbf{V} - p\mathbf{e}_x, \quad \phi = \partial_x \theta, \tag{3.1}$$

in which we write $\mathbf{W} = (W_x, W_\perp)$, and $W_\perp = (W_y, W_z)$. Using the equation (1.3) we obtain the formula for the pressure,

$$p = -\mu^{-1}\nabla_\perp \cdot V_\perp - W_x. \tag{3.2}$$

Then we write the system (1.1)-(1.3) in the form

$$\partial_x \mathbf{U} = \mathcal{L}_\mu \mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{3.3}$$

in which $\mathbf{U}$ is the 8-components vector

$$\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi),$$

and the operators $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ are linear and quadratic, respectively, defined by

$$\mathcal{L}_\mu \mathbf{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \theta\mathbf{e}_z - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp \theta - \mu V_z \end{pmatrix},$$

$$\mathcal{B}_\mu(\mathbf{U},\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathcal{P}^{-1}\big((V_\perp\cdot\nabla_\perp)V_x - V_x(\nabla_\perp\cdot V_\perp)\big) \\ \mathcal{P}^{-1}\big((V_\perp\cdot\nabla_\perp)V_\perp + \mu V_x W_\perp\big) \\ 0 \\ \mu\big((V_\perp\cdot\nabla_\perp)\theta + V_x\phi\big) \end{pmatrix}.$$

We look for solutions of (3.3) which are periodic in y and satisfy the boundary conditions (1.5) or (1.6). For such solutions we have

$$\frac{d}{dx}\int_{\Omega_{per}} V_x\,dy\,dz = -\int_{\Omega_{per}} \nabla_\perp\cdot V_\perp\,dy\,dz = -\int_{\partial\Omega_{per}} n\cdot V_\perp\,ds = 0,$$

where the subscript *per* means that the integration domain is restricted to one period. This property implies that the flux

$$\mathcal{F}(x) = \int_{\Omega_{per}} V_x\,dy\,dz$$

is constant, or, equivalently, that the dynamical system (3.3) leaves invariant the subspace orthogonal to the vector $\boldsymbol{\psi}_0 = (1,0,0,0,0,0,0,0)$. We restrict to this subspace, hence fixing the constant flux to 0. Including this property and the boundary conditions (1.5) in the definition of the phase space $\mathcal{X}$ of the dynamical system (3.3) we take

$$\mathcal{X} = \big\{\mathbf{U}\in(H^1_{per}(\Omega))^3\times(L^2_{per}(\Omega))^3\times H^1_{per}(\Omega)\times L^2_{per}(\Omega)\ ;$$
$$V_x = V_\perp = \theta = 0 \text{ on } z=0,1, \text{ and } \int_{\Omega_{per}} V_x\,dy\,dz = 0\big\}.$$

As in Section 2, $\Omega = \mathbb{R}\times(0,1)$ and the subscript *per* means that the functions are $2\pi/k_y$-periodic in y, for some fixed $k_y>0$. (In order to distinguish between periodicity in x and y, we add the subscript y in the notation of the wavenumber k.) The phase space $\mathcal{X}$ is a closed subspace of the Hilbert space

$$\widetilde{\mathcal{X}} = (H^1_{per}(\Omega))^3\times(L^2_{per}(\Omega))^3\times H^1_{per}(\Omega)\times L^2_{per}(\Omega),$$

so that it is a Hilbert space endowed with the usual scalar product of $\widetilde{\mathcal{X}}$. Accordingly, we define the domain of definition $\mathcal{Z}$ of the linear operator $\mathcal{L}_\mu$ by

$$\mathcal{Z} = \big\{\mathbf{U}\in\mathcal{X}\cap(H^2_{per}(\Omega))^3\times(H^1_{per}(\Omega))^3\times H^2_{per}(\Omega)\times H^1_{per}(\Omega)\ ;$$
$$\nabla_\perp\cdot V_\perp = W_\perp = \phi = 0 \text{ on } z=0,1\big\},$$

so that $\mathcal{L}_\mu$ is closed and its domain $\mathcal{Z}$ is dense and compactly embedded in $\mathcal{X}$. In particular, this latter property implies that $\mathcal{L}_\mu$ has purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicity.

The dynamical system (3.3) inherits the symmetries of the original system (1.1)-(1.5). As for the two-dimensional convection, horizontal translations $y\to y+a/k_y$ along the y direction give a one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ defined on $\mathcal{X}$ through

$$\boldsymbol{\tau}_a\mathbf{U}(y,z) = \mathbf{U}(y+a/k_y,z), \tag{3.4}$$

and which commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$. The reflection $x \mapsto -x$ now gives a reversibility symmetry

$$\mathbf{S}_1\mathbf{U}(y,z) = (-V_x, V_\perp, W_x, -W_\perp, \theta, -\phi)(y,z),$$

for $\mathbf{U} \in \mathcal{X}$, which anti-commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$, and the reflections $y \mapsto -y$ and $z \mapsto 1-z$ give the symmetries

$$\mathbf{S}_2\mathbf{U}(y,z) = (V_x, -V_y, V_z, W_x, -W_y, W_z, \theta, \phi)(-y,z),$$
$$\mathbf{S}_3\mathbf{U}(y,z) = (V_x, V_y, -V_z, W_x, W_y, -W_z, -\theta, -\phi)(y,1-z),$$

for $\mathbf{U} \in \mathcal{X}$, which both commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$. Notice that

$$\boldsymbol{\tau}_a\mathbf{S}_2 = \mathbf{S}_2\boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_0 = \boldsymbol{\tau}_{2\pi} = \mathbb{I},$$

so that the system (3.3) is $O(2)$-equivariant, and that $\mathbf{S}_3$ commutes with $\boldsymbol{\tau}_a$.

In addition to these symmetries inherited from the original system (1.1) -(1.5), the dynamical system (3.3) has a specific invariance due to the new variable $\mathbf{W} = (W_x, W_\perp)$ in (3.1). While $W_\perp$ satisfies the same boundary conditions as $V_\perp$, included in the domain of definition $\mathcal{Z}$ of the linear operator, there are no such conditions for W_x because the pressure p in the definition of W_x is only defined up to a constant. As a consequence, the dynamical system is invariant upon adding any constant to W_x, i.e., the vector field is invariant under the action of the one-parameter family of maps $(\boldsymbol{T}_b)_{b\in\mathbb{R}}$, defined on $\mathcal{X}$ through

$$\boldsymbol{T}_b\mathbf{U} = \mathbf{U} + b\boldsymbol{\varphi}_0, \quad \boldsymbol{\varphi}_0 = (0,0,0,1,0,0,0,0)^t. \tag{3.5}$$

This invariance introduces the vector $\boldsymbol{\varphi}_0$ in the kernel of $\mathcal{L}_\mu$ (see Lemma 4.1 below).

4 The bifurcation problem

As for the two-dimensional convection, we fix the Prandtl number $\mathcal{P}$ and take the square root μ of the Rayleigh number as bifurcation parameter.

4.1 Domain walls as heteroclinic solutions

The equilibria $\mathbf{U} \in \mathcal{Z}$ of the dynamical system (3.3) can be found as solutions $\mathbf{u} \in \mathcal{D}$ of the two-dimensional problem in Section 2, through the projection

$$\mathbf{u} = \Pi\mathbf{U} = (V_x, V_\perp, \theta). \tag{4.1}$$

The remaining components of an equilibrium $\mathbf{U}$ are obtained from (3.1),

$$(W_x, W_\perp, \phi) = (-p, 0, 0, 0),$$

with the pressure p determined, up to a constant, from the equation (1.1). In particular, for any $k_y = k > 0$ fixed close enough to k_c, the rolls in Section 2 give a circle of equilibria $\boldsymbol{\tau}_a(\mathbf{U}^*_{k,\mu})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, which bifurcate for $\mu > \mu_0(k)$ sufficiently close to $\mu_0(k)$, belong to $\mathcal{D}$, and satisfy

$$\mathbf{S}_1\mathbf{U}^*_{k,\mu} = \mathbf{S}_2\mathbf{U}^*_{k,\mu} = \mathbf{U}^*_{k,\mu}, \quad \mathbf{S}_3\mathbf{U}^*_{k,\mu} = \boldsymbol{\tau}_\pi\mathbf{U}^*_{k,\mu}. \tag{4.2}$$

Due to the rotation invariance of the three-dimensional problem (2.1), horizontally rotated rolls are solutions of (2.1) and also of the dynamical system (3.3). For any angle $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, we have the rotated rolls $\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}$, where the horizontal rotation $\boldsymbol{\mathcal{R}}_\alpha$ acts on the 4-components vector $\mathbf{u} = \Pi\mathbf{U}$ through

$$\boldsymbol{\mathcal{R}}_\alpha \mathbf{u}(x,y,z) = (\mathcal{R}_\alpha(V_x,V_y),V_z,\theta)(\mathcal{R}_{-\alpha}(x,y),z), \tag{4.3}$$

in which

$$\mathcal{R}_\alpha(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

(We do not need here the more complicated representation formula for the 8-components vector $\mathbf{U}$.) These rotated rolls are periodic functions in both x and y with wavenumbers $k\sin\alpha$ and $k\cos\alpha$, respectively. As solutions of the dynamical system (3.3), they belong to the phase space $\mathcal{X}$ provided $k_y = k\cos\alpha$, and in this case they are $2\pi/k\sin\alpha$-periodic solutions in x (see Figure 2.1(b) for a plot of the possible wavenumbers k_y in y for $\mu > \mu_c$ sufficiently close to μ_c). For the particular angles $\alpha = 0$ and $\alpha = \pi$ the rotated rolls are equilibria in the phase-space $\mathcal{X}$ with $k_y = k$. For the orthogonal angles $\alpha = \pi/2$ and $\alpha = 3\pi/2$, they are solutions $2\pi/k$-periodic in x, for any $k_y > 0$.

The invariance of $\mathbf{U}^*_{k,\mu}$ under the action of the symmetry $\mathbf{S}_2$ implies that rolls rotated by angles α and $\pi + \alpha$ coincide,

$$\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{\pi+\alpha} \mathbf{U}^*_{k,\mu}.$$

Upon rotation, rolls loose their invariance under the horizontal reflections $x \to -x$ and $y \to -y$, the actions of $\mathbf{S}_1$ and $\mathbf{S}_2$ on a roll rotated by an angle $\alpha \notin \{0,\pi\}$ gives the same roll but rotated by the opposite angle,

$$\mathbf{S}_1(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}(-x), \quad \mathbf{S}_2 \boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}.$$

These equalities imply that rotated rolls keep a reversibility symmetry,

$$\mathbf{S}_1\mathbf{S}_2(\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}(-x). \tag{4.4}$$

The last equality in (4.2) remains valid for angles $\alpha \notin \{\pi/2, 3\pi/2\}$, whereas for angles $\alpha = \pi/2$ and $\alpha = 3\pi/2$ the rotated rolls are invariant under the action of the entire family of linear maps $(\boldsymbol{\tau}_a)_{a\in\ \mathbb{R}/2\pi\mathbb{Z}}$.

We construct the domain walls as reversible heteroclinic solutions of the dynamical system (3.3) connecting two rotated rolls, $\boldsymbol{\mathcal{R}}_\alpha \mathbf{U}^*_{k,\mu}$ at $x = -\infty$ and $\boldsymbol{\mathcal{R}}_{-\alpha} \mathbf{U}^*_{k,\mu}$ at $x = \infty$. In the bifurcation problem, we will suitably fix $k_y \in (0, k_c)$ and take μ, close to μ_c, as bifurcation parameter. The next step of our analysis is to determine the purely imaginary eigenvalues of the linear operator $\mathcal{L}_{\mu_c}$.

4.2 Connection with the classical linear problem

Solutions $\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi) \in \mathcal{Z}$ of the eigenvalue problem

$$\mathcal{L}_\mu \mathbf{U} = i\omega \mathbf{U}, \tag{4.5}$$

are linear combinations of vectors of the form $\mathbf{U}_{\omega,n}(y,z) = e^{ink_y y}\widehat{\mathbf{U}}_{\omega,n}(z)$, with $n \in \mathbb{Z}$, due to periodicity in y. Projecting with Π given by (4.1), we obtain a solution

$$\mathbf{u}_{\omega,n}(x,y,z) = e^{i(\omega x + nk_y y)}\,\Pi\widehat{\mathbf{U}}_{\omega,n}(z),$$

of the linearized three-dimensional classical problem (2.1), and rotating by a suitable angle α we find a solution $e^{iky}\widehat{\mathbf{u}}_k(z)$ of the linear equation (2.2), with

$$k^2 = \omega^2 + n^2 k_y^2. \tag{4.6}$$

The angle α is determined by the equalities

$$\omega = k\sin\alpha, \quad nk_y = k\cos\alpha, \tag{4.7}$$

and we have the relationship

$$\Pi\widehat{\mathbf{U}}_{\omega,n}(z) = \boldsymbol{\mathcal{R}}_{-\alpha}\widehat{\mathbf{u}}_k(z).$$

Consequently, for a given $k_y > 0$, the eigenvectors $\mathbf{U}_{\omega,n}$ associated with purely imaginary eigenvalues $\nu = i\omega$ of $\mathcal{L}_\mu$ are obtained by rotating with $\boldsymbol{\mathcal{R}}_{-\alpha}$ the elements in the kernel of $\mathbf{L}_\mu$ given by (2.3), through the relationship (4.7) and

$$\Pi\mathbf{U}_{\omega,n}(y,z) = e^{ink_y y}\Pi\widehat{\mathbf{U}}_{\omega,n}(z) = e^{ink_y y}\boldsymbol{\mathcal{R}}_{-\alpha}\widehat{\mathbf{u}}_k(z). \tag{4.8}$$

This holds for all eigenvectors $\mathbf{U}_{\omega,n}$ such that $\Pi\mathbf{U}_{\omega,n} \neq 0$. We obtain in this way all purely imaginary eigenvalues of $\mathcal{L}_\mu$ with associated eigenvectors $\mathbf{U}$ such that $\Pi\mathbf{U} \neq 0$. Using the properties of the kernel of $\mathcal{L}_\mu$ in Section 2.1, we obtain the following result, for $\mu = \mu_0(k)$.

Lemma 4.1. *Assume that k_y and k are positive numbers. Then the linear operator $\mathcal{L}_{\mu_0(k)}$ has the complex conjugated purely imaginary eigenvalues*

$$\pm i\omega_n(k), \quad \omega_n(k) = \sqrt{k^2 - n^2k_y^2} > 0, \tag{4.9}$$

for any integer $0 \leqslant n < k/k_y$, and the following properties hold.[1]

(i) For $n = 0$, $\omega_0(k) = k$ and the complex conjugated eigenvalues $\pm ik$ are geometrically simple with associated eigenvector of the form

$$\mathbf{U}_{k,0}(y,z) = \widehat{\mathbf{U}}_{k,0}(z),$$

for the eigenvalue ik, and the complex conjugated vector for the eigenvalue $-ik$.

(ii) For $0 < n < k/k_y$, the complex conjugated eigenvalues $\pm i\omega_n(k)$ are geometrically double with associated eigenvectors of the form

$$\mathbf{U}_{\omega_n(k),\pm n}(y,z) = e^{\pm ink_y y}\widehat{\mathbf{U}}_{\omega_n(k),\pm n}(z),$$

for the eigenvalue $i\omega_n(k)$, and the complex conjugated vectors for the eigenvalue $-i\omega_n(k)$.

[1] If $k/k_y \in \mathbb{N}$, then the linear operator has an additional eigenvalue 0 which is geometrically triple. This situation is excluded from our bifurcation analysis.

(iii) If the derivative $\mu_0'(k)$ does not vanish, then the eigenvalues are semi-simple.

