Domain walls for the Bénard-Rayleigh convection problem with “rigid-free” boundary conditions

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Abstract

We prove the existence of domain walls for the Bénard-Rayleigh convection in the case of “rigid-free” boundary conditions. In the recent work [5], we studied this bifurcation problem in the cases of “rigid-rigid” and “free-free” boundary conditions. In the three cases, for the existence proof we use a spatial dynamics approach in which the governing equations are written as an infinite-dimensional dynamical system. A center manifold theorem shows that bifurcating domain walls lie on a 12-dimensional center manifold, and can be constructed as heteroclinic solutions connecting periodic solutions of the restriction of the dynamical system to this center manifold. The existence proof for these heteroclinic connections then relies upon a normal form analysis, the construction of a leading order heteroclinic connection, and the implicit function theorem. The main difference between the case of “rigid-free” boundary conditions considered here and the two cases in [5], is the loss of a vertical reflection symmetry of the governing equations. This symmetry was exploited in [5] to show that bifurcating domain walls lie on an 8-dimensional invariant submanifold of the center manifold. Consequently, the heteroclinic connections were found as solutions of an 8-dimensional, instead of a 12-dimensional, dynamical system.

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1 Introduction

The classical Bénard-Rayleigh convection problem is concerned with the flow of a viscous fluid filling the region between two horizontal planes and heated from below. We consider the Boussinesq approximation in which the dependency of the fluid density ρ on the temperature T is given by the relationship

$$\rho = \rho_0 \left(1 - \gamma(T - T_0)\right),$$

where γ is the (constant) volume expansion coefficient, T_0 and ρ_0 are the temperature and the density, respectively, at the lower plane. In Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$, the fluid occupies the domain $\mathbb{R}^2 \times (0, d)$ between the lower horizontal plane $z = 0$ and the upper horizontal plane $z = d$. The mathematical problem consists in solving the Navier-Stokes equations coupled with an equation for energy conservation, for the particle velocity $\boldsymbol{V} = (V_x, V_y, V_z)$, the pressure p, and the deviation of the temperature from the conduction profile θ. In dimensionless variables, the system reads

$$\begin{aligned} \partial_t \boldsymbol{V} &= \mathcal{R}^{-1/2} \Delta \boldsymbol{V} + \theta \boldsymbol{e}_z - \mathcal{P}^{-1} (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} - \nabla p, && (1.1) \\ 0 &= \nabla \cdot \boldsymbol{V}, && (1.2) \\ \partial_t \theta &= \mathcal{R}^{-1/2} \Delta \theta + V_z - (\boldsymbol{V} \cdot \nabla) \theta, && (1.3) \end{aligned}$$

in which $\boldsymbol{e}_z = (0, 0, 1)$ is the unit vertical vector. The dimensionless constants $\mathcal{R}$ and $\mathcal{P}$ are the Rayleigh and the Prandtl numbers, respectively, defined as

$$\mathcal{R} = \frac{\gamma g d^3 (T_0 - T_1)}{\nu \kappa}, \quad \mathcal{P} = \frac{\nu}{\kappa},$$

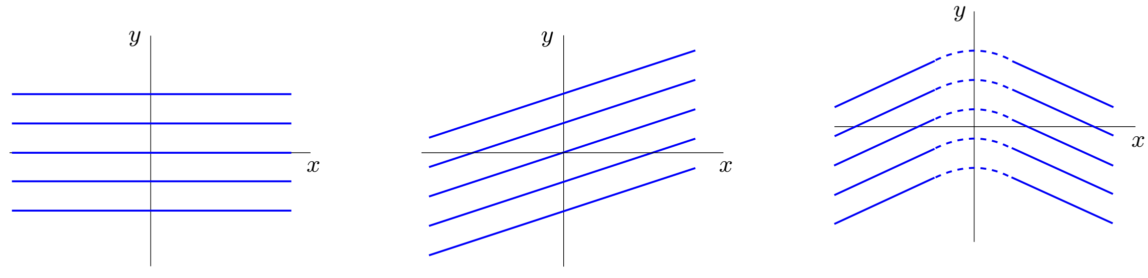
where ν is the kinematic viscosity, κ the thermal diffusivity, g the gravitational constant, and T_1 the temperature at the upper plane.

The equations (1.1)-(1.3) are completed by boundary conditions. Each of the two horizontal planar boundaries may be a rigid plane or a free boundary, hence leading to different possible types of boundary conditions: "rigid-rigid", "free-free" , "rigid-free", or "free-rigid". We consider here the case of a "rigid" boundary at $z = 0$ and a "free" boundary at $z = 1$,

$$V_x|_{z=0} = V_y|_{z=0} = 0, \quad \partial_z V_x|_{z=1} = \partial_z V_y|_{z=1} = 0, \quad V_z|_{z=0,1} = \theta|_{z=0,1} = 0. \qquad (1.4)$$

Together with these boundary conditions, the equations (1.1)-(1.3) are invariant under horizontal translations, rotations, and reflections in both x and y. While in the cases of "rigid-rigid" and "free-free" boundary conditions the system is also invariant under the vertical reflection $z \mapsto 1 - z$, this reflection changes the "rigid-free" boundary conditions above into "free-rigid" boundary conditions, with a "free" boundary at $z = 0$ and a "rigid" boundary at $z = 1$.

In the hydrodynamic problem, there is a critical value $\mathcal{R}_c$ of the Rayleigh number, below which the simple "conduction regime" is stable, the fluid is at rest and the temperature depends linearly on the vertical coordinate. Above this critical value, the "conduction regime" looses its stability and a "convective regime" appears. Steady patterns bifurcating in this "convective regime" have been extensively studied over the years. Mathematically, they are found as steady solutions of the Navier-Stokes-Boussinesq system (1.1)-(1.3). Already in the forties, Pellew and Southwell [10] computed the numerical value of $\mathcal{R}_c$ in the three cases of "rigid-rigid", "free-free", and "rigid-free" boundary conditions. The first existence results for regular patterns, such as rolls, hexagons, or squares, appeared in the sixties in the works by Yudovich et al [13, 15, 16, 17], Rabinowitz [11], Görtler et al [3]. We refer to [7] for further references on these types of patterns, and also to the recent work [2] on quasipatterns.



(a) **(b)** **(c)**

Figure 1.1: Projection in the (x, y)-horizontal plane of a **(a)** two-dimensional roll (periodic in y and constant in x); **(b)** rotated roll; **(c)** symmetric domain wall.

Besides regular convection patterns, different types of defects are regularly observed in experiments [1, 9]. Domain walls, or grain boundaries, are line defects which may occur between rolls with different orientations (see Figure 1.1), and are steady solutions of the system (1.1)-(1.3). In our previous work [5], we showed that symmetric domain walls bifurcate for the Navier-Stokes-Boussinesq system (1.1)-(1.3) with either "rigid-rigid" or "free-free" boundary conditions for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to the critical value $\mathcal{R}_c$. These domain walls are solutions of the steady system which are periodic in y, symmetric in x and their limits as $x \to \mp\infty$ are rolls rotated by opposite angles $\pm\alpha$ with $\alpha \in (0, \pi/3)$. Here, we prove that a similar result holds in the case of "rigid-free" boundary conditions. We point out that larger angles $\alpha \in [\pi/3, \pi/2)$ have been treated in the simpler case of the Swift-Hohenberg equation in [12] (see also [8]).

For the analysis of the bifurcation problem we use the spatial dynamics approach developed for the Swift-Hohenberg equation in [6] and adapted to the Navier-Stokes-Boussinesq system (1.1)-(1.3) in [5]. The steady system (1.1)-(1.3) is written as a dynamical system in which the evolutionary variable is the spatial variable x and a center manifold theorem is used to reduce this infinite dimensional dynamical system to a 12-dimensional system. The reduction holds for Rayleigh numbers $\mathcal{R}$ close to the critical value $\mathcal{R}_c$ and any fixed Prandtl number $\mathcal{P}$ and rotation angle $\alpha \in (0, \pi/3)$. Domain walls are constructed as heteroclinic solutions of this 12-dimensional system which connect suitably chosen periodic orbits. In [5], the vertical reflection symmetry has been used to further reduce this system to an 8-dimensional system. The loss of this symmetry in the case of "rigid-free" boundary conditions does not allow for this second reduction of dimension, here we have to analyze the full 12-dimensional system.