(iv) The vectors $\widehat{\mathbf{U}}_{k,0}(z)$ and $\widehat{\mathbf{U}}_{\omega_1(k),\pm 1}(z)$ are given by [2]

$$\widehat{\mathbf{U}}_{k,0}(z) = \begin{pmatrix} \frac{i}{k} DV_k \\ 0 \\ V_k \\ -\frac{1}{\mu_0(k)k^2} D^3 V_k \\ 0 \\ \frac{ik}{\mu_0(k)} V_k \\ \frac{1}{\mu_0(k)k^2}(D^2-k^2)^2 V_k \\ \frac{i}{\mu_0(k)k}(D^2-k^2)^2 V_k \end{pmatrix}, \quad \widehat{\mathbf{U}}_{\omega_1(k),\pm 1}(z) = \begin{pmatrix} \frac{i\omega_1(k)}{k^2} DV_k \\ \pm\frac{ik_y}{k^2} DV_k \\ V_k \\ -\frac{1}{\mu_0(k)k^2}(D^2-k_y^2) DV_k \\ \mp\frac{k_y\omega_1(k)}{\mu_0(k)k^2} DV_k \\ \frac{i\omega_1(k)}{\mu_0(k)} V_k \\ \frac{1}{\mu_0(k)k^2}(D^2-k^2)^2 V_k \\ \frac{i\omega_1(k)}{\mu_0(k)k^2}(D^2-k^2)^2 V_k \end{pmatrix},$$

where the function V_k is a real-valued solution of the boundary value problem

$$(D^2-k^2)^3 V_k + \mu_0(k)^2 k^2 V_k = 0, \quad V_k = DV_k = (D^2-k^2)^2 V_k = 0 \text{ in } z = 0, 1. \tag{4.10}$$

Proof. First, notice that for eigenvectors $\mathbf{U}$ with $\Pi\mathbf{U} = 0$, the eigenvalue problem (4.5) is reduced to the system

$$\begin{aligned} \mu W_\perp &= 0 \\ 0 &= i\omega W_x \\ -\nabla_\perp W_x &= 0 \\ \phi &= 0 \end{aligned}$$

for the variables $(W_x, W_\perp, \phi)$. The only nontrivial solution of this system is $(W_x, 0, 0, 0)$, with W_x a constant function, when $\omega = 0$. This implies that 0 is an eigenvalue of $\mathcal{L}_\mu$ with associated eigenvector $\boldsymbol{\varphi}_0$ given by (3.5), and that all other eigenvalues have associated eigenvectors $\mathbf{U}$ with $\Pi\mathbf{U} \neq 0$. In particular, nonzero purely imaginary eigenvalues of $\mathcal{L}_\mu$ and their associated eigenvectors are all determined from the properties of the kernel of the operator $\mathbf{L}_\mu$ in Section 2.1 through the equalities (4.6), (4.7), and (4.8).

For $\mu = \mu_0(k)$, we obtain the eigenvalues given by (4.9). The uniqueness, up to a multiplicative constant, of the element in the kernel of $\mathbf{L}_{\mu_0(k)}$ given by (2.3), implies that the eigenvalues $\pm ik$, for $n = 0$, are geometrically simple, and since opposite numbers $\pm n$ give the same pair of eigenvalues $\pm i\omega_n(k)$, for $n \neq 0$, these eigenvalues are geometrically double. This proves (i) and (ii). In Appendix A.2, we show that in the case $\mu_0'(k) \neq 0$ the algebraic multiplicity of each of these eigenvalues is equal to its geometric multiplicity, which proves (iii). Finally, the equalities (4.8) and (2.3), allow to compute the projections $\Pi\mathbf{U}_{k,0}$ and $\Pi\mathbf{U}_{\omega_n(k),\pm n}$ of the eigenvectors and the remaining components $(\mathbf{W}, \phi)$ are found from (3.1) and (3.2). We obtain the formulas in (iv), which completes the proof of the lemma. ∎

[2] For our purposes, we do not need the explicit formulas for $n > 1$.

4.3 The center spectrum of $\mathcal{L}_{\mu_c}$

Lemma 4.1 shows that the linear operator $\mathcal{L}_{\mu_c}$ has the purely imaginary eigenvalues

$$\pm i\sqrt{k_c^2 - n^2k_y^2},$$

for positive integers n such that $0 \leqslant n < k_c/k_y$. Upon decreasing k_y, the number of pairs of eigenvalues increases. For $k_y > k_c$, there is one pair of purely imaginary eigenvalues with $n = 0$, for $k_c \geqslant k_y > k_c/2$ there are two pairs with $n = 0, \pm 1$, and more generally for $k_c/N \geqslant k_y > k_c/(N+1)$ there are $N+1$ pairs with $n = 0, \pm 1, \dots, \pm N$. For the construction of domain walls we need at least one pair of purely imaginary eigenvalues with opposite Fourier modes $\pm n \neq 0$. We restrict here to the simplest situation when $k_c > k_y > k_c/2$ and $\mathcal{L}_{\mu_c}$ has two pairs of purely imaginary eigenvalues: $\pm ik_c$, for $n = 0$, and $\pm i\sqrt{k_c^2 - k_y^2}$, for $n = \pm 1$.

For notational convenience, we set

$$k_y = k_c \cos\alpha, \quad k_x = k_c \sin\alpha, \tag{4.11}$$

and take $\alpha \in (0, \pi/3)$. In the following lemma we give a complete description of the purely imaginary spectrum of the linear operator $\mathcal{L}_{\mu_c}$.

Lemma 4.2. *Assume that $k_y = k_c \cos\alpha$ with $\alpha \in (0, \pi/3)$. Then the center spectrum $\sigma_c(\mathcal{L}_{\mu_c})$ of the linear operator $\mathcal{L}_{\mu_c}$ consists of five eigenvalues,*

$$\sigma_c(\mathcal{L}_{\mu_c}) = \{0, \pm ik_c, \pm ik_x\}, \quad k_x = k_c \sin\alpha, \tag{4.12}$$

with the following properties.

(i) The eigenvalue 0 is simple with associated eigenvector $\boldsymbol{\varphi}_0$ given by (3.5), which is invariant under the actions of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$.

(ii) The complex conjugated eigenvalues $\pm ik_c$ are algebraically double and geometrically simple with associated generalized eigenvectors of the form

$$\boldsymbol{\zeta}_0(y,z) = \widehat{\mathbf{U}}_0(z), \quad \boldsymbol{\Psi}_0(y,z) = \widehat{\boldsymbol{\Psi}}_0(z),$$

for the eigenvalue ik_c, and the complex conjugated vectors for the eigenvalue $-ik_c$, such that

$$(\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\zeta}_0 = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\Psi}_0 = \boldsymbol{\zeta}_0,$$

and

$$\begin{aligned}
&\mathbf{S}_1\boldsymbol{\zeta}_0 = \overline{\boldsymbol{\zeta}_0}, \quad \mathbf{S}_2\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \mathbf{S}_3\boldsymbol{\zeta}_0 = -\boldsymbol{\zeta}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0,\\
&\mathbf{S}_1\boldsymbol{\Psi}_0 = -\overline{\boldsymbol{\Psi}_0}, \quad \mathbf{S}_2\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0, \quad \mathbf{S}_3\boldsymbol{\Psi}_0 = -\boldsymbol{\Psi}_0, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0.
\end{aligned}$$

(iii) The complex conjugated eigenvalues $\pm ik_x$ are algebraically quadruple and geometrically double with associated generalized eigenvectors of the form

$$\boldsymbol{\zeta}_\pm(y,z) = e^{\pm ik_y y}\widehat{\mathbf{U}}_\pm(z), \quad \boldsymbol{\Psi}_\pm(y,z) = e^{\pm ik_y y}\widehat{\boldsymbol{\Psi}}_\pm(z), \tag{4.13}$$

for the eigenvalue ik_x, and the complex conjugated vectors for the eigenvalue $-ik_x$, such that

$$(\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\zeta}_\pm = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\Psi}_\pm = \boldsymbol{\zeta}_\pm,$$

and

$$\begin{aligned}
&\mathbf{S}_1\boldsymbol{\zeta}_+ = \overline{\boldsymbol{\zeta}_-}, \quad \mathbf{S}_2\boldsymbol{\zeta}_+ = \boldsymbol{\zeta}_-, \quad \mathbf{S}_3\boldsymbol{\zeta}_+ = -\boldsymbol{\zeta}_+, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_+ = e^{ia}\boldsymbol{\zeta}_+,\\
&\mathbf{S}_1\boldsymbol{\zeta}_- = \overline{\boldsymbol{\zeta}_+}, \quad \mathbf{S}_2\boldsymbol{\zeta}_- = \boldsymbol{\zeta}_+, \quad \mathbf{S}_3\boldsymbol{\zeta}_- = -\boldsymbol{\zeta}_-, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_- = e^{-ia}\boldsymbol{\zeta}_-,\\
&\mathbf{S}_1\boldsymbol{\Psi}_+ = -\overline{\boldsymbol{\Psi}_-}, \quad \mathbf{S}_2\boldsymbol{\Psi}_+ = \boldsymbol{\Psi}_-, \quad \mathbf{S}_3\boldsymbol{\Psi}_+ = -\boldsymbol{\Psi}_+, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_+ = e^{ia}\boldsymbol{\Psi}_+,\\
&\mathbf{S}_1\boldsymbol{\Psi}_- = -\overline{\boldsymbol{\Psi}_+}, \quad \mathbf{S}_2\boldsymbol{\Psi}_- = \boldsymbol{\Psi}_+, \quad \mathbf{S}_3\boldsymbol{\Psi}_- = -\boldsymbol{\Psi}_-, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_- = e^{-ia}\boldsymbol{\Psi}_-.
\end{aligned}$$

Proof. The result in Lemma 4.1 shows that $\pm ik_c$ and $\pm ik_x$ are purely imaginary eigenvalues of $\mathcal{L}_{\mu_c}$ and the first part of its proof implies that 0 is an eigenvalue of $\mathcal{L}_{\mu_c}$. Since μ_c is the unique global minimum of $\mu_0(k)$, there are no other eigenvalues with zero real part. This proves the property (4.12). Furthermore, the eigenvalue 0 is geometrically simple, with associated eigenvector $\boldsymbol{\varphi}_0$ given by (3.5), and the eigenvalues $\pm ik_c$ and $\pm ik_x$ have geometric multiplicities one and two, respectively. The associated eigenvectors $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ are computed from the formulas in Lemma 4.1, by taking $n = 0$ and $n = \pm 1$, respectively, for $k = k_c$ and $k_y = k_c\cos\alpha$. We obtain

$$\boldsymbol{\zeta}_0(y,z) = \widehat{\mathbf{U}}_0(z), \quad \boldsymbol{\zeta}_\pm(y,z) = e^{\pm ik_y y}\widehat{\mathbf{U}}_\pm(z),$$

where

$$\widehat{\mathbf{U}}_0(z) = \begin{pmatrix} \frac{i}{k_c}DV \\ 0 \\ V \\ -\frac{1}{\mu_c k_c^2}D^3V \\ 0 \\ \frac{ik_c}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix}, \quad \widehat{\mathbf{U}}_\pm(z) = \begin{pmatrix} \frac{i\sin\alpha}{k_c}DV \\ \pm\frac{i\cos\alpha}{k_c}DV \\ V \\ -\frac{1}{\mu_c k_c^2}(D^2-k_c^2\cos^2\alpha)DV \\ \mp\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2-k_c^2)^2V \\ \frac{i\sin\alpha}{\mu_c k_c}(D^2-k_c^2)^2V \end{pmatrix}, \tag{4.14}$$

and the function V is a real-valued solution of the boundary value problem

$$(D^2-k_c^2)^3V + \mu_c^2k_c^2V = 0, \quad V = DV = (D^2-k_c^2)^2V = 0 \text{ in } z = 0, 1. \tag{4.15}$$

This boundary value problem being equivalent to (2.4)-(2.5) for $\mu = \mu_c$, the function V is positive and symmetric with respect to $z = 1/2$. The latter property and the explicit formulas above imply the symmetry properties of $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ in (*ii*) and (*iii*).

Next, the algebraic multiplicity of the eigenvalue 0 is directly determined by solving the equation

$$\mathcal{L}_{\mu_c}\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_0.$$

Up to an element in the kernel of $\mathcal{L}_{\mu_c}$, we find

$$\boldsymbol{\varphi}_1 = \Big(\frac{\mu_c}{2}z(1-z), 0, 0, 0, 0, 0, 0, 0\Big)^t.$$

The first component of $\boldsymbol{\varphi}_1$ does not satisfy the zero average condition in the definition of the phase space $\mathcal{X}$, which implies that $\boldsymbol{\varphi}_1 \notin \mathcal{X}$ and proves that 0 is an algebraically simple eigenvalue. The invariance of $\boldsymbol{\varphi}_0$ under the actions of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$ is easily checked, which completes the proof of part (i).

For the algebraic multiplicities of the nonzero eigenvalues $\pm ik_c$ and $\pm ik_x$, we use their continuation as eigenvalues of $\mathcal{L}_\mu$, for $\mu > \mu_c$ close to k_c. For any $\mu > \mu_c$ sufficiently close to μ_c, there are precisely two values k_1 and k_2 such that $\mu = \mu_0(k_1) = \mu_0(k_2)$ (see Figure 2.1(a)), and the spectrum close to the imaginary axis of $\mathcal{L}_\mu$ consists of the purely imaginary eigenvalues of the operators $\mathcal{L}_{\mu_0(k_1)}$ and $\mathcal{L}_{\mu_0(k_2)}$ in Lemma 4.1. Since $\mu_0'(k) \neq 0$ for k close to k_c, these eigenvalues are semi-simple, $\pm ik_1$ and $\pm ik_2$ which are algebraically simple, and $\pm i\omega_1(k_1)$ and $\pm i\omega_1(k_2)$ which are algebraically double. Taking the limit $\mu \to \mu_c$, the values k_1 and k_2 tend to k_c, and a standard continuation argument then shows that the eigenvalues $\pm ik_c$ and $\pm ik_x$ of $\mathcal{L}_{\mu_c}$ are algebraically double and quadruple, respectively.

Finally, we compute the generalized eigenvectors $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_\pm$ associated with the eigenvalues ik_c and ik_x, respectively, from the eigenvectors associated with the eigenvalues ik and $i\omega_1(k)$ of $\mathcal{L}_{\mu_0(k)}$ given in Lemma 4.1. Differentiating the eigenvalue problems

$$\mathcal{L}_{\mu_0(k)}\mathbf{U}_{k,0} = ik\mathbf{U}_{k,0}, \quad \mathcal{L}_{\mu_0(k)}\mathbf{U}_{\omega_1(k),\pm 1} = i\omega_1(k)\mathbf{U}_{\omega_1(k),\pm 1},$$

with respect to k at $k = k_c$, and using the properties

$$\mu_0'(k_c) = 0, \quad \omega_1'(k_c) = \frac{k_c}{\sqrt{k_c^2 - k_y^2}} = \frac{1}{\sin\alpha},$$

we obtain the equalities

$$(\mathcal{L}_{\mu_c} - ik_c)\left(\frac{d}{dk}\mathbf{U}_{k,0}\big|_{k=k_c}\right) = i\boldsymbol{\zeta}_0,$$

$$(\mathcal{L}_{\mu_c} - ik_x)\left(\frac{d}{dk}\mathbf{U}_{\omega_1(k),\pm 1}\big|_{k=k_c}\right) = \frac{i}{\sin\alpha}\boldsymbol{\zeta}_\pm.$$

Consequently, the generalized eigenvectors are given by

$$\boldsymbol{\Psi}_0 = -i\left(\frac{d}{dk}\mathbf{U}_{k,0}\big|_{k=k_c}\right), \quad \boldsymbol{\Psi}_\pm = -i\sin\alpha\left(\frac{d}{dk}\mathbf{U}_{\omega_1(k),\pm 1}\right)\big|_{k=k_c}. \tag{4.16}$$

In particular, they have the same form

$$\boldsymbol{\Psi}_0(y,z) = \widehat{\boldsymbol{\Psi}}_0(z), \quad \boldsymbol{\Psi}_\pm(y,z) = e^{\pm ik_y y}\widehat{\boldsymbol{\Psi}}_\pm(z),$$

as the eigenvectors $\mathbf{U}_{k,0}$ and $\mathbf{U}_{\omega_1(k),\pm1}$ given in Lemma 4.1. Furthermore, since the function V_k in the expressions of $\widehat{\mathbf{U}}_{k,0}(z)$ and $\widehat{\mathbf{U}}_{\omega_1(k),\pm1}(z)$ is symmetric with respect to $z = 1/2$, just as the function V in (4.15), the eigenvectors $\mathbf{U}_{k,0}$ and $\mathbf{U}_{\omega_1(k),\pm1}$ have the same symmetry properties as the eigenvectors $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$, respectively. Together with the formulas (4.16), this implies that $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_\pm$ have the symmetry properties given in (ii) and (iii), and completes the proof of the lemma. ■

5 Reduction of the nonlinear problem

The next step of our analysis is the center manifold reduction. Using the symmetries of the system (3.3), we identify an eight-dimensional invariant submanifold of the center manifold, which contains the heteroclinic solutions of (3.3) corresponding to domain walls.

5.1 Center manifold reduction

We set $\varepsilon = \mu - \mu_c$ and write the dynamical system (3.3) in the form

$$\partial_x \mathbf{U} = \mathcal{L}_{\mu_c}\mathbf{U} + \mathcal{R}(\ \mathbf{U}, \varepsilon), \tag{5.1}$$

where

$$\mathcal{R}(\mathbf{U}, \varepsilon) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c})\mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U})$$

is a smooth map from $\mathcal{Z} \times I_c$, $I_c = (-\mu_c, \infty)$, into $\mathcal{X}$. Furthermore,

$$\mathcal{R}(0, \varepsilon) = 0, \quad D_{\mathbf{U}}\mathcal{R}(0, 0) = 0,$$

so that $\mathcal{R}$ satisfies the hypotheses of the center manifold theorem (see [8, Section 2.3.1]). We also have to check two hypotheses on the linear operator $\mathcal{L}_{\mu_c}$. The first one requires that the center spectrum of $\mathcal{L}_{\mu_c}$ consists of finitely many purely imaginary eigenvalues with finite algebraic multiplicity and the result in Lemma 4.2 shows that this hypothesis holds. The second one is the estimate on the norm of resolvent of $\mathcal{L}_{\mu_c}$ obtained by taking $\mu = \mu_c$ in the lemma below.

Lemma 5.1. *For any $\mu > 0$, there exist positive constants C_μ and ω_μ such that*

$$\|(\mathcal{L}_\mu - i\omega)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leqslant \frac{C_\mu}{|\omega|}, \tag{5.2}$$

for any real number ω, with $|\omega| > \omega_\mu$.

Proof. We write $\mathcal{L}_\mu = \mathcal{A}_\mu + \mathcal{C}_\mu$, where

$$\mathcal{A}_\mu\mathbf{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp \theta \end{pmatrix}, \quad \mathcal{C}_\mu\mathbf{U} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\theta\mathbf{e}_z \\ 0 \\ -\mu V_z \end{pmatrix}.$$

Since the operator $\mathcal{C}_\mu$ is bounded in $\mathcal{X}$, the resolvent equality

$$(\mathcal{L}_\mu - i\omega)^{-1} = (\mathbb{I} + (\mathcal{A}_\mu - i\omega)^{-1}\mathcal{C}_\mu)(\mathcal{A}_\mu - i\omega)^{-1},$$

implies that it is enough to prove the result for $\mathcal{A}_\mu$. The action of $\mathcal{A}_\mu$ on the components $(\mathbf{V}, \mathbf{W})$ and (θ, ϕ) of $\mathbf{U}$ being decoupled, the operator is diagonal, $\mathcal{A}_\mu = \mathrm{diag}(\mathcal{A}_\mu^{\mathrm{St}}, \mathcal{A}_\mu^{\mathrm{so}})$, where $\mathcal{A}_\mu^{\mathrm{St}}$ acting on $(\mathbf{V}, \mathbf{W})$ is a Stokes operator and $\mathcal{A}_\mu^{\mathrm{so}}$ acting on (θ, ϕ) is a Laplace operator. The estimate (5.2) has been proved for the Stokes operator $\mathcal{A}_\mu^{\mathrm{St}}$ in [12, Appendix 2], and it is easily obtained for the Laplace operator $\mathcal{A}_\mu^{\mathrm{so}}$. This implies the result for $\mathcal{A}_\mu$ and completes the proof of the lemma. ■

Denote by $\mathcal{X}_c$ the spectral subspace associated with the center spectrum of $\mathcal{L}_{\mu_c}$, by $\mathcal{P}_c$ the corresponding spectral projection, and set $\mathcal{Z}_h = (\mathbb{I} - \mathcal{P}_c)\mathcal{Z}$. Applying the center manifold theorem [8, Section 2.3.1], for any arbitrary, but fixed, $k \geqslant 3$, there exists a map $\mathbf{\Phi} \in \mathcal{C}^k(\mathcal{X}_c \times I_c, \mathcal{Z}_h)$, with

$$\mathbf{\Phi}(0, \varepsilon) = 0, \quad D_{\mathbf{U}}\mathbf{\Phi}(0,0) = 0, \tag{5.3}$$

and a neighborhood $\mathcal{U}_1 \times \mathcal{U}_2$ of $(0,0)$ in $\mathcal{Z} \times I_c$ such that for any $\varepsilon \in \mathcal{U}_2$, the manifold

$$\mathcal{M}_c(\varepsilon) = \{\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon) \ ; \ \mathbf{U}_c \in \mathcal{X}_c\}, \tag{5.4}$$

has the following properties:

(i) $\mathcal{M}_c(\varepsilon)$ is locally invariant, i.e., if $\mathbf{U}$ is a solution of (5.1) satisfying $\mathbf{U}(0) \in \mathcal{M}_c(\varepsilon) \cap \mathcal{U}_1$ and $\mathbf{U}(x) \in \mathcal{U}_1$ for all $x \in [0, L]$, then $\mathbf{U}(x) \in \mathcal{M}_c(\varepsilon)$ for all $x \in [0, L]$;

(ii) $\mathcal{M}_c(\varepsilon)$ contains the set of bounded solutions of (5.1) staying in $\mathcal{U}_1$ for all $x \in \mathbb{R}$, i.e., if $\mathbf{U}$ is a solution of (5.1) satisfying $\mathbf{U}(x) \in \mathcal{U}_1$ for all $x \in \mathbb{R}$, then $\mathbf{U}(0) \in \mathcal{M}_c(\varepsilon)$;

(iii) the invariant dynamics on the center manifold is determined by the reduced system

$$\frac{d\mathbf{U}_c}{dx} = \mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c}\mathbf{U}_c + \mathcal{P}_c\mathcal{R}(\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon), \varepsilon) \stackrel{def}{=} f(\mathbf{U}_c, \varepsilon), \tag{5.5}$$

where

$$f(0, \varepsilon) = 0, \quad D_{\mathbf{U}_c} f(0,0) = \mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c};$$

(iv) the reduced system (5.5) inherits the symmetries of (5.1), i.e., the reduced vector field $f(\cdot, \varepsilon)$ anti-commutes with $\mathbf{S}_1$, commutes with $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$, and is invariant under the action of $\boldsymbol{T}_b$.

An immediate consequence of these properties is that the heteroclinic solutions of (5.1) representing domain walls belong to the center manifold $\mathcal{M}_c(\varepsilon)$, for sufficiently small ε, and can be constructed as solutions of the reduced system (5.5).

5.2 Reduced system

According to Lemma 4.2, the center space $\mathcal{X}_c$ has dimension 13 and we can write

$$\begin{aligned}\mathbf{U}_c &= w\boldsymbol{\varphi}_0 + A_0\boldsymbol{\zeta}_0 + B_0\boldsymbol{\Psi}_0 + A_+\boldsymbol{\zeta}_+ + B_+\boldsymbol{\Psi}_+ + A_-\boldsymbol{\zeta}_- + B_-\boldsymbol{\Psi}_- \\ &\quad + \overline{A_0\boldsymbol{\zeta}_0} + \overline{B_0\boldsymbol{\Psi}_0} + \overline{A_+\boldsymbol{\zeta}_+} + \overline{B_+\boldsymbol{\Psi}_+} + \overline{A_-\boldsymbol{\zeta}_-} + \overline{B_-\boldsymbol{\Psi}_-},\end{aligned} \tag{5.6}$$

where $w \in \mathbb{R}$ and $X = (A_0, B_0, A_+, B_+, A_-, B_-) \in \mathbb{C}^6$. Then the reduced system (5.5) takes the form

$$\frac{dw}{dx} = h(w, X, \overline{X}, \varepsilon), \tag{5.7}$$

$$\frac{dX}{dx} = F(w, X, \overline{X}, \varepsilon), \tag{5.8}$$

in which h is real-valued and $F = (f_0, g_0, f_+, g_+, f_-, g_-)$ has six complex-valued components. This system is completed by the complex conjugated equation of (5.8) for $\overline{X}$. Notice that the symmetries of the reduced system act on these variables through

$$\begin{aligned}
&\mathbf{S}_1(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w, \overline{A_0}, -\overline{B_0}, \overline{A_-}, -\overline{B_-}, \overline{A_+}, -\overline{B_+}),\\
&\mathbf{S}_2(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w, A_0, B_0, A_-, B_-, A_+, B_+),\\
&\mathbf{S}_3(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w, -A_0, -B_0, -A_+, -B_+, -A_-, -B_-),\\
&\boldsymbol{\tau}_a(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w, A_0, B_0, e^{ia}A_+, e^{ia}B_+, e^{-ia}A_-, e^{-ia}B_-),\\
&\boldsymbol{T}_b(w, A_0, B_0, A_+, B_+, A_-, B_-) = (w + b, A_0, B_0, A_+, B_+, A_-, B_-).
\end{aligned}$$

Using the last three symmetries above, we obtain the following result.

Lemma 5.2. *For any ε sufficiently small, the reduced system* (5.7)-(5.8) *has the following properties:*

(i) the reduced vector field (h, F) does not depend on w;

(ii) the components (f_0, g_0) of F are odd functions in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$ and even functions in the variables $(A_+, B_+, A_-, B_-, \overline{A_+}, \overline{B_+}, \overline{A_-}, \overline{B_-})$;

(iii) the components (f_+, g_+, f_-, g_-) of F are even functions in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$ and odd functions in the variables $(A_+, B_+, A_-, B_-, \overline{A_+}, \overline{B_+}, \overline{A_-}, \overline{B_-})$.

Proof. Due to the invariance of the reduced system (5.7)- (5.8) under the action of $\boldsymbol{T}_b$, the vector field (h, F) satisfies

$$(h, F)(w + b, X, \overline{X}, \varepsilon) = (h, F)(w, X, \overline{X}, \varepsilon),$$

for any real number b. This implies that (h, F) does not depend on w and proves (i).

Next, the vector field F, which only depends on X and $\overline{X}$, commutes with the symmetries $\boldsymbol{\tau}_\pi$ and $\mathbf{S}_3\boldsymbol{\tau}_\pi$ acting on these components through

$$\begin{aligned}
\boldsymbol{\tau}_\pi(A_0, B_0, A_+, B_+, A_-, B_-) &= (A_0, B_0, -A_+, -B_+, -A_-, -B_-),\\
\mathbf{S}_3\boldsymbol{\tau}_\pi(A_0, B_0, A_+, B_+, A_-, B_-) &= (-A_0, -B_0, A_+, B_+, A_-, B_-).
\end{aligned}$$

The first equality implies the parity properties of the components $(f_0, g_0, f_+, g_+, f_-, g_-)$ of F in the variables $(A_+, B_+, A_-, B_-, \overline{A_+}, \overline{B_+}, \overline{A_-}, \overline{B_-})$ and the second one implies the parity properties in the variables $(A_0, B_0, \overline{A_0}, \overline{B_0})$. This proves the properties (ii) and (iii). ■

An immediate consequence of the first property in the lemma above being that the two equations (5.7) and (5.8) are decoupled, we can first solve (5.8) for X, and then integrate (5.7) to determine w. We therefore restrict our existence analysis to the equation

$$\frac{dX}{dx} = F(X, \overline{X}, \varepsilon), \tag{5.9}$$

which together with the complex conjugate equation for $\overline{X}$ form a 12-dimensional system. For this system, the parity properties of the vector field F in Lemma 5.2, imply that there exist two invariant subspaces:

$$E_0 = \left\{(X, \overline{X}),\ X \in \mathbb{C}^6\ ;\ (A_+, B_+, A_-, B_-) = 0\right\},$$

which is 4-dimensional, and

$$E_\pm = \left\{(X, \overline{X}),\ X \in \mathbb{C}^6\ ;\ (A_0, B_0) = 0\right\},$$

which is 8-dimensional. Each of these subspaces gives an invariant submanifold of the center manifold.

Solutions in the submanifold associated with E_0 are invariant under the action of the family of maps $(\boldsymbol{\tau}_a)_{a\in\ \mathbb{R}/2\pi\mathbb{Z}}$ and therefore correspond to solutions of the full dynamical system (3.3) which do not depend on y. Solutions in the submanifold associated with $E_\pm$ are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$ and correspond to three-dimensional solutions of the full dynamical system (3.3). For the construction of domain walls we restrict to this 8-dimensional invariant submanifold.

6 Leading order dynamics

We determine the leading order dynamics of the restriction to $E_\pm$ of the reduced system (5.9) with the help of a normal forms transformation to cubic order, followed by suitable scalings of variables. For the resulting systems, we identify particular solutions which correspond to rotated rolls.

6.1 Cubic normal form of the reduced system

We write the reduced system (5.9) restricted to the invariant 8-dimensional subspace $E_\pm$ in the from

$$\frac{dY}{dx} = G(Y, \overline{Y}, \varepsilon), \tag{6.1}$$

in which $Y = (A_+, B_+, A_-, B_-) \in \mathbb{C}^4$. Taking into account the properties of the reduced system (5.5), the formula (5.6), and the choice for the generalized eigenvectors in Lemma 4.2, we find

$$G(0,0,\varepsilon) = 0, \quad D_Y G(0,0,0) = L_0, \quad D_{\overline{Y}} G(0,0,0) = 0,$$

where L_0 is a Jordan matrix acting on Y through

$$L_0 = \begin{pmatrix} ik_x & 1 & 0 & 0 \\ 0 & ik_x & 0 & 0 \\ 0 & 0 & ik_x & 1 \\ 0 & 0 & 0 & ik_x \end{pmatrix}. \tag{6.2}$$

Using a general normal forms theorem for parameter-dependent vector fields in the presence of symmetries (e.g., see [8, Chapter 3]), we determine a normal form of the system (6.1) up to cubic order.

Lemma 6.1. *For any $k \geqslant 3$, there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{C}^4$ and $\mathbb{R}$, respectively, such that for any $\varepsilon \in \mathcal{V}_2$, there is a polynomial $\boldsymbol{P}_\varepsilon : \mathbb{C}^4 \times \overline{\mathbb{C}^4} \to \mathbb{C}^4$ of degree 3 in the variables $(Z, \overline{Z})$, such that for $Z \in \mathcal{V}_1$, the polynomial change of variable*

$$Y = Z + \boldsymbol{P}_\varepsilon(Z, \overline{Z}), \tag{6.3}$$

transforms the equation (6.1) into the normal form

$$\frac{dZ}{dx} = L_0 Z + N(Z, \overline{Z}, \varepsilon) + \rho(Z, \overline{Z}, \varepsilon), \tag{6.4}$$

with the following properties:

(i) the map ρ belongs to $\mathcal{C}^k(\mathcal{V}_1 \times \overline{\mathcal{V}_1} \times \mathcal{V}_2, \mathbb{C}^4)$, and

$$\rho(Z, \overline{Z}, \varepsilon) = O(|\varepsilon|^2 \|Z\| + \varepsilon \|Z\|^3 + \|Z\|^5);$$

(ii) both $N(\cdot,\cdot,\varepsilon)$ and $\rho(\cdot,\cdot,\varepsilon)$ anti-commute with $\mathbf{S}_1$ and commute with $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$, for any $\varepsilon \in \mathcal{V}_2$;

(iii) the four components (N_+, M_+, N_-, M_-) of N are of the form

$$\begin{aligned} N_+ &= iA_+P_+ + A_-R_+ \\ M_+ &= iB_+P_+ + B_-R_+ + A_+Q_+ + iA_-S_+ \\ N_- &= iA_-P_- - A_+\overline{R_+} \\ M_- &= iB_-P_- - B_+\overline{R_+} + A_-Q_- - iA_+\overline{S_+} \end{aligned}$$

in which

$$
\begin{aligned}
P_+ &= \beta_0\varepsilon + \beta_1 A_+\overline{A_+} + i\beta_2(A_+\overline{B_+} - \overline{A_+}B_+) + \beta_3 A_-\overline{A_-} + i\beta_4(A_-\overline{B_-} - \overline{A_-}B_-)\\
P_- &= \beta_0\varepsilon + \beta_3 A_+\overline{A_+} + i\beta_4(A_+\overline{B_+} - \overline{A_+}B_+) + \beta_1 A_-\overline{A_-} + i\beta_2(A_-\overline{B_-} - \overline{A_-}B_-)\\
Q_+ &= b_0\varepsilon + b_1 A_+\overline{A_+} + ib_2(A_+\overline{B_+} - \overline{A_+}B_+) + b_3 A_-\overline{A_-} + ib_4(A_-\overline{B_-} - \overline{A_-}B_-)\\
Q_- &= b_0\varepsilon + b_3 A_+\overline{A_+} + ib_4(A_+\overline{B_+} - \overline{A_+}B_+) + b_1 A_-\overline{A_-} + ib_2(A_-\overline{B_-} - \overline{A_-}B_-)\\
R_+ &= \gamma_5(A_+\overline{B_-} - \overline{A_-}B_+), \quad S_+ = c_5(A_+\overline{B_-} - \overline{A_-}B_+),
\end{aligned}
$$

where (A_+, B_+, A_-, B_-) are the four components of Z and the coefficients β_j, b_j, γ_5 and c_5 are all real.