The first step in the construction of the heteroclinic connection for the reduced system consists in a normal form transformation. We prove a normal form theorem which is valid for general reversible and $O(2)$-equivariant 12-dimensional vector fields under a certain non-resonance condition. For our reduced system these symmetries are obtained from the invariance of the Navier-Stokes-Boussinesq system under reflections in the horizontal variables x and y and under translations in y, and the non-resonance condition is satisfied for angles $\alpha \neq \pi/6$. Next, similarly to [5], we scale variables in the normal form and identify a leading order system for which we construct a heteroclinic connection. Finally, using the implicit function theorem we prove the persistence of this leading order heteroclinic connection for the full 12-dimensional system. We obtain a heteroclinic solution connecting two periodic solutions of the 12-dimensional system. The two periodic solutions correspond to the same roll of the Navier-Stokes-Boussinesq system but rotated by opposite angles $\pm\alpha$ and the heteroclinic connection corresponds to a symmetric domain wall. The result holds under suitable conditions on two quotients g_1 and g_3 involving four coefficients of the normal form. While the conditions on the quotient g_3 are the same as the ones used in the persistence proof in [5], the conditions on the quotient g_1 are specific to the present case of "rigid-free" boundary conditions. Both g_1 and g_3 depend on the rotation angle α and the Prandtl number $\mathcal{P}$ through complicated analytical formulas. The validity of the required conditions is checked symbolically (Maple computations; see Figure 4.1). Our main result is summarized in the following theorem.

Theorem 1. *Consider the Navier-Stokes-Boussinesq system* (1.1)-(1.3) *with "rigid-free" boundary conditions* (1.4)*. Denote by $\mathcal{R}_c$ the critical Rayleigh number at which convective rolls with wavenumbers k_c bifurcate from the conduction state. Then for any angle $\alpha \in (0, \pi/3)$, $\alpha \neq \pi/6$, there exists $\mathcal{P}_*(\alpha) \geqslant 0$ such that, up to a finite set, for any Prandtl number $\mathcal{P} > \mathcal{P}_*(\alpha)$, a symmetric domain wall bifurcates for Rayleigh numbers $\mathcal{R} = \mathcal{R}_c + \epsilon$, with $\epsilon > 0$ sufficiently small. The domain wall connects two rotated rolls which are the rotations by opposite angles $\pm(\alpha + O(\epsilon))$ of a roll with wavenumber $k_c + O(\epsilon)$, continuously linked to the amplitude which is of order $O(\epsilon^{1/2})$.*

In contrast to the cases of "rigid-rigid" and "free-free" boundary conditions in [5], here we exclude the angle $\alpha = \pi/6$, because of the non-resonance condition in the normal form theorem, and for each angle $\alpha \in (0, \pi/3)$, $\alpha \neq \pi/6$, we exclude a finite number of values of the Prandtl number $\mathcal{P}$, at most, because of the presence of the additional four dimensions in the reduced system.

The analysis being similar to the one in [5], we focus in our presentation on the main differences, namely, the normal form analysis and the persistence proof. In Section 2 we recall the first steps of the approach which up to some computations are the same as in [5]. We give the formulation of the steady Navier-Stokes-Boussinesq system as a dynamical system in Section 2.1, briefly discuss the existence of rolls and rotated rolls in Section 2.2, and obtain the 12-dimensional reduced system in Section 2.3. In Section 3 we prove the general normal forms result for reversible and $O(2)$-equivariant 12-dimensional vector fields. The proof of Theorem 1 is completed in Section 4. In Section 4.1 we recall the connection between rotated rolls and periodic solutions of the normal form system. The leading order system is found in Section 4.2 and we construct the heteroclinic connection in Section 4.3. Finally, in Appendix A we compute the coefficients of the normal form and the quotients g_1 and g_3.

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2 Spatial dynamics approach

Following [5], we start in Section 2.1 by formulating the steady system (1.1)-(1.3) as a dynamical system. In Section 2.2 we recall the existence of rolls and rotated rolls, and in Section 2.3 we obtain the 12-dimensional reduced system by using the center manifold theorem. All these steps being similar to the ones in [5], we only recall the results and refer to [5] for further details.

As in [5], we replace the Rayleigh number $\mathcal{R}$ in the system (1.1)-(1.3) by its square root

$$\mu = \mathcal{R}^{1/2},$$

and denote by k_y the wavenumber in y of the solutions.

2.1 Formulation as a dynamical system

We introduce the new variables

$$\boldsymbol{W} = \mu^{-1}\partial_x \boldsymbol{V} - p\boldsymbol{e}_x, \quad \phi = \partial_x\theta,$$

and set

$$V_\perp = (V_y, V_z), \quad W_\perp = (W_y, W_z), \quad \boldsymbol{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi).$$

Then the steady system (1.1)-(1.3) is equivalent to the system

$$\partial_x \boldsymbol{U} = \mathcal{L}_\mu \boldsymbol{U} + \mathcal{B}_\mu(\boldsymbol{U}, \boldsymbol{U}), \tag{2.1}$$

in which the operators $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ are linear and quadratic, respectively, defined by

$$\mathcal{L}_\mu \boldsymbol{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \theta\boldsymbol{e}_z - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp\theta - \mu V_z \end{pmatrix},$$

$$\mathcal{B}_\mu(\boldsymbol{U}, \boldsymbol{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_x - V_x(\nabla_\perp \cdot V_\perp)\big) \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_\perp + \mu V_x W_\perp\big) \\ 0 \\ \mu\big((V_\perp \cdot \nabla_\perp)\theta + V_x\phi\big) \end{pmatrix}.$$

The choice of the phase space takes into account the boundary conditions (1.4), the periodicity in y of solutions, and the property that the flux

$$\mathcal{F}(x) = \int_{\Omega_{per}} V_x \, dy\, dz, \quad \Omega_{per} = (0, 2\pi/k_y) \times (0,1),$$

is constant. Fixing the constant flux to 0 we take the phase space

$$\mathcal{X} = \big\{ \boldsymbol{U} \in (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega) \ ; \ V_z = \theta = 0 \text{ on } z = 0, 1, \\ V_x = V_y = 0 \text{ on } z = 0, \text{ and } \int_{\Omega_{per}} V_x \, dy \, dz = 0 \big\},$$

where $\Omega = \mathbb{R} \times (0,1)$ and the subscript *per* means that the functions are $2\pi/k_y$-periodic in y. The remaining boundary conditions are included in the domain of definition of $\mathcal{L}_\mu$,

$$\mathcal{Z} = \big\{ \boldsymbol{U} \in \mathcal{X} \cap (H^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times H^2_{per}(\Omega) \times H^1_{per}(\Omega) \ ; \ W_z = \phi = 0 \text{ on } z = 0, 1, \\ \nabla_\perp \cdot V_\perp = W_y = 0 \text{ on } z = 0, \text{ and } \partial_z V_x = \partial_z V_y = 0 \text{ on } z = 1 \big\}.$$

The phase space $\mathcal{X}$ is a closed subspace of the Hilbert space

$$\widetilde{\mathcal{X}} = (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega),$$

so that it is a Hilbert space endowed with the usual scalar product of $\widetilde{\mathcal{X}}$ and with the choice above, the linear operator $\mathcal{L}_\mu$ is closed in $\mathcal{X}$ with dense and compactly embedded domain $\mathcal{Z}$. The latter property implies that $\mathcal{L}_\mu$ has purely point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities.