The proof of this lemma can be found in Appendix B.1. We point out that the result is valid for any system of the form (6.1) which has a linear part as in (6.2) and the symmetries $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\boldsymbol{\tau}_a$ given in Section 5.2.

6.2 Rotated rolls as periodic solutions

The normal form (6.4) truncated at cubic order has the property to leave invariant the two 4-dimensional subspaces

$$
E_+ = \left\{(Z,\overline{Z}),\ Z\in\mathbb{C}^4\ ;\ (A_-,B_-)=0\right\}, \quad E_- = \left\{(Z,\overline{Z}),\ Z\in\mathbb{C}^4\ ;\ (A_+,B_+)=0\right\},
$$

which is not the case for the full system (6.4). The systems obtained by restricting the normal form truncated at cubic order to E_+ and E_- being similar, we consider the one restricted to E_+,

$$
\frac{dA_+}{dx} = ik_xA_+ + B_+ + iA_+P_+ \tag{6.5}
$$

$$
\frac{dB_+}{dx} = ik_xB_+ + iB_+P_+ + A_+Q_+ \tag{6.6}
$$

with

$$
\begin{aligned}
P_+ &= \beta_0\varepsilon + \beta_1 A_+\overline{A_+} + i\beta_2(A_+\overline{B_+} - \overline{A_+}B_+),\\
Q_+ &= b_0\varepsilon + b_1 A_+\overline{A_+} + ib_2(A_+\overline{B_+} - \overline{A_+}B_+).
\end{aligned}
$$

Notice that (6.5)-(6.6) is the system found at cubic order in the case of the classical reversible $1:1$ resonance bifurcation, or reversible Hopf bifurcation. In our case, the reversibility symmetry is given by $\mathbf{S}_1\mathbf{S}_2$. This system is integrable and we refer to [8, Section 4.3.3] for a detailed discussion of its bounded solutions.

We consider here the periodic solutions of (6.5)-(6.6) with wavenumbers $k_x+\theta$ close to k_x, for small ε. According to [8, Section 4.3.3], these periodic solutions are determined, up to the action of $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ and to translations in x, by the reversible periodic solutions

$$
A_+ = r_0e^{i(k_x+\theta)x}, \quad B_+ = iq_0e^{i(k_x+\theta)x}, \tag{6.7}
$$

with real numbers $r_0 > 0$ and q_0 satisfying the equalities

$$\theta = \frac{q_0}{r_0} + \beta_0\varepsilon + \beta_1 r_0^2 + 2\beta_2 r_0 q_0,$$
$$0 = q_0^2 + r_0^2\left(b_0\varepsilon + b_1 r_0^2 + 2b_2 r_0 q_0\right),$$

obtained by replacing (6.7) into the system (6.5)-(6.6). Solving for q_0 and r_0, we find

$$q_0 = \frac{r_0\left(\theta - \beta_0\varepsilon - \beta_1 r_0^2\right)}{1 + 2\beta_2 r_0^2}, \quad r_0^2 = -\frac{b_0}{b_1}\varepsilon - \frac{1}{b_1}\theta^2 + O(|\varepsilon\theta| + |\varepsilon|^2 + |\theta|^3), \tag{6.8}$$

as $(\varepsilon, \theta) \to (0, 0)$. For ε such that $b_0\varepsilon/b_1 < 0$, the right hand side in the formula for r_0^2 is positive for small ε and θ small enough, and we have a solution (A_+, B_+) given by (6.7) for the system (6.5)-(6.6). Notice that θ must be $O(|\varepsilon|^{1/2})$-small when $b_1 > 0$, which, as we shall see later in this section, is the case here.

For the 8-dimensional normal form (6.4) truncated at cubic order we obtain the solutions $(A_+, B_+, 0, 0)$ which belong to the invariant subspace E_+. The persistence of these solutions for the full normal form (6.4) can be proved via the implicit function theorem, for instance, by adapting the method used in the case of reversible $1:1$ resonance bifurcations in [13, Section III.1]. For small ε such that $b_0\varepsilon/b_1 < 0$ and θ small enough, we obtain a family of reversible periodic solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ of the normal form (6.4), which are uniquely determined by their leading order part

$$(r_0 e^{i(k_x+\theta)x}, 0, 0, 0), \quad r_0^2 = -\frac{b_0}{b_1}\varepsilon - \frac{1}{b_1}\theta^2, \quad r_0 > 0. \tag{6.9}$$

This leading order part belongs to E_+, which is not the case for $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$, and it is the same as the one of the solutions (6.7) of the truncated system. As it follows from the implicit function theorem, the periodic solutions $\boldsymbol{\tau}_a(\widetilde{\mathbf{Z}}_{\varepsilon,\theta})$, $a \in \mathbb{R}/2\pi\mathbb{Z}$, are, up to translations in x, the only periodic solutions of the system (6.4) with leading order part of the form (6.9) in E_+ and wavenumbers $k_x + \theta$ sufficiently close to k_x, for sufficiently small ε. Notice that there are precisely two *reversible* solutions, $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ with $r_0 > 0$ and $\tau_\pi\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ with $r_0 < 0$. We show below that the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ correspond to solutions of dynamical system (3.3) which are rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, with k and μ sufficiently close to k_c and μ_c, respectively. We use this correspondence to compute the coefficients b_0 and b_1 of the normal form.

Consider the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, for $\mu > \mu_c$ close to μ_c, wavenumber k close to k_c such that

$$k \in (k_1, k_2), \quad \mu_0(k_1) = \mu_0(k_2) = \mu,$$

(see Figure 2.1), and rotation angle $\beta \in (0, \pi/2)$ chosen such that the rotated roll is a solution of the dynamical system (3.3), i.e., such that

$$k\cos\beta = k_y = k_c\cos\alpha. \tag{6.10}$$

The rotation angle $\beta \in (0, \pi/2)$ is uniquely determined through this formula, and from the Taylor expansion of $\mu_0(k)$,

$$\mu_0(k) = \mu_c + \frac{1}{2}\mu_0''(k_c)(k - k_c)^2 + O(|k - k_c|^3), \tag{6.11}$$

for k close to k_c, we find that the unique values k_1 and k_2 above are $O(|\mu-\mu_c|^{1/2})$-close to k_c. The rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ is periodic in x with wavenumber

$$k'_x = k\sin\beta = \sqrt{k^2-k_c^2\cos^2\alpha} = k_c\sin\alpha + \frac{1}{\sin\alpha}(k-k_c)+O(|k-k_c|^2), \tag{6.12}$$

where we used (6.10) to obtain the second equality, and has the reversibility symmetry (4.4). According to the formulas (2.8), (2.9), and (2.7) from Section 2, we have that

$$\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{\Pi}\mathbf{U}^*_{k,\mu}(x,y,z) = \delta e^{i(k'_x x+k_y y)}\boldsymbol{\mathcal{R}}_{-\beta}\widehat{\mathbf{u}}_k(z) + \delta e^{-i(k'_x x+k_y y)}\boldsymbol{\mathcal{R}}_{-\beta}\overline{\widehat{\mathbf{u}}_k(z)} + O(\delta^2), \tag{6.13}$$

where $\delta>0$ is the small parameter in (2.8) and $\widehat{\mathbf{u}}_k(z)$ is given by (2.3). Furthermore, from (4.8) we obtain

$$e^{ik_y y}\boldsymbol{\mathcal{R}}_{-\beta}\widehat{\mathbf{u}}_k(z) = \mathbf{\Pi}\mathbf{U}_{\omega_1(k),1}(y,z) = \mathbf{\Pi}\boldsymbol{\zeta}_+(y,z)+O(|k-k_c|), \tag{6.14}$$

where $\mathbf{U}_{\omega_1(k),1}$ and $\boldsymbol{\zeta}_+$ are the eigenvectors in Lemma 4.1 and Lemma 4.2, respectively.

For $\mu=\mu_c+\varepsilon$, the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ is a solution of the dynamical system (5.1), which is the same as (3.3). From (2.8) and (6.11) we obtain the relationship

$$\varepsilon = (\mu-\mu_0(k)) + (\mu_0(k)-\mu_c) = \mu_2\delta^2 + \frac{1}{2}\mu_0''(k_c)(k-k_c)^2 + O(|\delta|^3+|k-k_c|^3), \tag{6.15}$$

implying that $\delta = O(\varepsilon^{1/2})$ and $|k-k_c| = O(\varepsilon^{1/2})$, since the values μ_2 and $\mu_0''(k_c)$ given by (2.12) and (2.6), respectively, are positive. In particular, the rotated roll $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ has small amplitude of order $O(\varepsilon^{1/2})$ and therefore belongs to the center manifold (5.4) of (5.1), provided ε is sufficiently small. Furthermore, we saw in Section 4.1 that for rotation angles $\beta\in(0,\pi/2)$, the rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$. This implies that $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ belongs to the center submanifold associated to $E_\pm$ found in Section 5.2. Consequently, it provides a periodic solution of the reduced system (6.1), from which we obtain a periodic solution for the normal form system (6.4) through the change of variables (6.3). These periodic solutions inherit the reversibility symmetry (4.4) of the rotated rolls.

We set

$$\theta = k'_x - k_x = k'_x - k_c\sin\alpha = \frac{1}{\sin\alpha}(k-k_c)+O(|k-k_c|^2), \tag{6.16}$$

where k'_x is the wavenumber given by (6.12), and denote by $\mathbf{Z}_{\varepsilon,\theta}$ the periodic solution of the normal form (6.4) corresponding to $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$. The parameters (ε,θ) are related to (k,μ) through the equalities $\varepsilon=\mu-\mu_c$ and (6.16), which define a one-to-one map $(k,\mu)\to(\varepsilon,\theta)$, for k in a neighborhood of k_c and any μ. Comparing the expressions of $\mathbf{\Pi}\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ given by (6.13) and by the formulas (5.4) and (5.6) for the solutions on the center manifold, using the equalities (6.14) and (6.16), we obtain the expansion

$$\mathbf{Z}_{\varepsilon,\theta}(x) = \left(\delta e^{i(k_x+\theta)x},0,0,0\right) + O(|\delta||\theta|+|\delta|^2), \tag{6.17}$$

with $\delta>0$ determined through (6.15) and (6.16),

$$\delta^2 = \frac{1}{\mu_2}\varepsilon - \frac{\mu_0''(k_c)\sin^2\alpha}{2\mu_2}\,\theta^2 + O(|\varepsilon|^{3/2}+|\varepsilon|^{1/2}|\theta|^2+|\theta|^3). \tag{6.18}$$

The existence and the above properties of the periodic solutions $\mathbf{Z}_{\varepsilon,\theta}$ of the normal form system (6.4) are directly obtained from the existence and properties of the rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$, without using the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ found from the periodic solutions (6.7) of the truncated system. With $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$, the solutions $\mathbf{Z}_{\varepsilon,\theta}$ share the property of being reversible periodic solutions of the system (6.4) with leading order parts in E_+ and wavenumbers $k_x+\theta$ sufficiently close to k_x, for sufficiently small ε. The solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ and $\tau_\pi\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ being the only ones with these properties, taking into account that δ in (6.17) and r_0 in (6.9) are both positive, we deduce that $\mathbf{Z}_{\varepsilon,\theta}$ and $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$ are the same solutions of the system (6.4), for sufficiently small ε and θ. In particular, their leading order parts are the same. Identifying the leading order part of δ^2 in (6.18) with r_0^2 in (6.9), we can compute the coefficients

$$b_0 = -\frac{2}{\mu_0''(k_c)\sin^2\alpha} < 0, \quad b_1 = \frac{2\mu_2}{\mu_0''(k_c)\sin^2\alpha} > 0. \tag{6.19}$$

The signs of these two coefficients are needed in the subsequent arguments.

Remark 6.2. *As usual in this type of approach, the coefficient b_0 can be determined from the property that the eigenvalues of the matrix obtained by linearizing the normal form (6.4) at $Z=0$ are equal to the continuation of the eigenvalues $\pm ik_x$ of $\mathcal{L}_{\mu_c}$ as eigenvalues of $\mathcal{L}_\mu$ for $\mu=\mu_c+\varepsilon$ and sufficiently small ε. In the proof of Lemma 4.2 we saw that the latter eigenvalues are the purely imaginary eigenvalues $\pm i\omega_1(k_1)$ and $\pm i\omega_1(k_2)$ given by (4.9), with $k_1<k_c<k_2$ such that $\mu=\mu_0(k_1)=\mu_0(k_2)$. Computing the eigenvalues of the normal form (6.4) we obtain*

$$i\omega_1(k_1) = i\left(k_x - \sqrt{-b_0\varepsilon} + O(\varepsilon)\right),$$

whereas from (4.9) we find

$$i\omega_1(k_1) = i\sqrt{k_1^2 - k_c^2\cos^2\alpha} = i\left(k_c\sin\alpha + \frac{1}{\sin\alpha}(k_1-k_c) + O(|k_1-k_c|^2)\right).$$

These two equalities and the Taylor expansion (6.11) of $\mu_0(k)$, taken at $k=k_1$, give the value of b_0 in (6.19). Furthermore, by replacing the expansions (6.17) and (6.18) with $\theta=0$ into the equation for B_+ of the normal form (6.4) and identifying the coefficients of the terms of order $O(\varepsilon^{3/2})$, we easily obtain that $b_1=-\mu_2 b_0$. These arguments give an alternative way for the computation of b_0 and b_1, without using the solutions $\widetilde{\mathbf{Z}}_{\varepsilon,\theta}$.

6.3 Leading order system

From now on we restrict to $\varepsilon>0$, which corresponds to values $\mu>\mu_c$ for which rolls exist. We further transform the normal form (6.4) by introducing new variables

$$\widehat{x} = |b_0\varepsilon|^{1/2}x, \quad A_\pm(x) = \left|\frac{b_0\varepsilon}{b_1}\right|^{1/2} e^{ik_x x}C_\pm(\widehat{x}), \quad B_\pm(x) = \frac{|b_0\varepsilon|}{|b_1|^{1/2}}e^{ik_x x}D_\pm(\widehat{x}). \tag{6.20}$$

Taking into account the signs of b_0 and b_1 in (6.19), we obtain the first order system,

$$C'_+ = D_+ + \widehat{f}_+(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.21}$$

$$D'_+ = \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+ + \widehat{g}_+(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.22}$$

$$C'_- = D_- + \widehat{f}_-(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.23}$$

$$D'_- = (-1 + g|C_+|^2 + |C_-|^2) C_- + \widehat{g}_-(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.24}$$

in which g is the quotient

$$g = \frac{b_3}{b_1}, \tag{6.25}$$

and $\widehat{f}_\pm$, $\widehat{g}_\pm$ are C^k-functions in their arguments of the form

$$\begin{aligned}
&\widehat{f}_\pm = \widehat{f}_{\pm,0} + \widehat{f}_{\pm,1}, \quad \widehat{g}_\pm = \widehat{g}_{\pm,0} + \widehat{g}_{\pm,1},\\
&\widehat{f}_{\pm,0} = \widehat{f}_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{f}_{\pm,1} = \widehat{f}_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon^{3/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{g}_{\pm,0} = \widehat{g}_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&\widehat{g}_{\pm,1} = \widehat{g}_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x\widehat{x}/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon(|C_\pm| + |D_\pm|)).
\end{aligned}$$

Solving the equations (6.21) and (6.23) for D_+ and D_-, respectively, we rewrite the first order system (6.21)-(6.24) as a second order system,

$$C''_+ = \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+ + h_+(C_\pm, C'_\pm, \overline{C_\pm}, \overline{C'_\pm}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.26}$$

$$C''_- = \left(-1 + g|C_+|^2 + |C_-|^2\right) C_- + h_-(C_\pm, C'_\pm, \overline{C_\pm}, \overline{C'_\pm}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}), \tag{6.27}$$

where we replaced $\widehat{x}$ by x, for notational convenience, and $h_\pm$ are C^k-functions in their arguments of the form

$$\begin{aligned}
&h_\pm = h_{\pm,0} + h_{\pm,1},\\
&h_{\pm,0} = h_{\pm,0}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, \varepsilon^{1/2}) = O(\varepsilon^{1/2}(|C_\pm| + |D_\pm|)),\\
&h_{\pm,1} = h_{\pm,1}(C_\pm, D_\pm, \overline{C_\pm}, \overline{D_\pm}, e^{\pm ik_x x/|b_0\varepsilon|^{1/2}}, \varepsilon^{1/2}) = O(\varepsilon(|C_\pm| + |D_\pm|)).
\end{aligned}$$

Notice that both systems above inherit the symmetries of the normal form (6.4).

Through the change of variables (6.20), after rescaling θ, from the periodic solutions $\mathbf{Z}_{\varepsilon,\theta}$ of the normal form (6.4) we obtain a family of solutions $\mathbf{P}_{\varepsilon,\theta}$ of the second order system (6.26)-(6.27). The properties below are easily obtained from the ones found for $\mathbf{Z}_{\varepsilon,\theta}$ in Section 6.2.

Lemma 6.3. *For any $\varepsilon > 0$ and θ sufficiently small, the system (6.26)-(6.27) possesses a two-parameter family of solutions $\mathbf{P}_{\varepsilon,\theta}$ with the following properties:*

(i) $e^{-i\theta x}\mathbf{P}_{\varepsilon,\theta}$ is periodic in x with wavenumber $\theta + k_x/|b_0\varepsilon|^{1/2}$;

(ii) $\mathbf{S}_1\mathbf{S}_2(\mathbf{P}_{\varepsilon,\theta}(x)) = \mathbf{P}_{\varepsilon,\theta}(-x)$*, for all* $x \in \mathbb{R}$*;*

(iii) $\mathbf{P}_{\varepsilon,\theta}(x) = \big((1-\theta^2)^{1/2}e^{i\theta x}, 0\big) + O(\varepsilon^{1/2})$*, as* $(\varepsilon, \theta) \to (0,0)$*;*

(iv) $\mathbf{P}_{\varepsilon,\theta}$ *corresponds to a solution of the system* (3.3) *which is a rotated roll* $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}^*_{k,\mu}$ *with*

$$\cos\beta = k_y/k, \quad \mu = \mu_c + \varepsilon, \quad k = k_c + |b_0\varepsilon|^{1/2}\theta\sin\alpha + O(\varepsilon\theta^2). \tag{6.28}$$

Notice that $\mathbf{P}_{\varepsilon,\theta}$ is periodic in x when $\theta = 0$, whereas for $\theta \neq 0$ it is a quasiperiodic function. This comes from the change of variables (6.20) where in the expressions of $A_\pm$ and $B_\pm$ we only factored out the exponential $e^{ik_x x}$, instead of the exponential $e^{i(k_x+\theta)x}$ which would have preserved periodicity. This lack of periodicity does not pose any problem for the remaining arguments, in which we only use the properties *(ii)-(iv)* above.

The second property in Lemma 6.3 shows that the solutions $\mathbf{P}_{\varepsilon,\theta}$ are reversible, the reversibility symmetry being $\mathbf{S}_1\mathbf{S}_2$. Using the reversibility symmetry $\mathbf{S}_1$, we obtain a second family of solutions of the system (6.26)-(6.27),

$$\mathbf{Q}_{\varepsilon,\theta}(x) = \mathbf{S}_1(\mathbf{P}_{\varepsilon,\theta}(-x)) = \Big(0, (1-\theta^2)^{1/2}e^{i\theta x}\Big) + O(\varepsilon^{1/2}). \tag{6.29}$$

These solutions have the properties *(i)* and *(ii)* in Lemma 6.3 and correspond to the rotated rolls $\boldsymbol{\mathcal{R}}_\beta\mathbf{U}^*_{k,\mu}$ satisfying (6.28). In addition, the family of maps $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ provides the circles of solutions $\boldsymbol{\tau}_a(\mathbf{P}_{\varepsilon,\theta})$ and $\boldsymbol{\tau}_a(\mathbf{Q}_{\varepsilon,\theta})$, $a \in \mathbb{R}/2\pi\mathbb{Z}$.

The existence proof in the next section requires that the quotient g in (6.25) takes values in the interval $(1, 4+\sqrt{13})$. The lemma below shows that this property holds at least for small angles α.

Lemma 6.4. *For any Prandtl number* $\mathcal{P}$*, there exists an angle* $\alpha_*(\mathcal{P}) \in (0, \pi/3]$ *such that* $1 < g < 4+\sqrt{13}$*, for any* $\alpha \in (0, \alpha_*(\mathcal{P}))$*.*

Proof. We compute the coefficient g in Appendix B.2. The result in formula (B.12) shows that the limit as α tends to 0 of g is equal to 2, which proves the result. ■

A symbolic computation, using the package Maple, of g shows that the inequality $g > 1$ holds for any Prandtl number $\mathcal{P} > 0$ and any angle $\alpha \in (0, \pi/3)$, and that the inequality $g < 4+\sqrt{13}$ holds in a region of the $(\alpha, \mathcal{P})$-plane which includes all positive values of the Prandtl number $\mathcal{P}$, for sufficiently small angles $\alpha \leqslant \alpha_*$, with $\alpha_* \approx \pi/9.112$, and all angles $\alpha \in (0, \pi/3)$, for sufficiently large Prandtl numbers $\mathcal{P} \geqslant \mathcal{P}_*$, with $\mathcal{P}_* \approx 0.126$ (see Figure 6.1).

7 Existence of domain walls

We construct domain walls as reversible heteroclinic solutions of (6.26)-(6.27) connecting the solutions $\mathbf{Q}_{\varepsilon,\theta}$ as $x \to -\infty$ with $\mathbf{P}_{\varepsilon,\theta}$ as $x \to \infty$, for a suitable $\theta = \theta(\varepsilon^{1/2})$ and $\varepsilon > 0$ sufficiently small. While the asymptotic solutions $\mathbf{P}_{\varepsilon,\theta}$ and $\mathbf{Q}_{\varepsilon,\theta}$ have the reversibility symmetry $\mathbf{S}_1\mathbf{S}_2$, the heteroclinic solutions will have the reversibility symmetry $\mathbf{S}_1$.

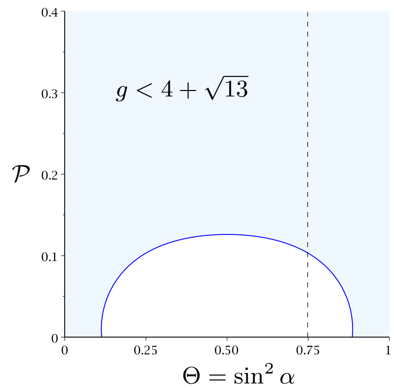


Figure 6.1: "Rigid-rigid" case. In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2\alpha$, Maple plot of the curve along which $g = 4+\sqrt{13}$, for $\Theta \in (0,1)$. The inequality $g < 4+\sqrt{13}$ holds in the shaded regions, whereas the inequality $g > 1$ holds everywhere. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$.

Following the approach developed in [10], we start by constructing a heteroclinic solution for the leading order system obtained at $\varepsilon = 0$ and then using the implicit function theorem we show that it persists for the full system. In contrast to the reduced system in [10] which was 12-dimensional, we have here an 8-dimensional system, only. This simplifies a part of the proof of Lemma 7.3 below. On the other hand, the quotient g takes here different values depending on the Prandtl number $\mathcal{P}$ and the angle α (see Figure 6.1), whereas $g = 2$ in [10]. We therefore need to extend the arguments from [10] to more general values g. We obtain a persistence result for $g \in (1, 4+\sqrt{13})$.

7.1 Leading order heteroclinic

Consider the leading order system

$$C_+'' = \left(-1 + |C_+|^2 + g|C_-|^2\right) C_+, \tag{7.1}$$

$$C_-'' = \left(-1 + g|C_+|^2 + |C_-|^2\right) C_-, \tag{7.2}$$

obtained by setting $\varepsilon = 0$ in (6.26)-(6.27). According to Lemma 6.3, this system has the solutions

$$\mathbf{P}_{0,\theta}(x) = \left((1-\theta^2)^{1/2} e^{i\theta x}, 0\right), \quad \mathbf{Q}_{0,\theta}(x) = \left(0, (1-\theta^2)^{1/2} e^{i\theta x}\right),$$

with θ sufficiently small. The leading order heteroclinic is constructed for $\theta = 0$, as a real-valued solution of (7.1)-(7.2) connecting the equilibrium $\mathbf{Q}_{0,0} = (0,1)$ as $x \to -\infty$ with the equilibrium $\mathbf{P}_{0,0} = (1,0)$ as $x \to \infty$.

Under the assumption that $g > 1$ [3], the existence of such a heteroclinic solution has been proved in [28]. According to [28, Theorem 5], for any $g > 1$, the system (7.1)-(7.2) possesses

[3] It turns out that this condition is necessary and sufficient.

a heteroclinic solution (C_+^*, C_-^*), where $C_\pm^*$ are smooth real-valued functions defined on $\mathbb{R}$ and have the following properties:

$$
\begin{aligned}
&(i) \quad \lim_{x\to-\infty}(C_+^*(x), C_-^*(x)) = (0,1) \text{ and } \lim_{x\to\infty}(C_+^*(x), C_-^*(x)) = (1,0);\\
&(ii) \quad C_+^*(x) = C_-^*(-x),\ \forall\, x \in \mathbb{R};\\
&(iii) \quad C_+^*(x)^2 + C_-^*(x)^2 \leqslant 1 \text{ and } C_+^*(x) + C_-^*(x) \geqslant \min(1, 2/\sqrt{g+1}),\ \forall\, x \in \mathbb{R};\\
&(iv) \quad (C_+^{*\prime}(x))^2 + (C_-^{*\prime}(x))^2 = \frac{1}{2}\left(C_+^*(x)^2 + C_-^*(x)^2 - 1\right)^2 + (g-1)C_+^*(x)^2 C_-^*(x)^2,\ \forall\, x \in \mathbb{R}.
\end{aligned}
$$

The second property above shows that (C_+^*, C_-^*) is reversible, with reversibility symmetry $\mathbf{S}_1$. The last property is a consequence of the Hamiltonian structure of the system (7.1)-(7.2), which was one of the key ingredients in the existence proof in [28]. Notice that the equilibria $(1,0)$ and $(0,1)$ of the system (7.1)-(7.2) are both saddles having a two-dimensional stable manifold and a two-dimensional unstable manifold. The heteroclinic connection (C_+^*, C_-^*) belongs to the intersection of the two-dimensional stable manifold of $(1,0)$ with the two-dimensional unstable manifold of $(0,1)$.

In addition to these properties, in the proof of Lemma 7.3 below we need the two results in the following lemma.

Lemma 7.1. *Consider the heteroclinic solution (C_+^*, C_-^*) of the system* (7.1)-(7.2)*.*

(i) For any $g > 1$, the functions C_+^* and C_-^* have the asymptotic behavior

$$C_+^*(x) = \alpha_* e^{\sqrt{g-1}\,x} + O(e^{(\sqrt{g-1}+\delta_*)x}), \quad C_-^*(x) = 1 - \beta_* e^{d_* x} + O(e^{(d_*+\delta_*)x}), \tag{7.3}$$

as $x \to -\infty$, for some positive constants α_*, d_*, δ_* and $\beta_* \geqslant 0$.

(ii) For any $g \in (1, 4+\sqrt{13})$, the functions C_+^* and C_-^* satisfy the inequality

$$3C_+^{*2}(x) + gC_-^{*2}(x) > 1, \quad \forall\, x \in \mathbb{R}. \tag{7.4}$$

Proof. (i) The heteroclinic connection (C_+^*, C_-^*) being included in the unstable manifold of the equilibrium $(0,1)$, the functions C_+^* and $1 - C_-^*$ decay exponentially to 0, as $x \to -\infty$. This implies the behavior of C_-^* and by taking into account the behavior of the different terms in the equation (7.1), we obtain the result for C_+^*.

(ii) For $g \in (3/2, 4+\sqrt{13})$ the property (7.4) is an immediate consequence of the inequality

$$C_+^*(x) + C_-^*(x) \geqslant \min(1, 2/\sqrt{g+1}), \quad \forall\, x \in \mathbb{R},$$

given above. We set

$$f_g(x) = 3C_+^{*2}(x) + gC_-^{*2}(x) - 1,$$

so that f_g is a smooth function defined on $\mathbb{R}$ and f_g is positive for any $g \in (3/2, 4+\sqrt{13})$. Assuming that there exists $g \in (1, 3/2]$ such that (7.4) does not hold, since f_g has positive limits at $x = \pm\infty$,

$$\lim_{x\to-\infty} f_g(x) = g - 1 > 0, \quad \lim_{x\to\infty} f_g(x) = 2,$$

and since the property holds for any $g \in (3/2, 4+\sqrt{13})$, there exists $g \in (1, 3/2]$ and $x_* \in \mathbb{R}$ such that

$$f_g(x_*) = 0, \quad f_g'(x_*) = 0, \quad f_g''(x_*) \geqslant 0, \tag{7.5}$$

i.e., f_g vanishes at a local minimum x_*.

For notational simplicity, we set

$$U = C_+^{*2}(x_*), \quad V = C_-^{*2}(x_*), \quad X = (C_+'(x_*))^2, \quad Y = (C_-'(x_*))^2.$$

Then the two equalities in (7.5) imply,

$$3U + gV = 1, \quad 9UX = g^2VY,$$

and from the property (iv) above we find that

$$X + Y = \frac{1}{2}(U+V-1)^2 + (g-1)UV.$$

Consequently, we can write V, X, Y as functions of U,

$$\begin{aligned} V &= \frac{1}{g}(1-3U), \\ X &= \frac{1}{2}\frac{(1-3U)((5g^2-9)U^2 + 6(1-g)U - (g-1)^2)}{g(3(g-3)U-g)}, \\ Y &= \frac{9}{2}\frac{U((5g^2-9)U^2 + 6(1-g)U - (g-1)^2)}{g^2(3(g-3)U-g)}, \end{aligned}$$

and then compute

$$\begin{aligned} f_g''(x_*) &= 2(3X + gY + 3U(-1+U+gV) + gV(-1+gU+V)) \\ &= \big(18(g-1)(g^2-9)U^3 + (12g(9-g^2) - 27(3+g^2))U^2 \\ &\quad +2g(g^2+6g-9)U + (g-1)(g-3)\big) / (g(g-3(g-3)U)). \end{aligned}$$

For $g \in (1, 3/2)$ and $U \in (0,1)$ we find that $f_g''(x_*) < 0$, which proves the result. ■

Remark 7.2. *(i) As pointed out in [28], the system* (7.1)-(7.2) *is integrable in the case* $g = 3$*, and the heteroclinic solution* (C_+^*, C_-^*) *can be explicitly computed in this case. We find that*

$$C_\pm^*(x) = \frac{1}{2}\left(1 \pm \tanh\left(\frac{x}{\sqrt{2}}\right)\right).$$

These formulas allow to easily check the properties in Lemma 7.1, and also the ones in Lemma 7.3 below, in this particular case.