The dynamical system (2.1) is reversible and $O(2)$-equivariant. The reflection $x \mapsto -x$ gives the reversibility symmetry

$$\boldsymbol{S}_1 \boldsymbol{U}(y,z) = (-V_x, V_\perp, W_x, -W_\perp, \theta, -\phi)(y,z), \quad \boldsymbol{U} \in \mathcal{X},$$

which anti-commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$, the reflection $y \mapsto -y$ gives the symmetry

$$\boldsymbol{S}_2 \boldsymbol{U}(y,z) = (V_x, -V_y, V_z, W_x, -W_y, W_z, \theta, \phi)(-y,z), \quad \boldsymbol{U} \in \mathcal{X},$$

which commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$, and the horizontal translations $y \to y + a/k_y$ along the y direction give a one-parameter family of linear maps $(\boldsymbol{\tau}_a)_{a \in \mathbb{R}/2\pi\mathbb{Z}}$,

$$\boldsymbol{\tau}_a \boldsymbol{U}(y,z) = \boldsymbol{U}(y + a/k_y, z), \quad \boldsymbol{U} \in \mathcal{X},$$

which commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ and satisfy

$$\boldsymbol{\tau}_a \boldsymbol{S}_2 = \boldsymbol{S}_2 \boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_0 = \boldsymbol{\tau}_{2\pi} = \mathbb{I}.$$

In addition, the pressure p in the system (1.1)-(1.4) being only defined up to a constant, the dynamical system is invariant upon adding any constant to the new variable W_x, i.e., it is invariant under the action of the one-parameter family of maps $(\boldsymbol{T}_b)_{b \in \mathbb{R}}$ defined by

$$\boldsymbol{T}_b \boldsymbol{U} = \boldsymbol{U} + b \boldsymbol{\varphi}_0, \quad \boldsymbol{\varphi}_0 = (0,0,0,1,0,0,0,0)^t, \quad \boldsymbol{U} \in \mathcal{X}. \tag{2.2}$$

2.2 Rolls, rotated rolls and domain walls

The existence of convection rolls for the steady system (1.1)-(1.3) with any combination of "rigid"/"free" boundary conditions is well-known (e.g., see [5, Section 2.2]). We consider here rolls which are constant in x and periodic in y with wavenumber k (see Figure 1.1a). According to classical theory, convection rolls bifurcate supercritically at the instability threshold $\mu = \mu_c$ of the conduction state, for wavenumbers k close to a critical value k_c and $\mu > \mu_0(k)$ such that $\mu_0(k_c) = \mu_c$. The value $\mu_0(k)$ is the smallest value μ for which the boundary value problem

$$(D^2 - k^2)^2 V = \mu k^2 \theta, \quad V = DV = 0 \text{ in } z = 0,\ V = D^2 V = 0 \text{ in } z = 1, \tag{2.3}$$

$$(D^2 - k^2)\theta = -\mu V, \quad \theta = 0 \text{ in } z = 0, 1, \tag{2.4}$$

where $D = d/dz$ denotes the derivative with respect to z, possesses nontrivial real-valued solutions $(V, \theta) = (V_k(z), \theta_k(z))$. The map $k \mapsto \mu_0(k)$ is analytical in k and has a strict global minimum at $k = k_c$.[1]

These solutions being independent of x, they are equilibria of the dynamical system (2.1). For any $k_y = k > 0$ fixed close enough to k_c, we obtain a circle of equilibria $\boldsymbol{\tau}_a(\boldsymbol{U}^*_{k,\mu})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, which bifurcate for $\mu > \mu_0(k)$ sufficiently close to $\mu_0(k)$, belong to $\mathcal{Z}$, and satisfy

$$\boldsymbol{S}_1 \boldsymbol{U}^*_{k,\mu} = \boldsymbol{S}_2 \boldsymbol{U}^*_{k,\mu} = \boldsymbol{U}^*_{k,\mu}.$$

Furthermore, we have the expansions (see [5, Sections 2.2 and 4.1])

$$\boldsymbol{U}^*_{k,\mu}(y,z) = \delta e^{ik_y y} \widehat{\boldsymbol{U}}_k(z) + \delta e^{-ik_y y} \overline{\widehat{\boldsymbol{U}}_k(z)} + O(\delta^2), \quad \mu = \mu_0(k) + \mu_2 \delta^2 + O(\delta^3), \tag{2.5}$$

in which $\delta > 0$ is sufficiently small, $\mu_2 > 0$, and

$$\widehat{\boldsymbol{U}}_k(z) = \left(0, \frac{i}{k} DV_k(z), V_k(z), -p_k(z), 0, 0, \theta_k(z), 0\right)^T,$$

where (V_k, θ_k) is a solution of the boundary value problem (2.3)-(2.4) for $\mu = \mu_0(k)$ and the pressure p_k is determined up to a constant from the stationary part of the equation (1.1).

The system (1.1)-(1.4) is invariant under horizontal rotations acting through

$$\boldsymbol{\mathcal{R}}_\alpha(V_x, V_y, V_z, \theta)(x, y, z) = (\mathcal{R}_\alpha(V_x, V_y), V_z, \theta)(\mathcal{R}_{-\alpha}(x, y), z), \tag{2.6}$$

where

$$\mathcal{R}_\alpha(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha),$$

and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ is the rotation angle, and this rotation invariance is inherited by the dynamical system (2.1).[2] Consequently, the rotated rolls $\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu}$ are solutions of both systems. As solutions of the dynamical system (2.1), they are $2\pi/k \sin\alpha$-periodic solutions in x which belong to the phase space $\mathcal{X}$ for $k_y = k\cos\alpha$. For the particular angles $\alpha \in \{0, \pi\}$ the rotated rolls are

[1] Pellew and Southwell [10] computed the numerical values of k_c and $\mu_c = \mu_0(k_c)$ and found in the case of "rigid-free" boundary conditions that $k_c \approx 2.682$ and $\mu_c \approx 33.176$.

[2] We do not need here the more complicated representation formula for the 8-components vector $\boldsymbol{U}$ in (2.1).

equilibria in the phase-space $\mathcal{X}$ with $k_y = k$, whereas for $\alpha \in \{\pi/2, 3\pi/2\}$ they are $2\pi/k$-periodic solutions in x, for any $k_y > 0$. Notice that rolls rotated by angles α and $\pi + \alpha$ coincide,

$$\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{\pi+\alpha} \boldsymbol{U}^*_{k,\mu},$$

and that the actions of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ on a roll rotated by an angle $\alpha \notin \{0, \pi\}$ gives the same roll but rotated by the opposite angle,

$$\boldsymbol{S}_1(\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_{-\alpha} \boldsymbol{U}^*_{k,\mu}(-x), \quad \boldsymbol{S}_2 \boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu} = \boldsymbol{\mathcal{R}}_{-\alpha} \boldsymbol{U}^*_{k,\mu}. \tag{2.7}$$

In particular, rotated rolls keep a reversibility symmetry,

$$\boldsymbol{S}_1 \boldsymbol{S}_2(\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu}(x)) = \boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu}(-x). \tag{2.8}$$

Heteroclinic solutions of the dynamical system (2.1) connecting two rolls rotated by different angles are domain walls. We construct our symmetric domain walls as $\boldsymbol{S}_1$-reversible heteroclinic solutions of the dynamical system (2.1) which connect the rotated rolls $\boldsymbol{\mathcal{R}}_\alpha \boldsymbol{U}^*_{k,\mu}$ at $x = -\infty$ and $\boldsymbol{\mathcal{R}}_{-\alpha} \boldsymbol{U}^*_{k,\mu}$ at $x = \infty$, for rotation angles $\alpha \in (0, \pi/3)$.

2.3 Reduced bifurcation problem

Following the bifurcation analysis in [5, Section 4], we consider the dynamical system (2.1) in the phase $\mathcal{X}$ with

$$k_y = k_c \cos\alpha, \quad \alpha \in (0, \pi/3),$$

fix the Prandtl number $\mathcal{P}$ and take $\mu = \mu_c + \varepsilon$ close to μ_c as bifurcation parameter. We apply a center manifold theorem to construct a 12-dimensional invariant manifold which contains the small bounded solutions of the dynamical system (2.1).

We write the dynamical system (2.1) in the form

$$\partial_x \boldsymbol{U} = \mathcal{L}_{\mu_c} \boldsymbol{U} + \mathcal{R}(\boldsymbol{U}, \varepsilon), \tag{2.9}$$

where

$$\mathcal{R}(\boldsymbol{U}, \varepsilon) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c})\boldsymbol{U} + \mathcal{B}_\mu(\boldsymbol{U}, \boldsymbol{U})$$

is a smooth map from $\mathcal{Z} \times I_c$, $I_c = (-\mu_c, \infty)$, into $\mathcal{X}$, and

$$\mathcal{R}(0, \varepsilon) = 0, \quad D_{\boldsymbol{U}} \mathcal{R}(0, 0) = 0.$$

The hypotheses of the center manifold theorem are checked in the same way as in [5, Section 5.1]. The only differences are the explicit formulas of the eigenvectors and generalized eigenvectors which are slightly changed because the boundary conditions are different, and the absence of a symmetry due to the loss of the vertical reflection symmetry $z \mapsto 1 - z$ for the "rigid-free" boundary conditions.

As shown in [5, Lemma 4.2], the center spectrum $\sigma_c(\mathcal{L}_{\mu_c})$ of the linear operator $\mathcal{L}_{\mu_c}$ consists of five eigenvalues,

$$\sigma_c(\mathcal{L}_{\mu_c}) = \{0, \pm ik_c, \pm ik_x\}, \quad k_x = k_c \sin\alpha, \tag{2.10}$$

with the following properties.

(i) The eigenvalue 0 is simple with associated eigenvector φ_0 given by (2.2), which is invariant under the actions of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{\tau}_a$.

(ii) The complex conjugated eigenvalues $\pm ik_c$ are algebraically double and geometrically simple with associated generalized eigenvectors of the form

$$\boldsymbol{\zeta}_0(y,z) = \widehat{\boldsymbol{U}}_0(z), \quad \boldsymbol{\Psi}_0(y,z) = \widehat{\boldsymbol{\Psi}}_0(z),$$

for the eigenvalue ik_c, and the complex conjugated vectors for the eigenvalue $-ik_c$, such that

$$(\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\zeta}_0 = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_c)\boldsymbol{\Psi}_0 = \boldsymbol{\zeta}_0,$$

and

$$\begin{aligned} &\boldsymbol{S}_1\boldsymbol{\zeta}_0 = \overline{\boldsymbol{\zeta}_0}, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \\ &\boldsymbol{S}_1\boldsymbol{\Psi}_0 = -\overline{\boldsymbol{\Psi}_0}, \quad \boldsymbol{S}_2\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_0. \end{aligned}$$

(iii) The complex conjugated eigenvalues $\pm ik_x$ are algebraically quadruple and geometrically double with associated generalized eigenvectors of the form

$$\boldsymbol{\zeta}_\pm(y,z) = e^{\pm ik_y y}\widehat{\boldsymbol{U}}_\pm(z), \quad \boldsymbol{\Psi}_\pm(y,z) = e^{\pm ik_y y}\widehat{\boldsymbol{\Psi}}_\pm(z),$$

for the eigenvalue ik_x, and the complex conjugated vectors for the eigenvalue $-ik_x$, such that

$$(\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\zeta}_\pm = \mathbf{0}, \quad (\mathcal{L}_{\mu_c} - ik_x)\boldsymbol{\Psi}_\pm = \boldsymbol{\zeta}_\pm,$$

and

$$\begin{aligned} &\boldsymbol{S}_1\boldsymbol{\zeta}_+ = \overline{\boldsymbol{\zeta}_-}, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_+ = \boldsymbol{\zeta}_-, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_+ = e^{ia}\boldsymbol{\zeta}_+, \\ &\boldsymbol{S}_1\boldsymbol{\zeta}_- = \overline{\boldsymbol{\zeta}_+}, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_- = \boldsymbol{\zeta}_+, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_- = e^{-ia}\boldsymbol{\zeta}_-, \\ &\boldsymbol{S}_1\boldsymbol{\Psi}_+ = -\overline{\boldsymbol{\Psi}_-}, \quad \boldsymbol{S}_2\boldsymbol{\Psi}_+ = \boldsymbol{\Psi}_-, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_+ = e^{ia}\boldsymbol{\Psi}_+, \\ &\boldsymbol{S}_1\boldsymbol{\Psi}_- = -\overline{\boldsymbol{\Psi}_+}, \quad \boldsymbol{S}_2\boldsymbol{\Psi}_- = \boldsymbol{\Psi}_+, \quad \boldsymbol{\tau}_a\ \boldsymbol{\Psi}_- = e^{-ia}\boldsymbol{\Psi}_-. \end{aligned}$$

The explicit formulas of the eigenvectors $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ needed for our computations are given in Appendix A.

As a consequence of the center manifold theorem, the small bounded solutions of the dynamical system (2.9), and in particular the heteroclinic solutions, belong to a center manifold

$$\mathcal{M}_c(\varepsilon) = \{\boldsymbol{U}_c + \boldsymbol{\Phi}(\boldsymbol{U}_c,\varepsilon)\ ;\ \boldsymbol{U}_c \in \mathcal{X}_c\}, \tag{2.11}$$

where $\mathcal{X}_c$ is the spectral subspace associated with the center spectrum (2.10) of $\mathcal{L}_{\mu_c}$ and $\boldsymbol{\Phi} \in \mathcal{C}^k(\mathcal{X}_c \times I_c, \mathcal{Z}_h)$, for any arbitrary, but fixed, $k \geqslant 4$, with $\mathcal{Z}_h = (\mathbb{I} - \mathcal{P}_c)\mathcal{Z}$ and $\mathcal{P}_c$ the spectral projection onto $\mathcal{X}_c$. The invariant dynamics on the center manifold is determined by the reduced system

$$\frac{d\boldsymbol{U}_c}{dx} = \mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c}\boldsymbol{U}_c + \mathcal{P}_c\mathcal{R}(\boldsymbol{U}_c + \boldsymbol{\Phi}(\boldsymbol{U}_c,\varepsilon),\varepsilon) \stackrel{def}{=} f(\boldsymbol{U}_c,\varepsilon), \tag{2.12}$$

where

$$f(0,\varepsilon)=0,\quad D_{\boldsymbol{U}_c}f(0,0)=\mathcal{L}_{\mu_c}\big|_{\mathcal{X}_c},$$

and this reduced system inherits the symmetries of (2.9), i.e., the reduced vector field $f(\cdot,\varepsilon)$ anti-commutes with $\boldsymbol{S}_1$, commutes with $\boldsymbol{S}_2$ and $\boldsymbol{\tau}_a$, and is invariant under the action of $\boldsymbol{T}_b$.

Writing

$$\begin{aligned}\boldsymbol{U}_c \;&=\; w\boldsymbol{\varphi}_0+A_0\boldsymbol{\zeta}_0+B_0\boldsymbol{\Psi}_0+A_+\boldsymbol{\zeta}_++B_+\boldsymbol{\Psi}_++A_-\boldsymbol{\zeta}_-+B_-\boldsymbol{\Psi}_- \\ &\quad+\overline{A_0\boldsymbol{\zeta}_0}+\overline{B_0\boldsymbol{\Psi}_0}+\overline{A_+\boldsymbol{\zeta}_+}+\overline{B_+\boldsymbol{\Psi}_+}+\overline{A_-\boldsymbol{\zeta}_-}+\overline{B_-\boldsymbol{\Psi}_-},\end{aligned}\tag{2.13}$$

with $w\in\mathbb{R}$ and $X=(A_0,B_0,A_+,B_+,A_-,B_-)\in\mathbb{C}^6$, the reduced system (2.12) leads to the system

$$\frac{dw}{dx}=h(w,X,\overline{X},\varepsilon),\tag{2.14}$$

$$\frac{dX}{dx}=F(w,X,\overline{X},\varepsilon),\tag{2.15}$$

together with the complex conjugated equation of (2.15) for $\overline{X}$, in which h is real-valued and $F=(f_0,g_0,f_+,g_+,f_-,g_-)$ has six complex-valued components. The invariance of the reduced system under the action of $\boldsymbol{T}_b$,

$$\boldsymbol{T}_b(w,A_0,B_0,A_+,B_+,A_-,B_-)=(w+b,A_0,B_0,A_+,B_+,A_-,B_-),$$

implies that the reduced vector field (h,F) does not depend on w, so that we can first solve (2.15) for X, and then determine w by integrating (2.14). Consequently, we can restrict to solving the equation

$$\frac{dX}{dx}=F(X,\overline{X},\varepsilon),\tag{2.16}$$

which together with the complex conjugate equation for $\overline{X}$ form a 12-dimensional system. Notice that the symmetries of the reduced system act on these variables through

$$\boldsymbol{S}_1(A_0,B_0,A_+,B_+,A_-,B_-)=(\overline{A_0},-\overline{B_0},\overline{A_-},-\overline{B_-},\overline{A_+},-\overline{B_+}),\tag{2.17}$$

$$\boldsymbol{S}_2(A_0,B_0,A_+,B_+,A_-,B_-)=(A_0,B_0,A_-,B_-,A_+,B_+),\tag{2.18}$$

$$\boldsymbol{\tau}_a(A_0,B_0,A_+,B_+,A_-,B_-)=(A_0,B_0,e^{ia}A_+,e^{ia}B_+,e^{-ia}A_-,e^{-ia}B_-).\tag{2.19}$$

3 A cubic normal form for 12-dimensional vector fields

In this section we prove a normal form theorem for reversible and $O(2)$-equivariant 12-dimensional vector fields having the same linear part and symmetries as the one in (2.16).