(ii) The heteroclinic connection (C_+^*, C_-^*) *being real-valued, it is in fact a solution of the 4-dimensional system obtained by restricting* (7.1)-(7.2) *to the invariant subspace of real-valued solutions. As a solution of the (complex) 8-dimensional system, it belongs to the circle of heteroclinic solutions* $\boldsymbol{\tau}_a(C_+^*, C_-^*)$*, for* $a \in \mathbb{R}/2\pi\mathbb{Z}$*, and all these heteroclinic solutions are reversible. Notice that such a property does not hold for the circle of solutions* $\boldsymbol{\tau}_a(\mathbf{P}_{\varepsilon,\theta})$ *found in Section 6.3, the reason being that the reversibility symmetries are different,* $\mathbf{S}_1$ *for* (C_+^*, C_-^*) *and* $\mathbf{S}_1\mathbf{S}_2$ *for* $\mathbf{P}_{\varepsilon,\theta}$*.*

7.2 Persistence of the heteroclinic

The heteroclinic solution (C_+^*, C_-^*) is a particular reversible solution of the system (6.26)-(6.27) for $\varepsilon = 0$. Its persistence for small $\varepsilon > 0$ is proved by applying the implicit function theorem in a space of reversible exponentially decaying functions,

$$\mathcal{X}_\eta^r = \{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in \mathcal{X}_\eta \ ; \ C_+(x) = \overline{C_-}(-x), \ x \in \mathbb{R}\}, \tag{7.6}$$

where, for $\eta > 0$,

$$\mathcal{X}_\eta = \left\{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in (L_\eta^2)^4\right\}, \quad L_\eta^2 = \left\{f : \mathbb{R} \to \mathbb{C} \ ; \ \int_{\mathbb{R}} e^{2\eta|x|}|f(x)|^2 < \infty\right\}.$$

A key step of the proof is the analysis of the operator obtained by linearizing the leading order system (7.1)-(7.2), together with the complex conjugated equations, at (C_+^*, C_-^*), i.e., the linear operator $\mathcal{L}_*$ acting on (C_+, C_-) through

$$\mathcal{L}_* \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_+'' - \left(-1 + 2C_+^{*2} + gC_-^{*2}\right) C_+ - C_+^{*2}\overline{C_+} - gC_+^* C_-^* (C_- + \overline{C_-}) \\ C_-'' - \left(-1 + gC_+^{*2} + 2C_-^{*2}\right) C_- - C_-^{*2}\overline{C_-} - gC_+^* C_-^* (C_+ + \overline{C_+}) \end{pmatrix}.$$

In the space of exponentially decaying functions $\mathcal{X}_\eta$, the operator $\mathcal{L}_*$ is closed with dense domain

$$\mathcal{Y}_\eta = \left\{(C_+, C_-, \overline{C_+}, \overline{C_-}) \in (H_\eta^2)^4\right\}, \quad H_\eta^2 = \left\{f : \mathbb{R} \to \mathbb{C} \ ; \ f, f', f'' \in L_\eta^2\right\}, \tag{7.7}$$

and the subspace $\mathcal{X}_\eta^r$ of reversible functions is invariant under the action of $\mathcal{L}_*$, due to the reversibility of both the system (6.26)-(6.27) and the heteroclinic (C_+^*, C_-^*). The following lemma extends the result in [10, Lemma 4.1] to values $g \in (1, 4 + \sqrt{13})$.

Lemma 7.3. *Assume that $g \in (1, 4 + \sqrt{13})$. For any $\eta > 0$ sufficiently small, the operator $\mathcal{L}_*$ acting in $\mathcal{X}_\eta^r$ is Fredholm with index -1. The kernel of $\mathcal{L}_*$ is trivial, and the one-dimensional kernel of its L^2-adjoint is spanned by $(\mathrm{i}C_+^*, -\mathrm{i}C_-^*, -\mathrm{i}C_+^*, \mathrm{i}C_-^*)$.*

Proof. Taking as new variables the real and imaginary parts of $C_\pm$,

$$U_\pm = \frac{1}{2}(C_\pm + \overline{C_\pm}), \quad V_\pm = \frac{1}{2i}(C_\pm - \overline{C_\pm}),$$

we obtain the matrix operator

$$\mathcal{M}_* = \begin{pmatrix} \mathcal{M}_r & 0 \\ 0 & \mathcal{M}_i \end{pmatrix},$$

with

$$\mathcal{M}_r \begin{pmatrix} U_+ \\ U_- \end{pmatrix} = \begin{pmatrix} U_+'' - \left(-1 + 3C_+^{*2} + gC_-^{*2}\right) U_+ - 2gC_+^* C_-^* U_- \\ U_-'' - \left(-1 + gC_+^{*2} + 3C_-^{*2}\right) U_- - 2gC_+^* C_-^* U_+ \end{pmatrix},$$

$$\mathcal{M}_i \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \begin{pmatrix} V_+'' - \left(-1 + C_+^{*2} + gC_-^{*2}\right) V_+ \\ V_-'' - \left(-1 + gC_+^{*2} + C_-^{*2}\right) V_- \end{pmatrix},$$

acting in, respectively,

$$X_\eta^r = \left\{(U_+,U_-) \in (L_\eta^2)^2 \; ; \; U_+(x) = U_-(-x), \; x \in \mathbb{R}\right\},$$
$$X_\eta^i = \left\{(V_+,V_-) \in (L_\eta^2)^2 \; ; \; V_+(x) = -V_-(-x), \; x \in \mathbb{R}\right\}.$$

The properties of $\mathcal{L}_*$ are found from the ones of $\mathcal{M}_r$ and $\mathcal{M}_i$. In the case $g = 2$, the operator $\mathcal{M}_r$ has been studied in [9, Lemma 4.6] and the operator $\mathcal{M}_i$ in [10, Lemma 4.1]. Using the same arguments, it is straightforward to show that, for any $g > 1$, the operator $\mathcal{M}_r$ is Fredholm with index 0, whereas the operator $\mathcal{M}_i$ is Fredholm with index -1, has a trivial kernel, and the one-dimensional kernel of its L^2-adjoint is spanned by $(C_+^*, -C_-^*)$. To complete the proof it remains to show that the kernel of $\mathcal{M}_r$ is trivial. In this part of the proof, we use the two properties given in Lemma 7.1, the second one leading to the restriction $g \in (1, 4 + \sqrt{13})$.

Elements in the kernel of $\mathcal{M}_r$ are couples of functions $(U_+, U_-) \in X_\eta^r$, solving the linear system

$$U_+'' = \left(-1 + 3C_+^{*2} + gC_-^{*2}\right) U_+ + 2gC_+^* C_-^* U_-, \tag{7.8}$$
$$U_-'' = \left(-1 + gC_+^{*2} + 3C_-^{*2}\right) U_- + 2gC_+^* C_-^* U_+. \tag{7.9}$$

Due to the translation invariance of the leading order system (7.1)-(7.2), the derivative $(C_+^{*\prime}, C_-^{*\prime})$ is a solution of this linear system, but it does not satisfy the reversibility condition $U_+(x) = U_-(-x)$, and therefore it does not belong to the kernel of $\mathcal{M}_r$. We show below that the space of bounded solutions of this linear system is one-dimensional, hence spanned by the derivative $(C_+^{*\prime}, C_-^{*\prime})$ of the heteroclinic solution. This implies that the kernel of $\mathcal{M}_r$ is trivial and proves the result.

In the limit $x = -\infty$, the system (7.8)-(7.9) is autonomous, and the equations are decoupled,

$$U_+'' = (g-1)U_+, \quad U_-'' = 2U_-.$$

Consequently, the set of solutions of (7.8)-(7.9) which are bounded as $x \to -\infty$ is a two-dimensional vector space consisting of pairs (U_+, U_-) of exponentially decaying functions. Taking into account the exponential decay of solutions of the autonomous system and the asymptotic behavior of the heteroclinic solution in (7.3) we obtain that

$$U_+(x) = \alpha_+ e^{\sqrt{g-1}\,x} + O(e^{(\sqrt{g-1}+\delta_*)x}), \tag{7.10}$$

as $x \to -\infty$, for some $\alpha_+ \in \mathbb{R}$ and $\delta_* > 0$. We show below that $\alpha_+ \neq 0$, which implies that the space of bounded solutions of this linear system is one-dimensional. Indeed, assuming that there are two linearly independent solutions of (7.8)-(7.9), then a suitable linear combination of these solutions gives a solution with $\alpha_+ = 0$, which contradicts the property $\alpha_+ \neq 0$.

Assume that $\alpha_+ = 0$. Then the exponential decay of U_+ is given to leading order by the coupling term $2gC_+^* C_-^* U_-$ in (7.8). The product $2gC_+^* C_-^*$ being positive, this implies that U_+ and U_- have the same sign as $x \to -\infty$. Since both functions decay exponentially as $x \to -\infty$, they have constant signs on an interval $(-\infty, m)$, for some real number m. Assume, for instance, that they are both positive for x in $(-\infty, m)$, and take the first local maximum x_* of U_-, hence satisfying

$$U_-(x_*) > 0, \quad U_-'(x_*) = 0, \quad U_-''(x_*) \leqslant 0, \quad U_-(x) > 0, \ \forall\ x < x_*.$$

From the equation (7.9) we find

$$2gC_+^*(x_*)C_-^*(x_*)U_+(x_*) \leqslant -\left(-1 + gC_+^{*2}(x_*) + 3C_-^{*2}(x_*)\right) U_-(x_*),$$

which together with the property (7.4) in Lemma 7.1 and the positivity of $U_-(x_*)$, C_+^*, and C_-^*, implies that $U_+(x_*) < 0$. We claim that $U_+(x) < 0$, for all $x \leqslant x_*$. Indeed, assuming that U_+ is not negative, there exists a local maximum at some point $\widetilde{x}_* < x_*$ such that

$$U_+(\widetilde{x}_*) \geqslant 0, \quad U_+'(\widetilde{x}_*) = 0, \quad U_+''(\widetilde{x}_*) \leqslant 0.$$

Using now the equation (7.8), and arguing as above we obtain that $U_-(\widetilde{x}_*) \leqslant 0$, which contradicts the positivity of U_- for $x < x_*$. This implies that U_+ and U_- cannot have the same signs as $x \to -\infty$, which contradicts the assumption $\alpha_+ = 0$, and completes the proof. ■

The remaining part of the persistence proof consists in applying the implicit function theorem to show the existence of a heteroclinic solution for the full system (6.26)-(6.27), connecting $\mathbf{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\mathbf{P}_{\varepsilon,\theta}$, as $x \to \infty$. The operator $\mathcal{L}_*$ being Fredholm with index -1, the presence of the parameter θ is essential in these last arguments. In the proof, θ plays the role of an additional unknown which is determined as a function of ε when applying the implicit function theorem.

Theorem 2. *Assume that $g \in (1, 4+\sqrt{13})$. For any $\varepsilon > 0$ sufficiently small, there exists $\theta = O(\varepsilon^{1/2})$, continuously depending on $\varepsilon^{1/2}$, such that the system* (6.26)-(6.27) *possesses a reversible heteroclinic solution $\mathbf{C}_\varepsilon = (C_{+,\varepsilon}, C_{-,\varepsilon})$ connecting the solutions $\mathbf{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\mathbf{P}_{\varepsilon,\theta}$, as $x \to \infty$.*

Proof. We follow the proofs in [10, Theorem 2] and [26, Theorem 2].

The system (6.26)-(6.27) together with the complex conjugated equations is of the form

$$\mathcal{F}(\mathbf{C}, \overline{\mathbf{C}}, \varepsilon^{1/2}) = 0, \quad \mathbf{C} = (C_+, C_-), \tag{7.11}$$

and it has the particular solutions $\mathbf{P}_{\varepsilon,\theta}$ and $\mathbf{Q}_{\varepsilon,\theta}$ found in Section 6.3, for sufficiently small θ and $\varepsilon > 0$, and the heteroclinic solution $\mathbf{C}^* = (C_+^*, C_-^*)$ from Section 7.1, for $\varepsilon = 0$. We set

$$\widetilde{\mathbf{P}}_{\varepsilon,\theta} = \mathbf{P}_{\varepsilon,\theta} - (1,0)\, e^{i\theta x}, \quad \widetilde{\mathbf{Q}}_{\varepsilon,\theta} = \mathbf{Q}_{\varepsilon,\theta} - (0,1)\, e^{i\theta x},$$

and take a smooth function $\chi : \mathbb{R} \to [0,1]$ such that

$$\chi(x) = 1, \text{ if } x \geqslant M, \quad \chi(x) = 0, \text{ if } x \leqslant m,$$

for some positive constants $m < M$. We look for solutions of (7.11) of the form

$$\mathbf{C}(x) = e^{i\theta x}\mathbf{C}^*(x) + \chi(x)\widetilde{\mathbf{P}}_{\varepsilon,\theta}(x) + \chi(-x)\widetilde{\mathbf{Q}}_{\varepsilon,\theta}(x) + \mathbf{V}(x), \tag{7.12}$$

with $(\mathbf{V},\overline{\mathbf{V}}) \in \mathcal{Y}_\eta^r = \mathcal{Y}_\eta \cap \mathcal{X}_\eta^r$, where $\mathcal{X}_\eta^r$ and $\mathcal{Y}_\eta$ are defined in (7.6) and (7.7), respectively. Notice that the difference $\mathbf{C} - \mathbf{P}_{\epsilon,\theta}$ (resp. $\mathbf{C} - \mathbf{Q}_{\epsilon,\theta}$) decays exponentially to 0, as $x \to \infty$ (resp. $x \to -\infty$), with the same decay rate as $\mathbf{V}$, and that $\mathbf{C}$ and $\mathbf{V}$ have the same reversibility symmetry $\mathbf{S}_1$.

Substituting (7.12) into (7.11) we obtain an equation of the form

$$\mathcal{T}(\mathbf{V},\overline{\mathbf{V}},\theta,\varepsilon^{1/2}) = 0.$$

As shown in [10, Theorem 2], $\mathcal{T}(\mathbf{V},\overline{\mathbf{V}},\theta,\varepsilon^{1/2}) \in \mathcal{X}_\eta^r$, for any $(\mathbf{V},\overline{\mathbf{V}}) \in \mathcal{Y}_\eta^r$ and $(\theta,\varepsilon^{1/2})$ sufficiently small, and from the properties of $h_\pm$ in (6.26)-(6.27) we find that

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1, \quad \mathcal{T}_1 = O(\varepsilon), \tag{7.13}$$

with $\mathcal{T}_0$ continuously differentiable and $\mathcal{T}_1$ continuous and continuously differentiable with respect to $(\mathbf{V},\overline{\mathbf{V}},\theta)$. Furthermore,

$$\mathcal{T}(0,0,0,0) = \mathcal{F}(\mathbf{C}^*,\overline{\mathbf{C}^*},0) = 0,$$

and a direct calculation shows that

$$D_{\mathbf{V}}\mathcal{T}(0,0,0,0) = \mathcal{L}_*, \quad D_\theta\mathcal{T}(0,0,0,0) = \mathcal{L}_* \begin{pmatrix} ix\mathbf{C}^* \\ -ix\mathbf{C}^* \end{pmatrix} = \begin{pmatrix} 2i\mathbf{C}^{*\prime} \\ -2i\mathbf{C}^{*\prime} \end{pmatrix}.$$

According to Lemma 7.3, the operator $\mathcal{L}_*$ is Fredholm with index -1, injective, and its range is L^2-orthogonal to $(iC_+^*, -iC_-^*, -iC_+^*, iC_-^*)$. The L^2-scalar product of this vector with the differential $D_\theta\mathcal{T}(0,0,0,0)$ is given by

$$2\int_{\mathbb{R}} \left(2C_+^{*\prime}(x)C_+^*(x) - 2C_-^{*\prime}(x)C_-^*(x)\right) \mathrm{d}x = 2\int_{\mathbb{R}} \left(C_+^{*2}(x) - C_-^{*2}(x)\right)' \mathrm{d}x = 4 \neq 0, \tag{7.14}$$

which implies that $D_\theta\mathcal{T}(0,0,0,0)$ does not belong to the range of $\mathcal{L}^*$. Consequently, the differential $D_{(\mathbf{V},\theta)}\mathcal{T}(0,0,0,0)$ is bijective, and the result in the lemma follows from the implicit function theorem [5, Theorems 10.1.1 and 10.1.2] and (7.13). ■

Going back to the Bénard-Rayleigh problem, the result in this theorem, together with Lemma 6.3, implies the existence of a symmetric domain wall connecting two rotated rolls, $\boldsymbol{\mathcal{R}}_\beta\mathbf{U}_{k,\mu}^*$, as $x \to -\infty$, to $\boldsymbol{\mathcal{R}}_{-\beta}\mathbf{U}_{k,\mu}^*$, as $x \to \infty$, with $k = k_c + O(\varepsilon)$ and $\beta = \alpha + O(\varepsilon)$, for positive $\varepsilon = \mu - \mu_c$ sufficiently small. The family of maps $(\boldsymbol{\tau}_a)_{a\in\,\mathbb{R}/2\pi\mathbb{Z}}$ provides the circle of reversible heteroclinic solutions $\boldsymbol{\tau}_a(C_{+,\varepsilon}, C_{-,\varepsilon})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, which corresponds to translations in y of the symmetric domain wall. This proves Theorem 1 in the case of "rigid-rigid" boundary conditions. Notice that $\epsilon = \mathcal{R} - \mathcal{R}_c$ in Theorem 1 is linked to $\varepsilon = \mu - \mu_c$ in Theorem 2 through $\mathcal{R}^{1/2} = \mu$ and $\mathcal{R}_c^{1/2} = \mu_c$.

8 Discussion

This approach can also be used for other boundary conditions, when one, or both, of the rigid boundaries is replaced by a free boundary. It turns out that the arguments remain the same when both boundaries are free, but a major difference occurs in the case of one rigid and one free boundary. We briefly discuss these two cases below.

8.1 "Free-free" boundary conditions

In the case of two free boundaries, the "rigid-rigid" boundary conditions (1.5) are replaced by the "free-free" boundary conditions (1.6), the horizontal components (V_x, V_y) of the velocity field $\mathbf{V}$ satisfying now Neumann boundary conditions along the horizontal boundaries $z = 0, 1$, instead of Dirichlet boundary conditions. The equations in the system (1.1)-(1.3) are the same, and with these boundary conditions the system has exactly the same symmetries as in the case of "rigid-rigid" boundary conditions.

In the classical two-dimensional convection, the existence of rolls is shown as in Section 2.2. The sequence of parameter values $\mu_0(k) < \mu_1(k) < \mu_2(k) < \dots$ has the same properties as in Section 2.1, the difference being that in the boundary value problem (2.4)-(2.5) the equality $DV = 0$ is replaced by $D^2V = 0$. This changes the formula for $\mu_0(k)$, which is now explicit (see [22]),

$$\mu_0(k) = \frac{1}{|k|}\left(k^2 + \pi^2\right)^{3/2},$$

from which we easily obtain the numerical values

$$k_c = \frac{\pi}{\sqrt{2}}, \quad \mu_c = \frac{3\sqrt{3}}{2}\pi^2.$$

The solution V of the boundary value problem (2.4)-(2.5) is also explicit, $V(z) = \sin(\pi z)$.

In our approach, we replace the spaces $\mathcal{X}$ and $\mathcal{Z}$ in the spatial dynamics formulation (3.3) by

$$\begin{aligned}\mathcal{X} = \big\{\mathbf{U} \in (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega)\ ;\\ V_z = \theta = 0 \text{ on } z = 0, 1, \text{ and } \int_{\Omega_{per}} V_x\, dy\, dz = 0\big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Z} = \big\{\mathbf{U} \in \mathcal{X} \cap (H^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times H^2_{per}(\Omega) \times H^1_{per}(\Omega)\ ;\\ \partial_z V_x = \partial_z V_y = W_z = \phi = 0 \text{ on } z = 0, 1\big\}.\end{aligned}$$

The equations in (3.3) and the symmetries $\boldsymbol{\tau}_a$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{T}_b$ in Section 3 do not change, and the results and arguments in Sections 4-7, including the existence result in Theorem 2, remain valid. The only differences are at the computational level, in the different boundary value problems involving the component V_z of the velocity field, the equality $DV_z = 0$ being replaced by $D^2V_z = 0$ (for instance, the boundary value problem for V in the proof of Lemma 4.2).

The explicit formulas for $\mu_0(k)$ and for the solution V of the boundary value problem (2.4)-(2.5) given above, make the computation of the quotient g in Section B.2 much simpler in this case. We obtain an explicit formula for b_{31} in (B.12),

$$b_{31}(\Theta) = \frac{18\sqrt{3}\pi^8(1-\Theta)^2}{\ell_\Theta}\left((\Theta+2)^2 + \frac{9}{2}\Theta\,\mathcal{P}^{-1} + 3\Theta(\Theta+2)\mathcal{P}^{-2}\right),$$

and a Maple computation of the quotient g gives the result in Figure 8.1. This proves the result in Theorem 1 in the case of "free-free" boundary conditions.

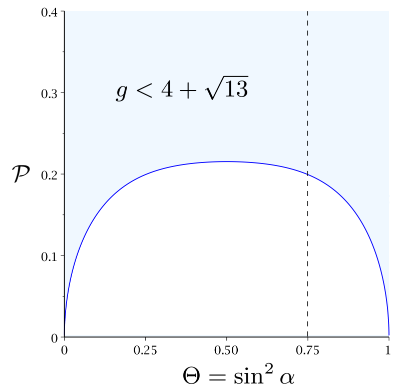


Figure 8.1: "Free-free" case. In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2\alpha \in (0,1)$, Maple plot of the curve along which $g = 4+\sqrt{13}$, in the case of "free-free" boundary conditions. The inequality $g < 4+\sqrt{13}$ holds in the shaded regions, whereas the inequality $g > 1$ holds everywhere. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$.

8.2 "Rigid-free" boundary conditions

In the case of one rigid and one free boundaries, the boundary conditions (1.5) are replaced by the "rigid-free" boundary conditions

$$V_x|_{z=0} = V_y|_{z=0} = 0, \quad \partial_z V_x|_{z=1} = \partial_z V_y|_{z=1} = 0, \quad V_z|_{z=0,1} = \theta|_{z=0,1} = 0, \tag{8.1}$$

and, as in the previous case, the equations (1.1)-(1.3) remain the same. In contrast to the "rigid-rigid" and "free-free" boundary conditions, these "rigid-free" boundary conditions are asymmetric and the system looses its reflection symmetry in the vertical coordinate z. As an immediate consequence, in the spatial dynamics formulation, the system (3.3) is not equivariant under the action of the symmetry $\mathbf{S}_3$ anymore. While the spectral properties of the linear operator $\mathcal{L}_{\mu_c}$ in Section 4 and the center manifold reduction in Section 5 remain valid, the parity properties of the reduced vector field in Lemma 5.2 do not hold. Consequently, in this case we do not have an invariant 8-dimensional center submanifold, and we have to treat the full 12-dimensional reduced system. This leads to additional difficulties.

First, the normal forms analysis in Section 6 becomes more complicated since it has to be done for 12-dimensional vector fields instead of 8-dimensional vector fields. As a result, the leading order normal form leads to the following system of three second order ODEs

$$C_0'' = \big(a_0 + a_1|C_0|^2 + a_2(|C_+|^2 + |C_-|^2)\big)\, C_0, \tag{8.2}$$

$$C_+'' = \big(b_0 + a_3|C_0|^2 + b_1|C_+|^2 + b_3|C_-|^2\big)\, C_+, \tag{8.3}$$

$$C_-'' = \big(b_0 + a_3|C_0|^2 + b_3|C_+|^2 + b_1|C_-|^2\big)\, C_-, \tag{8.4}$$

similar to the one found in [10] for the Swift-Hohenberg equation. The arguments in Section 6.2 remain valid showing that $b_0 < 0$, $b_1 > 0$, and assuming that $b_3/b_1 > 1$, we obtain a heteroclinic solution $(0, C_+^*, C_-^*)$, as in Section 7.1. Next, the persistence proof from [10], which has been done for particular values of the coefficients in the leading order system, has to be extended to more general systems of the form (8.2)-(8.4). This leads to additional conditions, to be determined, on the coefficients in the system (8.2)-(8.4). Checking these conditions requires further, and much longer, computations. This case is the object of future work.

A Some properties of linear operators

A.1 Adjoint operator

The explicit, but not so obvious, expression of the adjoint of operator $\mathcal{L}_\mu$ given below is necessary for computing the algebraic multiplicities of eigenvalues and the coefficients of the normal form.

Denote by $\langle\cdot,\cdot\rangle$ the scalar product in $(L^2_{per}(\Omega))^8$ and consider the closed subspace

$$\mathcal{H}_0 = \left\{\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi) \in (L^2_{per}(\Omega))^8 \;;\; \int_{\Omega_{per}} V_x \, dy\, dz = 0\right\} \subset (L^2_{per}(\Omega))^8,$$

which is the closure in $(L^2_{per}(\Omega))^8$ of both $\mathcal{X}$ and the domain of definition $\mathcal{Z}$ of the operator $\mathcal{L}_\mu$. We compute the adjoint $\mathcal{L}^*_\mu$ of $\mathcal{L}_\mu$ from the scalar product $\langle \mathcal{L}_\mu \mathbf{U}, \mathbf{U}'\rangle$, for $\mathbf{U} \in \mathcal{Z}$, and choose $\mathbf{U}' \in \mathcal{H}_0$ such that $\mathbf{U} \mapsto \langle \mathcal{L}_\mu \mathbf{U}, \mathbf{U}'\rangle$ is a linear continuous form on $\mathcal{H}_0$. We obtain the linear operator

$$\mathcal{L}^*_\mu \mathbf{U} = \begin{pmatrix} -\mu^{-1}\left(\Delta_\perp W_x - \langle \Delta_\perp W_x\rangle\right) \\ \nabla_\perp V_x - \mu^{-1}\Delta_\perp W_\perp - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot W_\perp) - \mu\phi\mathbf{e}_z \\ \nabla_\perp \cdot W_\perp \\ \mu V_\perp \\ -W_z - \Delta_\perp \phi \\ \theta \end{pmatrix},$$

where

$$\langle \Delta_\perp W_x\rangle = \int_{\Omega_{per}} \Delta_\perp W_x(y,z)\, dy\, dz.$$

The operator $\mathcal{L}^*_\mu$ is closed in the space $\mathcal{X}^*$ defined by

$$\begin{aligned}\mathcal{X}^* = \big\{\mathbf{U} \in (L^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times L^2_{per}(\Omega) \times H^1_{per}(\Omega) \;;\\ W_x = W_\perp = \phi = 0 \text{ on } z = 0,1, \text{ and } \int_{\Omega_{per}} V_x\, dy\, dz = 0\big\},\end{aligned}$$

with domain

$$\begin{aligned}\mathcal{Z}^* = \big\{\mathbf{U} \in \mathcal{X}^* \cap (H^1_{per}(\Omega))^3 \times (H^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times H^2_{per}(\Omega) \;;\\ V_\perp = \nabla_\perp \cdot W_\perp = \theta = 0 \text{ on } z = 0,1\big\}.\end{aligned}$$

The adjoint operator $\mathcal{L}^*_\mu$ has the same center spectrum as the operator $\mathcal{L}_\mu$. For our purposes we need to compute its kernel, an eigenvector associated with the eigenvalue $-ik$ of $\mathcal{L}^*_{\mu_0(k)}$, and one of the eigenvectors associated with the eigenvalue $-ik_x$ of $\mathcal{L}^*_{\mu_c}$. The kernel of $\mathcal{L}^*_\mu$ is easily computed by solving the equation $\mathcal{L}^*_\mu \mathbf{U} = 0$, and we find that it is spanned by the vector

$$\boldsymbol{\varphi}_0^* = (0, 0, 0, z(1-z), 0, 0, 0, 0,)^t .$$

We use this vector in the computation of the coefficients of the cubic normal form in Appendix B.2.

Next, for $\mu = \mu_0(k)$, the operator $\mathcal{L}^*_{\mu_0(k)}$ has the geometrically simple eigenvalues $\pm ik$, just as the operator $\mathcal{L}_{\mu_0(k)}$. In Appendix A.2 we need the expression of an eigenvector $\boldsymbol{\Psi}^*_{k,0}$ associated with the eigenvalue $-ik$. A direct calculation gives

$$\boldsymbol{\Psi}^*_{k,0}(y,z) = \widehat{\boldsymbol{\Psi}}^*_{k,0}(z), \quad \widehat{\boldsymbol{\Psi}}^*_{k,0}(z) = \begin{pmatrix} -\frac{1}{\mu_0(k)k^2}\left(D^3V_k - \langle D^3V_k\rangle\right) \\ 0 \\ \frac{ik}{\mu_0(k)}V_k \\ -\frac{i}{k}DV_k \\ 0 \\ -V_k \\ -ik\phi_k \\ \phi_k \end{pmatrix}, \tag{A.1}$$

where

$$\langle D^3V_k\rangle = \int_{\Omega_{per}} D^3V_k(z)\,dy\,dz,$$

V_k is the solution of the boundary value problem (4.10), and ϕ_k is the unique solution of the boundary value problem

$$(D^2 - k^2)\phi_k = V_k, \quad \phi_k|_{z=0,1} = 0.$$

Notice that the function ϕ_k is related to the function θ in the boundary value problem (2.4)-(2.5) through the equality $\theta = -\mu_0(k)\phi_k$.

Finally, in the computations in Appendix B.2 we also need an eigenvector associated with the eigenvalue $-ik_x$ of $\mathcal{L}^*_{\mu_c}$ which is of the form

$$\boldsymbol{\Psi}^*_+(y,z) = \widehat{\boldsymbol{\Psi}}^*_+(z)e^{ik_y y}.$$

We obtain that

$$\widehat{\boldsymbol{\Psi}}^*_+(z) = \begin{pmatrix} -\frac{1}{\mu_c k_c^2}(D^2 - k_c^2\cos^2\alpha)DV \\ -\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ -\frac{i\sin\alpha}{k_c}DV \\ -\frac{i\cos\alpha}{k_c}DV \\ -V \\ -ik_c(\sin\alpha)\phi \\ \phi \end{pmatrix},$$

where V is the solution of the boundary value problem (4.15), and ϕ is the unique solution of the boundary value problem

$$(D^2 - k_c^2)\phi = V, \quad \phi|_{z=0,1} = 0. \tag{A.2}$$

A.2 Algebraic multiplicities of $\pm ik$ and $\pm i\omega_1(k)$

Consider the geometrically simple eigenvalues $\pm ik$ and the geometrically double eigenvalues $\pm i\omega_1(k)$ of the operator $\mathcal{L}_{\mu_0(k)}$ given in Lemma 4.1. We assume that $\mu_0'(k) \neq 0$, and show that the algebraic multiplicities of these eigenvalues are equal to their geometric multiplicities. We prove the result for the eigenvalue ik, the arguments being the same for the eigenvalue $i\omega_1(k)$.

Assuming that the algebraic multiplicity of the eigenvalue ik is larger than its geometric multiplicity, there exists a vector $\mathbf{\Psi}_{k,0}$ such that

$$(\mathcal{L}_{\mu_0(k)} - ik)\mathbf{\Psi}_{k,0} = \mathbf{U}_{k,0}. \tag{A.3}$$

Differentiating the eigenvalue problem

$$\mathcal{L}_{\mu_0(k)}\mathbf{U}_{k,0} = ik\mathbf{U}_{k,0},$$

with respect to k leads to the equality

$$(\mathcal{L}_{\mu_0(k)} - ik)\left(\frac{d}{dk}\mathbf{U}_{k,0}\right) = \left(i - \mu_0'(k)\,\frac{\partial}{\partial\mu}\mathcal{L}_\mu\big|_{\mu=\mu_0(k)}\right)\mathbf{U}_{k,0}.$$

Since $\mu_0'(k) \neq 0$, this identity and the equality (A.3) imply that there is a solution $\mathbf{\Phi}_{k,0}$ of the linear equation

$$(\mathcal{L}_{\mu_0(k)} - ik)\mathbf{\Phi}_{k,0} = \frac{\partial}{\partial\mu}\mathcal{L}_\mu\big|_{\mu=\mu_0(k)}\mathbf{U}_{k,0}. \tag{A.4}$$

As a consequence, the vector in the right hand side of the above equation is orthogonal to the kernel of the adjoint operator $(\mathcal{L}^*_{\mu_0(k)} + ik)$, and in particular to the eigenvector $\mathbf{\Psi}^*_{k,0}$ given by (A.1). A direct computation shows that their scalar product is equal to the positive number

$$\frac{1}{\mu_0^2(k)k^2}\left(\|D^2V_k\|^2 + 2k^2\|DV_k\|^2 + k^4\|V_k\|^2\right) + \|D\phi_k\|^2 + k^2\|\phi_k\|^2 > 0.$$

This contradicts the orthogonality condition, and proves that the algebraic multiplicity of the eigenvalue ik is equal to its geometric multiplicity.

B Cubic normal form

B.1 Proof of Lemma 6.1

Proof. The existence of the polynomial $\boldsymbol{P}_\varepsilon$ and the first two properties in Lemma 6.1 follow from the general normal form theorems in [8, Sections 3.2.1, 3.3.1, and 3.3.2]. In addition, $N(\cdot,\cdot,\varepsilon)$ is an odd polynomial of degree 3 such that $N(0,0,\varepsilon) = 0$ and the identity

$$D_Z N(Z,\overline{Z},\varepsilon)L_0^*Z + D_{\overline{Z}}N(Z,\overline{Z},\varepsilon)\overline{L_0^*Z} = L_0^*N(Z,\overline{Z},\varepsilon), \tag{B.1}$$

in which L_0^* is the adjoint of L_0, holds for any $Z \in \mathbb{C}^4$ and $\varepsilon \in \mathcal{V}_2$. We write

$$N(Z, \overline{Z}, \varepsilon) = N_1(Z, \overline{Z})\varepsilon + N_3(Z, \overline{Z}),$$

where N_1 and N_3 denote the linear and cubic terms, respectively, of N. It is now straightforward to check that the linear part N_1 has the form in Lemma 6.1 (*iii*), and it remains to check the cubic terms N_3.