Theorem 2. *Consider a system of ordinary differential equations*

$$\frac{dX}{dx}=G(X,\overline{X},\varepsilon),\tag{3.1}$$

in which $X=(A_0,B_0,A_+,B_+,A_-,B_-)\in\mathbb{C}^6$ *and the vector field* G *is of class* C^k*, for some* $k\geqslant 4$*, in a neighborhood* $\mathcal{U}_1\times\overline{\mathcal{U}_1}\times\mathcal{U}_2\subset\overline{\mathbb{C}^6}\times\mathbb{C}^6\times\mathbb{R}$ *of the origin. Assume that*

$$G(0,0,\varepsilon)=0,\quad D_XG(0,0,0)=L_0,\quad D_{\overline{X}}G(0,0,0)=0,$$

where L_0 *is a Jordan matrix acting on* X *through*

$$L_0=\begin{pmatrix} B_1 & 0 & 0\\ 0 & B_2 & 0\\ 0 & 0 & B_2\end{pmatrix},\quad B_1=\begin{pmatrix} ik_c & 1\\ 0 & ik_c\end{pmatrix},\quad B_2=\begin{pmatrix} ik_x & 1\\ 0 & ik_x\end{pmatrix},$$

with $1<k_c/k_x\neq 2$*, and that* $G(\cdot,\cdot,\varepsilon)$ *anti-commutes with* $\boldsymbol{S}_1$ *given by* (2.17) *and commutes with* $\boldsymbol{S}_2$ *and* $\boldsymbol{\tau}_a$ *given by* (2.18) *and* (2.19)*, respectively.*

There exist neighborhoods $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *of* 0 *in* $\mathbb{C}^6$ *and* $\mathbb{R}$*, respectively, such that for any* $\varepsilon\in\mathcal{V}_2$*, there is a polynomial* $P(\cdot,\cdot,\varepsilon):\mathbb{C}^6\times\overline{\mathbb{C}^6}\to\mathbb{C}^6$ *of degree* 3 *in the variables* $(Z,\overline{Z})$*, such that for* $Z\in\mathcal{V}_1$*, the change of variable*

$$X=Z+P(Z,\overline{Z},\varepsilon),$$

transforms the equation (3.1) into the normal form

$$\frac{dZ}{dx}=L_0Z+N(Z,\overline{Z},\varepsilon)+\rho(Z,\overline{Z},\varepsilon),\tag{3.2}$$

with the following properties:

(i) the map ρ *belongs to* $\mathcal{C}^k(\mathcal{V}_1\times\overline{\mathcal{V}_1}\times\mathcal{V}_2,\mathbb{C}^6)$*, and*

$$\rho(Z,\overline{Z},\varepsilon)=O(|\varepsilon|^2\|Z\|+\varepsilon\|Z\|^3+\|Z\|^4);$$

(ii) both $N(\cdot,\cdot,\varepsilon)$ *and* $\rho(\cdot,\cdot,\varepsilon)$ *anti-commute with* $\boldsymbol{S}_1$ *and commute with* $\boldsymbol{S}_2$ *and* $\boldsymbol{\tau}_a$*, for any* $\varepsilon\in\mathcal{V}_2$*;*

(iii) the six components $(N_0,M_0,N_+,M_+,N_-,M_-)$ *of* N *are of the form*

$$\begin{aligned}
N_0&=iA_0P_0+\alpha_5(A_+u_7+A_-u_8),\\
M_0&=iB_0P_0+A_0Q_0+\alpha_5(B_+u_7+B_-u_8)+ia_5(A_+u_7+A_-u_8),\\
N_+&=iA_+P_++\beta_7A_0\overline{u_7}+\beta_8A_-u_9,\\
M_+&=iB_+P_++A_+Q_++\beta_7B_0\overline{u_7}+ib_7A_0\overline{u_7}+\beta_8B_-u_9+ib_8A_-u_9,\\
N_-&=iA_-P_-+\beta_7A_0\overline{u_8}-\beta_8A_+\overline{u_9},\\
M_-&=iB_-P_-+A_-Q_-+\beta_7B_0\overline{u_8}+ib_7A_0\overline{u_8}-\beta_8B_+\overline{u_9}-ib_8A_+\overline{u_9},
\end{aligned}$$

with

$$P_0 = \alpha_0\varepsilon + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3(u_3+u_5) + \alpha_4(u_4+u_6),$$
$$Q_0 = a_0\varepsilon + a_1 u_1 + a_2 u_2 + a_3(u_3+u_5) + a_4(u_4+u_6),$$
$$P_+ = \beta_0\varepsilon + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5 + \beta_6 u_6,$$
$$Q_+ = b_0\varepsilon + b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 + b_5 u_5 + b_6 u_6,$$
$$P_- = \beta_0\varepsilon + \beta_1 u_1 + \beta_2 u_2 + \beta_5 u_3 + \beta_6 u_4 + \beta_3 u_5 + \beta_4 u_6,$$
$$Q_- = b_0\varepsilon + b_1 u_1 + b_2 u_2 + b_5 u_3 + b_6 u_4 + b_3 u_5 + b_4 u_6,$$

where $(A_0, B_0, A_+, B_+, A_-, B_-)$ *are the six components of* Z*, the coefficients* α_j*,* a_j*,* β_j*,* b_j *are all real, and*

$$u_1 = A_0\overline{A_0}, \quad u_2 = i(A_0\overline{B_0} - \overline{A_0}B_0),$$
$$u_3 = A_+\overline{A_+}, \quad u_4 = i(A_+\overline{B_+} - \overline{A_+}B_+), \quad u_5 = A_-\overline{A_-}, \quad u_6 = i(A_-\overline{B_-} - \overline{A_-}B_-),$$
$$u_7 = (A_0\overline{B_+} - \overline{A_+}B_0), \quad u_8 = (A_0\overline{B_-} - \overline{A_-}B_0), \quad u_9 = (A_+\overline{B_-} - \overline{A_-}B_+).$$

Proof. From general normal form theorems (e.g., see [4, Sections 3.2.1, 3.3.1, and 3.3.2]), we obtain the existence of two polynomials $P(\cdot,\cdot,\varepsilon)$ and $N(\cdot,\cdot,\varepsilon)$ of degree 3 in the variables $(Z,\overline{Z})$ such that the properties (i) and (ii) hold, the polynomial N is of the form

$$N(Z,\overline{Z},\varepsilon) = \widetilde{N}_1(Z,\overline{Z})\varepsilon + N_2(Z,\overline{Z}) + \widetilde{N}_2(Z,\overline{Z})\varepsilon + N_3(Z,\overline{Z}), \tag{3.3}$$

with N_p and $\widetilde{N}_p$ homogeneous polynomials of degree p in $(Z,\overline{Z})$, and satisfies the identity

$$D_Z N(Z,\overline{Z},\varepsilon)L_0^* Z + D_{\overline{Z}}N(Z,\overline{Z},\varepsilon)\overline{L_0^* Z} = L_0^* N(Z,\overline{Z},\varepsilon), \quad \forall\ (Z,\varepsilon)\in\mathbb{C}^6\times\mathcal{V}_2, \tag{3.4}$$

in which L_0^* is the adjoint of L_0. Due to the equivariance of the normal form under the action of the symmetry $\boldsymbol{S}_2$, it is enough to determine the first four components (N_0, M_0, N_+, M_+) of N, the result for (N_-, M_-) being obtained by exchanging the indices $+$ and $-$ in the expressions of (N_+, M_+).