We set $N_3 = (\widetilde{N}_+, \widetilde{M}_+, \widetilde{N}_-, \widetilde{M}_-)$. Then the identity (B.1) becomes

$$\begin{aligned}(\mathcal{D}^* + ik_x)\widetilde{N}_+ = 0, &\quad (\mathcal{D}^* + ik_x)\widetilde{M}_+ = \widetilde{N}_+,\\ (\mathcal{D}^* + ik_x)\widetilde{N}_- = 0, &\quad (\mathcal{D}^* + ik_x)\widetilde{M}_- = \widetilde{N}_-,\end{aligned}$$

in which

$$\begin{aligned}\mathcal{D}^* \;=\; & -ik_xA_+\frac{\partial}{\partial A_+} + (A_+ - ik_xB_+)\frac{\partial}{\partial B_+} - ik_xA_-\frac{\partial}{\partial A_-} + (A_- - ik_xB_-)\frac{\partial}{\partial B_-}\\ & + ik_x\overline{A_+}\frac{\partial}{\partial \overline{A_+}} + (\overline{A_+} + ik_x\overline{B_+})\frac{\partial}{\partial \overline{B_+}} + ik_x\overline{A_-}\frac{\partial}{\partial \overline{A_-}} + (\overline{A_-} + ik_x\overline{B_-})\frac{\partial}{\partial \overline{B_-}}.\end{aligned}$$

Due to the equivariance of the normal form under the action of the symmetry $\mathbf{S}_2$, it is enough to determine $(\widetilde{N}_+, \widetilde{M}_+)$, the components $(\widetilde{N}_-, \widetilde{M}_-)$ being obtained by switching the indices $+$ and $-$ in the expressions of $(\widetilde{N}_+, \widetilde{M}_+)$.

Cubic monomials are of the form

$$A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-},$$

with nonnegative exponents such that

$$p_+ + q_+ + r_+ + s_+ + p_- + q_- + r_- + s_- = 3. \tag{B.2}$$

We claim that the cubic monomials in $\widetilde{N}_+$ and $\widetilde{M}_+$ also satisfy

$$p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_- = 1. \tag{B.3}$$

Indeed, for any monomial as above we have

$$\begin{aligned}&\mathcal{D}^*\left(A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-^{s_-}}\right) =\\ &\quad -ik_x\left(p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_-\right)A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-}\\ &\quad + r_+A_+^{p_++1}\overline{A_+}^{q_+}B_+^{r_+-1}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-}\\ &\quad + s_+A_+^{p_+}\overline{A_+}^{q_++1}B_+^{r_+}\overline{B_+}^{s_+-1}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-}\\ &\quad + r_-A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-+1}\overline{A_-}^{q_-}B_-^{r_--1}\overline{B_-}^{s_-}\\ &\quad + s_-A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-+1}B_-^{r_-}\overline{B_-}^{s_--1},\end{aligned}$$

implying that the subspace of monomials for which the sum in the left hand side of (B.3) is constant is invariant under the action of $\mathcal{D}^*$. Ordering the monomials by decreasing exponents

p_+, q_+, r_+, s_+, p_-, q_-, r_-, and s_-, this action is represented by a lower triangular matrix with equal elements on the diagonal given by

$$-ik_x\left(p_+ - q_+ + r_+ - s_+ + p_- - q_- + r_- - s_-\right).$$

Consequently, the polynomials $\widetilde{N}_+$ and $\widetilde{M}_+$, which belong to the kernel and generalized kernel of $\mathcal{D}_* + ik_x$, respectively, belong to the subspace for which (B.3) holds. This proves the claim. Furthermore, the commutativity of N_3 and $\boldsymbol{\tau}_a$, implies that monomials in $(\widetilde{N}_+, \widetilde{M}_+)$ also satisfy

$$p_+ - q_+ + r_+ - s_+ - p_- + q_- - r_- + s_- = 1. \tag{B.4}$$

Collecting all possible monomials in $(\widetilde{N}_+, \widetilde{M}_+)$ for which the conditions (B.2)-(B.4) hold, we compute:

$$\begin{aligned}
&(\mathcal{D}^* + ik_x)(A_+^2\overline{A_+}) = 0,\\
&(\mathcal{D}^* + ik_x)(A_+^2\overline{B_+}) = (\mathcal{D}^* + ik_x)(A_+\overline{A_+}B_+) = A_+^2\overline{A_+},\\
&(\mathcal{D}^* + ik_x)(A_+B_+\overline{B_+}) = A_+^2\overline{B_+} + A_+\overline{A_+}B_+, \quad (\mathcal{D}^* + ik_x)(\overline{A_+}B_+^2) = 2A_+\overline{A_+}B_+,\\
&(\mathcal{D}^* + ik_x)(B_+^2\overline{B_+}) = 2A_+B_+\overline{B_+} + \overline{A_+}B_+^2,
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{D}^* + ik_x)(A_+A_-\overline{A_-}) = 0,\\
&(\mathcal{D}^* + ik_x)(A_+A_-\overline{B_-}) = (\mathcal{D}^* + ik_x)(A_+\overline{A_-}B_-) = (\mathcal{D}^* + ik_x)(B_+A_-\overline{A_-}) = A_+A_-\overline{A_-}\\
&(\mathcal{D}^* + ik_x)(A_+B_-\overline{B_-}) = A_+A_-\overline{B_-} + A_+\overline{A_-}B_-,\\
&(\mathcal{D}^* + ik_x)(B_+A_-\overline{B_-}) = A_+A_-\overline{B_-} + B_+A_-\overline{A_-},\\
&(\mathcal{D}^* + ik_x)(B_+\overline{A_-}B_-) = A_+\overline{A_-}B_- + B_+A_-\overline{A_-},\\
&(\mathcal{D}^* + ik_x)(B_+B_-\overline{B_-}) = A_+B_-\overline{B_-} + B_+A_-\overline{B_-} + B_+\overline{A_-}B_-.
\end{aligned}$$

Since $\widetilde{N}_+$ and $\widetilde{M}_+$ are necessarily linear combinations of these 14 monomials, the equalities above imply that they are of the form

$$\begin{aligned}
\widetilde{N}_+ &= A_+\widetilde{P}_+(u_1, u_2, u_3, u_4) + A_-\widetilde{R}_+(u_5),\\
\widetilde{M}_+ &= B_+\widetilde{P}_+(u_1, u_2, u_3, u_4) + B_-\widetilde{R}_+(u_5) + A_+\widetilde{Q}_+(u_1, u_2, u_3, u_4) + A_-\widetilde{S}_+(u_5),
\end{aligned}$$

with $\widetilde{P}_+, \widetilde{R}_+, \widetilde{Q}_+, \widetilde{S}_+$ linear in their arguments, which are the quadratic expressions

$$\begin{aligned}
&u_1 = A_+\overline{A_+}, \quad u_2 = i(A_+\overline{B_+} - \overline{A_+}B_+), \quad u_3 = A_-\overline{A_-},\\
&u_4 = i(A_-\overline{B_-} - \overline{A_-}B_-), \quad u_5 = (A_+\overline{B_-} - \overline{A_-}B_+).
\end{aligned}$$

This proves the expressions of the cubic terms of N_+ and M_+ in (iii). Finally, taking into account the action of the reversibility $\mathbf{S}_1$, it is straightforward to check that the coefficients β_j, b_j, γ_5, and c_5 are real. ■

B.2 Computation of the quotient $g = b_3/b_1$

For the computation of the coefficients b_1 and b_3, we follow the method in [8, Section 3.4.1]. We restrict to the 8-dimensional center manifold

$$\mathcal{M}_\pm(\varepsilon) = \{\mathbf{U}_c + \mathbf{\Phi}(\mathbf{U}_c, \varepsilon)\ ;\ \mathbf{U}_c \in E_\pm\}.$$

Recall that solutions on this submanifold are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$. Combining the transformations from the center manifold reduction in Section 5.1 and the normal form in Lemma 6.1, we write

$$\begin{aligned}\mathbf{U} \;=\;& A_+\boldsymbol{\zeta}_+ + B_+\mathbf{\Psi}_+ + A_-\boldsymbol{\zeta}_- + B_-\mathbf{\Psi}_- + \overline{A_+\boldsymbol{\zeta}_+} + \overline{B_+\mathbf{\Psi}_+} + \overline{A_-\boldsymbol{\zeta}_-} + \overline{B_-\mathbf{\Psi}_-}\\ &+ \widetilde{\mathbf{\Phi}}(A_+, B_+, A_-, B_-, \overline{A_+}, \overline{B_+}, \overline{A_-}, \overline{B_-}, \varepsilon),\end{aligned}$$

in which $Z = (A_+, B_+, A_-, B_-)$ satisfies the normal form (6.4). Substituting $\mathbf{U}$ given by this formula in the dynamical system (3.3), and using the expressions of the derivatives of A_+, B_+, A_-, B_- given by the normal form in Lemma 6.1, we obtain an equality for the variables A_+, B_+, A_-, B_- and their complex conjugates. We find the coefficients of the normal form, and in particular b_1 and b_3, by identifying the coefficients of suitably chosen monomials in this equality.

We denote by $\mathbf{\Phi}_{rstu}$ the coefficient of the monomial $A_+^r\overline{A_+}^s A_-^t\overline{A_-}^u$ in the expansion of $\widetilde{\mathbf{\Phi}}$. Identifying successively the coefficients of the monomials $A_+^2\overline{A_+}$, $A_+A_-\overline{A_-}$, and then A_+^2, $A_+\overline{A_+}$, A_+A_-, $A_+\overline{A_-}$, $A_-\overline{A_-}$, we find the equalities

$$\begin{aligned}&i\beta_1\boldsymbol{\zeta}_+ + b_1\mathbf{\Psi}_+ = (\mathcal{L}_{\mu_c} - ik_x)\mathbf{\Phi}_{2100} + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{2000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1100}, \boldsymbol{\zeta}_+),\\ &i\beta_3\boldsymbol{\zeta}_+ + b_3\mathbf{\Psi}_+ = (\mathcal{L}_{\mu_c} - ik_x)\mathbf{\Phi}_{1011} + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{0011}, \boldsymbol{\zeta}_+),\end{aligned}$$

and

$$(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{2000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_+), \tag{B.5}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{1100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_+}), \tag{B.6}$$

$$(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{1010} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_-), \tag{B.7}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{1001} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_-}), \tag{B.8}$$

$$\mathcal{L}_{\mu_c}\mathbf{\Phi}_{0011} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_-, \overline{\boldsymbol{\zeta}_-}). \tag{B.9}$$

We determine the coefficients b_1 and b_3 by taking the scalar product of the first two equalities above with the vector $\mathbf{\Psi}_+^*$ in the kernel of the adjoint operator $(\mathcal{L}_{\mu_c} - ik_x)^*$ computed in Appendix A.1,

$$b_1\langle\mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle = \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{2000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^*\rangle, \tag{B.10}$$

$$b_3\langle\mathbf{\Psi}_+, \mathbf{\Psi}_+^*\rangle = \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{1001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{0011}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^*\rangle, \tag{B.11}$$

where $\mathbf{\Phi}_{2000}$, $\mathbf{\Phi}_{1100}$, $\mathbf{\Phi}_{1010}$, $\mathbf{\Phi}_{1001}$, and $\mathbf{\Phi}_{0011}$ are solutions of the linear equations (B.5)-(B.9).

In the equations (B.5) and (B.7), the linear operator $(\mathcal{L}_{\mu_c} - 2ik_x)$ is invertible, except in the case $\alpha = \pi/6$ when $2k_x = k_c$. Nevertheless, we only have to solve the equations in the subspace of vectors which are invariant under the action of $\mathbf{S}_3\boldsymbol{\tau}_\pi$ and the restriction of $(\mathcal{L}_{\mu_c} - ik_c)$ to this subspace is invertible, since its two-dimensional kernel is spanned by $\boldsymbol{\zeta}_0$ and $\overline{\boldsymbol{\zeta}_0}$ which do not belong to this subspace. Consequently, $\boldsymbol{\Phi}_{2000}$ and $\boldsymbol{\Phi}_{1010}$ are uniquely determined. In the equations (B.6), (B.8) and (B.9), the linear operator $\mathcal{L}_{\mu_c}$ has a one-dimensional kernel spanned by the vector $\boldsymbol{\varphi}_0$ in Lemma 4.2 (i), and the kernel of its adjoint is spanned by the vector $\boldsymbol{\varphi}_0^*$ in Appendix A.1. The solvability condition is easily checked in all cases, so that we can solve these equations up to an element in the kernel of $\mathcal{L}_\mu$. The choice of this element in the kernel does not influence the result from (B.10)-(B.11), since $\mathcal{B}_\mu$ is invariant upon adding a multiple of $\boldsymbol{\varphi}_0$.

After long and intricate computations we obtain that

$$g = \frac{b_3}{b_1} = \frac{b_{31}(\sin^2\alpha) + b_{31}(\cos^2\alpha) + b_{31}(0)}{\frac{1}{2}b_{31}(1) + b_{31}(0)}, \tag{B.12}$$

in which

$$b_{31}(\Theta) = A_{31}(\Theta) + B_{31}(\Theta)\mathcal{P}^{-1} + C_{31}(\Theta)\mathcal{P}^{-2},$$

with

$$\begin{aligned} A_{31}(\Theta) &= 2\mu_c^3\langle (D^2 - 4k_c^2\Theta)^2 V_1, R_1\rangle, \\ B_{31}(\Theta) &= 4\mu_c^3\Theta\left(\langle V_1, R_2\rangle + \langle V_2, R_1\rangle\right), \\ C_{31}(\Theta) &= -\frac{2\mu_c\Theta}{k_c^2}\langle (D^2 - 4k_c^2\Theta)V_2, R_2\rangle, \end{aligned}$$

where

$$R_1 = VD\phi + (1-2\Theta)\phi DV, \quad R_2 = \left(D^2 - 4k_c^2(1-\Theta)\right)(VDV) - 4\Theta(DV)(D^2V),$$

and V_1, V_2 are the unique solutions of the boundary value problems

$$\begin{aligned} &(D^2 - 4k_c^2\Theta)^3 V_1 + 4k_c^2\mu_c^2\Theta\, V_1 = R_1, \\ &V_1 = DV_1 = (D^2 - 4k_c^2\Theta)^2 V_1 = 0 \text{ in } z = 0, 1, \end{aligned}$$

and

$$\begin{aligned} &(D^2 - 4k_c^2\Theta)^3 V_2 + 4k_c^2\mu_c^2\Theta\, V_2 = R_2, \\ &V_2 = (D^2 - 4k_c^2\Theta)V_2 = (D^2 - 4k_c^2\Theta)DV_2 = 0 \text{ in } z = 0, 1, \end{aligned}$$

respectively. Recall that V and ϕ are the unique symmetric solutions of the boundary value problems (4.15) and (A.2), respectively. Notice that $g \to 2$, as $\alpha \to 0$, which was the value of g in the case of the Swift-Hohenberg equation in [10].

Remark B.1. *In this way we can also compute the coefficient b_0. By identifying the coefficients of the terms εA_+, and then taking the scalar product with $\boldsymbol{\Psi}_+^*$ we obtain*

$$b_0\langle \boldsymbol{\Psi}_+, \boldsymbol{\Psi}_+^*\rangle = \langle \mathcal{L}^{(1)}\boldsymbol{\zeta}_+, \boldsymbol{\Psi}_+^*\rangle,$$

in which $\mathcal{L}^{(1)}$ *is the derivative with respect to* μ *of the operator* $\mathcal{L}_\mu$ *in* (A.4) *taken at* $\mu = \mu_c$. *A direct computation gives*

$$b_0\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle = \frac{1}{\mu_c^2 k_c^2}\left(\|D^2V\|^2 + 2k_c^2\|DV\|^2 + k_c^4\|V\|^2\right) + \|D\phi\|^2 + k_c^2\|\phi\|^2 > 0, \tag{B.13}$$

and implies that $\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle < 0$*, since* $b_0 < 0$*. We point out that it is not obvious to determine the sign of this scalar product directly from the explicit formulas of* $\mathbf{\Psi}_+$ *and* $\mathbf{\Psi}_+^*$.

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