Monomials in N_0 are M_0 are of the form

$$A_0^{p_0}\overline{A_0}^{q_0}B_0^{r_0}\overline{B_0}^{s_0}A_+^{p_+}\overline{A_+}^{q_+}B_+^{r_+}\overline{B_+}^{s_+}A_-^{p_-}\overline{A_-}^{q_-}B_-^{r_-}\overline{B_-}^{s_-}, \tag{3.5}$$

with nonnegative exponents such that

$$p_0+q_0+r_0+s_0+p_++q_++r_++s_++p_-+q_-+r_-+s_- = m, \quad m\in\{1,2,3\}. \tag{3.6}$$

From the commutativity of N and $\boldsymbol{\tau}_a$, we obtain that their exponents also satisfy the equality

$$(p_+-q_++r_+-s_+) - (p_--q_-+r_--s_-) = 0, \tag{3.7}$$

and we claim that we also have the equalities

$$p_0-q_0+r_0-s_0 = 1, \quad (p_+-q_++r_+-s_+) + (p_--q_-+r_--s_-) = 0, \tag{3.8}$$

when $k_c/k_x \neq 2$.

Indeed, the identity (3.4) implies that N_0 and M_0 satisfy the equalities

$$(\mathcal{D}^* + ik_c)N_0 = 0, \quad (\mathcal{D}^* + ik_c)M_0 = N_0,$$

in which

$$\begin{aligned}\mathcal{D}^* \;=\; & -ik_cA_0\frac{\partial}{\partial A_0} + (A_0 - ik_cB_0)\frac{\partial}{\partial B_0} + ik_c\overline{A_0}\frac{\partial}{\partial \overline{A_0}} + (\overline{A_0} + ik_c\overline{B_0})\frac{\partial}{\partial \overline{B_0}} \\ & -ik_xA_+\frac{\partial}{\partial A_+} + (A_+ - ik_xB_+)\frac{\partial}{\partial B_+} + ik_x\overline{A_+}\frac{\partial}{\partial \overline{A_+}} + (\overline{A_+} + ik_x\overline{B_+})\frac{\partial}{\partial \overline{B_+}} \\ & -ik_xA_-\frac{\partial}{\partial A_-} + (A_- - ik_xB_-)\frac{\partial}{\partial B_-} + ik_x\overline{A_-}\frac{\partial}{\partial \overline{A_-}} + (\overline{A_-} + ik_x\overline{B_-})\frac{\partial}{\partial \overline{B_-}},\end{aligned}$$

is a linear map which preserves the degree of homogeneous polynomials. For a fixed degree m, taking a basis of monomials ordered by decreasing exponents p_0, q_0, r_0, s_0, p_+, q_+, r_+, s_+, p_-, q_-, r_-, and s_-, the action of $\mathcal{D}^*$ is represented by a lower triangular matrix with equal elements on the diagonal given by

$$-ik_c\left(p_0 - q_0 + r_0 - s_0\right) - ik_x\left(p_+ - q_+ + r_+ - s_+\right) - ik_x\left(p_- - q_- + r_- - s_-\right).$$

Consequently, the polynomials N_0 and M_0, which belong to the kernel and generalized kernel of $\mathcal{D}_* + ik_c$, respectively, belong to the subspace spanned by monomials for which the quantity above is equal to $-ik_c$. Taking into account that $k_c/k_x > 1$ and the properties (3.6)-(3.7), we conclude that (3.8) holds when $k_c/k_x \neq 2$. This proves the claim.

Next, taking $m = 1$ in (3.6) and using (3.6)-(3.8) it is straightforward to check that the first two components of $\widetilde{N}_1$ in (3.3) have the form given in (iii). For even integers m, and in particular for $m = 2$, from the equalities (3.6) and (3.7) we obtain that $p_0 - q_0 + r_0 - s_0$ must be an even integer. This contradicts the first equality in (3.7). Consequently, there are no monomials of even degree in the first two components of N. It remains to consider the cubic monomials, $m = 3$. Collecting all cubic monomials satisfying (3.6)-(3.8), we directly compute the action of $(\mathcal{D}^* + ik_c)$ on all these monomials. Then we identify a basis for the kernel of $(\mathcal{D}^* + ik_c)$ which gives the result for N_0, and a basis for the generalized kernel of $(\mathcal{D}^* + ik_c)$ which gives the result for M_0.

For the components N_+ and M_+ of N the result is obtained in the same way. We only point out that for these polynomials the exponents of the monomials (3.5) satisfy (3.6), the equality

$$(p_+ - q_+ + r_+ - s_+) - (p_- - q_- + r_- - s_-) = 1,$$

replacing (3.7), and

$$p_0 - q_0 + r_0 - s_0 = 0, \quad (p_+ - q_+ + r_+ - s_+) + (p_- - q_- + r_- - s_-) = 1,$$

instead of (3.8).

Finally, taking into account the action of the reversibility $\boldsymbol{S}_1$, it is straightforward to check that the coefficients α_j, a_j, β_j and b_j are real. This completes the proof of the theorem. ∎

Remark 3.1. *In the resonant case $k_c/k_x = 2$, the two polynomials N_0 and M_0 contain the additional quadratic terms $\alpha_6 A_+ A_-$ and $i\alpha_6(B_+A_- + A_+B_-) + a_6 A_+A_-$, respectively, with real coefficients α_6 and c_6.*

Applying the result in Theorem 2 to our reduced system (2.16) we obtain its normal form (3.2) for $k_c/k_x \neq 2$, i.e., for any angle $\alpha \in (0, \pi/3)$, $\alpha \neq \pi/6$.

4 Existence of domain walls

In this section, we prove the existence of a heteroclinic connection for the normal form system (3.2). Following the analysis from [5], we focus on the main differences and refer to [5] for the technical details which remain the same.

We restrict to $\varepsilon > 0$, which corresponds to values $\mu > \mu_c$ for which rolls exist and exclude the resonant angle $\alpha = \pi/6$ which requires a different analysis.

4.1 Rotated rolls and coefficients of the normal form

The rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\boldsymbol{U}^*_{k,\mu}$ with rotation angle $\beta \in (0, \pi/2)$ are $2\pi/k\sin\beta$-periodic solutions of the dynamical system (2.1) and belong to the phase $\mathcal{X}$ when their wavenumber in y is equal to k_y,

$$k\cos\beta = k_y = k_c\cos\alpha.$$

For (k,μ) close enough to (k_c,μ_c) they are small bounded solutions which belong to the center manifold (2.11) of (2.9). Projected on the center space $\mathcal{X}_c$ they become $2\pi/k\sin\beta$-periodic solutions of the reduced system (3.1) and also of the normal form system (3.2). Comparing the formulas (2.5) and (2.6) for $\boldsymbol{\mathcal{R}}_{-\beta}\boldsymbol{U}^*_{k,\mu}$ with the formulas (2.11) and (2.13) for the solutions on the center manifold, we obtain a family $\boldsymbol{Z}_{\varepsilon,\theta}$ of $2\pi/k\sin\beta$-periodic solutions of the normal form system, where the parameters (ε,θ) are related to (k,μ) through the equalities

$$\varepsilon = \mu - \mu_c, \quad \theta = k\sin\beta - k_x = k\sin\beta - k_c\sin\alpha = \frac{1}{\sin\alpha}(k - k_c) + O(|k-k_c|^2).$$

These solutions have the expansion

$$\boldsymbol{Z}_{\varepsilon,\theta}(x) = \Big(0, 0, \delta e^{i(k_x+\theta)x}, 0, 0, 0\Big) + O(|\delta||\theta| + |\delta|^2),$$

with $\delta > 0$ as in (2.5), and satisfying

$$\delta^2 = \frac{1}{\mu_2}\varepsilon - \frac{\mu_0''(k_c)\sin^2\alpha}{2\mu_2}\theta^2 + O(|\varepsilon|^{3/2} + |\varepsilon|^{1/2}|\theta|^2 + |\theta|^3).$$

As shown in [5, Section 6.2], we can use this family of solutions to determine two coefficients of the normal form system

$$b_0 = -\frac{2}{\mu_0''(k_c)\sin^2\alpha} < 0, \quad b_3 = \frac{2\mu_2}{\mu_0''(k_c)\sin^2\alpha} > 0, \tag{4.1}$$

where $\mu_0''(k) > 0$ is the second order derivative of $\mu_0(k)$ with respect to k and $\mu_2 > 0$ is determined by the second equality in (2.5). The symmetry properties (2.7) and (2.8) of rotated rolls are preserved so that $\boldsymbol{Z}_{\varepsilon,\theta}$ is $\boldsymbol{S}_1\boldsymbol{S}_2$-reversible and the rolls $\boldsymbol{\mathcal{R}}_\beta \boldsymbol{U}^*_{k,\mu}$ rotated by the opposite angle β give the periodic solutions $\boldsymbol{S}_2\boldsymbol{Z}_{\varepsilon,\theta}$. We refer to [5, Section 6.2] for more details.

Similarly, by taking the rotation angle $\beta = \pi/2$, from the rotated rolls $\boldsymbol{\mathcal{R}}_{\pi/2}\boldsymbol{U}^*_{k,\mu}$, which are constant in y and therefore $2\pi/k$-periodic solutions of the dynamical system (2.1), we obtain a second family of periodic solutions for the normal form system. With the help of these solutions we can compute two other coefficients of the normal form system,

$$a_0 = -\frac{2}{\mu_0''(k_c)} < 0, \quad a_1 = \frac{2\mu_2}{\mu_0''(k_c)} > 0.$$

Notice that we have the following relationship between coefficients:

$$a_0 = b_0 \sin^2\alpha < 0, \quad a_1 = b_3 \sin^2\alpha > 0. \tag{4.2}$$

4.2 Leading order dynamics

We consider the new variables

$$\widehat{x} = |b_0\varepsilon|^{1/2}x, \quad A_0(x) = \left|\tfrac{b_0\varepsilon}{b_3}\right|^{1/2} e^{ik_c x}C_0(\widehat{x}), \quad B_0(x) = \tfrac{|b_0\varepsilon|}{|b_3|^{1/2}}e^{ik_c x}D_0(\widehat{x}), \tag{4.3}$$

$$A_\pm(x) = \left|\tfrac{b_0\varepsilon}{b_3}\right|^{1/2} e^{ik_x x}C_\pm(\widehat{x}), \quad B_\pm(x) = \tfrac{|b_0\varepsilon|}{|b_3|^{1/2}}e^{ik_x x}D_\pm(\widehat{x}). \tag{4.4}$$

Replacing these variables into the normal form system (3.2) and taking into account the signs of b_0 and b_3 in (4.1) and the relationship (4.2), we obtain the new system

$$C_0' = D_0 + O(\varepsilon^{1/2}), \tag{4.5}$$

$$D_0' = \sin^2\alpha\left(-1 + |C_0|^2 + g_1(|C_+|^2 + |C_-|^2)\right)C_0 + O(\varepsilon^{1/2}), \tag{4.6}$$

$$C_+' = D_+ + O(\varepsilon^{1/2}), \tag{4.7}$$

$$D_+' = \left(-1 + g_2|C_0|^2 + |C_+|^2 + g_3|C_-|^2\right)C_+ + O(\varepsilon^{1/2}), \tag{4.8}$$

$$C_-' = D_- + O(\varepsilon^{1/2}), \tag{4.9}$$

$$D_-' = \left(-1 + g_2|C_0|^2 + g_3|C_+|^2 + |C_-|^2\right)C_- + O(\varepsilon^{1/2}), \tag{4.10}$$

in which

$$g_1 = \frac{a_3}{a_1}, \quad g_2 = \frac{b_1}{b_3}, \quad g_3 = \frac{b_5}{b_3}.$$

Solving the equations (4.5), (4.7) and (4.9) for D_0, D_+ and D_-, respectively, we rewrite the first order system (4.5)-(4.10) as a second order system,

$$C_0'' = \sin^2\alpha\left(-1 + |C_0|^2 + g_1(|C_+|^2 + |C_-|^2)\right)C_0 + O(\varepsilon^{1/2}), \tag{4.11}$$

$$C_+'' = \left(-1 + g_2|C_0|^2 + |C_+|^2 + g_3|C_-|^2\right)C_+ + O(\varepsilon^{1/2}), \tag{4.12}$$

$$C_-'' = \left(-1 + g_2|C_0|^2 + g_3|C_+|^2 + |C_-|^2\right)C_- + O(\varepsilon^{1/2}), \tag{4.13}$$

in which the $O(\varepsilon^{1/2})$-terms are continuous in $(C_0, C_\pm, \epsilon^{1/2})$ and continuously differentiable in $(C_0, C_\pm)$.[3]

From the periodic solutions $\boldsymbol{Z}_{\varepsilon,\theta}$ and $\boldsymbol{S}_2\boldsymbol{Z}_{\varepsilon,\theta}$ of the normal form system we obtain the $\boldsymbol{S}_1\boldsymbol{S}_2$-reversible solutions $\boldsymbol{P}_{\varepsilon,\theta}$ and $\boldsymbol{Q}_{\varepsilon,\theta} = \boldsymbol{S}_2\boldsymbol{P}_{\varepsilon,\theta}$, respectively, for the system (4.11)-(4.13) with expansions

$$\boldsymbol{P}_{\varepsilon,\theta}(x) = \Big(0, (1-\theta^2)^{1/2}e^{i\theta x}, 0\Big) + O(\varepsilon^{1/2}), \quad \boldsymbol{Q}_{\varepsilon,\theta}(x) = \Big(0, 0, (1-\theta^2)^{1/2}e^{i\theta x}\Big) + O(\varepsilon^{1/2});$$

(see also [5, Lemma 6.3]). Notice that these solutions represent the rotated rolls $\boldsymbol{\mathcal{R}}_{-\beta}\boldsymbol{U}^*_{k,\mu}$ and $\boldsymbol{\mathcal{R}}_{\beta}\boldsymbol{U}^*_{k,\mu}$. While they are periodic at leading order, the terms $O(\varepsilon^{1/2})$ are quasiperiodic in x due to the presence of the exponentials e^{ik_cx} and e^{ik_xx} in the change of variables (4.3)-(4.4).

An important property of the leading order system obtained by setting $\varepsilon = 0$ in (4.11)-(4.13) is that it leaves the four-dimensional subspace $\{(C_0, C_+, C_-)\ ;\ C_0 = 0\}$ invariant. Restricting to this subspace we recover the leading order system from [5, Section 7.1],

$$C_+'' = \left(-1 + |C_+|^2 + g_3|C_-|^2\right) C_+, \tag{4.14}$$

$$C_-'' = \left(-1 + g_3|C_+|^2 + |C_-|^2\right) C_-. \tag{4.15}$$

The existence of a heteroclinic solution for this system has been proved in [14]. According to [14, Theorem 5], for any $g_3 > 1$, the system (4.14)-(4.15) possesses a heteroclinic solution (C_+^*, C_-^*), with $C_\pm^*$ smooth real-valued functions defined on $\mathbb{R}$, which is $\boldsymbol{S}_1$-reversible and connects the equilibrium $(0,1)$ as $x \to -\infty$ with the equilibrium $(1,0)$ as $x \to \infty$. This heteroclinic solution represents the leading order part of the domain walls constructed in [5].

The invariance of the subspace $\{(C_0, C_+, C_-)\ ;\ C_0 = 0\}$ implies that the leading order system from (4.11)-(4.13) possesses the $\boldsymbol{S}_1$-reversible heteroclinic solution $(0, C_+^*, C_-^*)$, with $C_\pm^*$ as above, which connects the equilibrium $(0,0,1)$ as $x \to -\infty$ with the equilibrium $(0,1,0)$ as $x \to \infty$. Notice that the equilibria $(0,0,1)$ and $(0,1,0)$ are equal to $\boldsymbol{Q}_{0,0}$ and $\boldsymbol{P}_{0,0}$ and therefore represent the rotated rolls with wavenumber k_c, $\boldsymbol{\mathcal{R}}_{\alpha}\boldsymbol{U}^*_{k_c,\mu}$ and $\boldsymbol{\mathcal{R}}_{-\alpha}\boldsymbol{U}^*_{k_c,\mu}$, respectively.

4.3 Existence of heteroclinic solutions

The existence of domain walls is obtained by proving that the $\boldsymbol{S}_1$-reversible heteroclinic solution $(0, C_+^*, C_-^*)$ found for $\varepsilon = 0$ persists for $\varepsilon > 0$. More precisely, we have the following result.

Theorem 3. *Assume that $g_1 > 1$ and $g_3 \in (1, 4+\sqrt{13})$. Except for at most a sequence of values g_1 converging to 1, for any $\varepsilon > 0$ sufficiently small, there exists $\theta = O(\varepsilon^{1/2})$, continuously depending on $\varepsilon^{1/2}$, such that the system (4.12)-(4.13) possesses a reversible heteroclinic solution $\boldsymbol{C}_\varepsilon = (C_{0,\varepsilon}, C_{+,\varepsilon}, C_{-,\varepsilon})$ connecting the solutions $\boldsymbol{Q}_{\varepsilon,\theta}$, as $x \to -\infty$, to $\boldsymbol{P}_{\varepsilon,\theta}$, as $x \to \infty$.*

[3] This property of the $O(\varepsilon^{1/2})$-terms is less precise than the one from [5, Section 6.3] but it is enough for our purposes. It is used when applying the implicit function theorem in the proof of Theorem 3. We point out that the $O(\varepsilon^{1/2})$-terms which are only continuous in $\epsilon^{1/2}$ could be excluded by considering a normal form to order 5, the remaining terms being then continuously differentiable in $(C_0, C_\pm, \epsilon^{1/2})$.

The quotients g_1 and g_3 depend on the angle α and the Prandtl number $\mathcal{P}$ through the complicated analytical formulas (A.1) and (A.2) computed in Appendix A. Taking $\alpha = 0$ in (A.2) we obtain that the limit as $\alpha \to 0$ of g_3 is equal to 2, just as in [5]. Consequently, the condition $g_3 \in (1, 4+\sqrt{13})$ holds at least for small angles $\alpha \in (0, \alpha_*(\mathcal{P}))$, for some positive $\alpha_*(\mathcal{P})$. For g_1, we have the same property when $\alpha = \pi/2$, but these angles are excluded from our analysis and it seems difficult to check analytically the inequality $g_1 > 1$ for angles $\alpha \in (0, \pi/3)$. Instead, we compute g_3 symbolically, using Maple, and obtain that $g_3 > 1.3$, for any positive Prandtl number $\mathcal{P}$ and any angle $\alpha \in (0, \pi/2)$, hence the inequality $g_3 > 1$ is always satisfied. By comparing the formulas (A.1) and (A.2) we then obtain that the inequality $g_1 > 1.3$ always holds, as well. This implies that we only have to exclude a finite number of values g_1 in Theorem 3. The same Maple computation also allows to determine the values $(\alpha, \mathcal{P})$ for which the second condition on g_3 is satisfied, i.e., $g_3 < 4 + \sqrt{13}$. We summarize these properties in Figure 4.1.

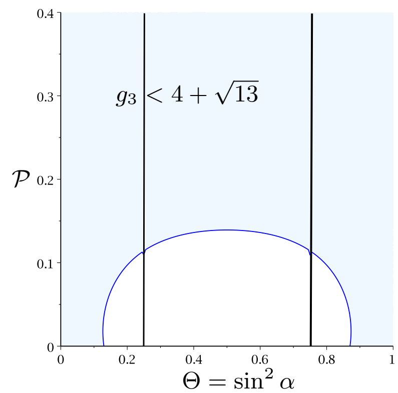

Figure 4.1: In the $(\Theta, \mathcal{P})$-plane, with $\Theta = \sin^2\alpha$, Maple plot of the curve along which $g_3 = 4+\sqrt{13}$, for $\Theta \in (0,1)$. The inequality $g_3 < 4+\sqrt{13}$ holds in the shaded region, whereas $g_3 > 1.3$ and $g_1 > 1.3$, for any positive Prandtl number $\mathcal{P}$ and any angle $\alpha \in (0, \pi/2)$. Domain walls are constructed in the shaded region situated to the left of the vertical line $\Theta = \sin^2(\pi/3) = 0.75$, except for the values on the vertical line $\Theta = \sin^2(\pi/6) = 0.25$ which correspond to the resonant case $k_c/k_x = 2$, and perhaps a finite number of curves corresponding to the finite number of (unknown) values g_1 excluded by Theorem 3.

The solutions $\boldsymbol{Q}_{\varepsilon,\theta}$ and $\boldsymbol{P}_{\varepsilon,\theta}$ in Theorem 3 representing the rotated rolls $\boldsymbol{\mathcal{R}}_\beta \boldsymbol{U}^*_{k,\mu}$ and $\boldsymbol{\mathcal{R}}_{-\beta} \boldsymbol{U}^*_{k,\mu}$, respectively, the result in Theorem 1 is an immediate consequence of Theorem 3 and the properties of g_1 and g_3 in Figure 4.1.

The proof of Theorem 3 is based on the implicit function theorem applied in the space of $\boldsymbol{S}_1$-reversible exponentially decaying functions,

$$\mathcal{X}^r_\eta = \{(C_0, C_+, C_-, \overline{C_0}, \overline{C_+}, \overline{C_-}) \in \mathcal{X}_\eta \ ; \ C_0(x) = \overline{C_0}(-x),\ C_+(x) = \overline{C_-}(-x),\ x \in \mathbb{R}\},$$

where, for $\eta > 0$,

$$\mathcal{X}_\eta = \left\{(C_0, C_+, C_-, \overline{C_0}, \overline{C_+}, \overline{C_-}) \in (L^2_\eta)^4\right\}, \quad L^2_\eta = \left\{ f : \mathbb{R} \to \mathbb{C} \ ; \ \int_{\mathbb{R}} e^{2\eta|x|} |f(x)|^2 < \infty \right\}.$$

It turns out that the linear operator needed in the implicit function theorem has exactly the same properties as the one in [5, Lemma 7.3]. Therefore, the implicit function theorem can be applied in the same way as done in [5, Theorem 2] and we omit these details of proof here. We focus on the properties of this linear operator, which is obtained by linearizing the system (4.11)-(4.13) with $\varepsilon = 0$, together with the complex conjugated equations, at $(0, C_+^*, C_-^*)$, i.e., the linear operator $\mathcal{L}_*$ acting on (C_0, C_+, C_-) through

$$\mathcal{L}_* \begin{pmatrix} C_0 \\ C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_0'' - \sin^2\alpha \left(-1 + g_1(C_+^{*2} + C_-^{*2})\right) C_0 \\ C_+'' - \left(-1 + 2C_+^{*2} + g_3 C_-^{*2}\right) C_+ - C_+^{*2}\overline{C_+} - g_3 C_+^* C_-^* (C_- + \overline{C_-}) \\ C_-'' - \left(-1 + g_3 C_+^{*2} + 2C_-^{*2}\right) C_- - C_-^{*2}\overline{C_-} - g_3 C_+^* C_-^* (C_+ + \overline{C_+}) \end{pmatrix}.$$

We prove that this operator has the same properties as in [5, Lemma 7.3], which completes the proof of Theorem 3.

Lemma 4.1. *Assume that $g_1 > 1$ and $g_3 \in (1, 4+\sqrt{13})$. Except for at most a sequence of values g_1 converging to 1, for any $\eta > 0$ sufficiently small, the operator $\mathcal{L}_*$ acting in $\mathcal{X}_\eta^r$ is Fredholm with index -1. The kernel of $\mathcal{L}_*$ is trivial and the one-dimensional kernel of its L^2-adjoint is spanned by $(0, \mathrm{i}C_+^*, -\mathrm{i}C_-^*, 0, -\mathrm{i}C_+^*, \mathrm{i}C_-^*)$.*

Proof. The action of the operator $\mathcal{L}_*$ on the C_0-component being decoupled from the action on the (C_+, C_-)-components, we may write

$$\mathcal{L}_* = \begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & \mathcal{L}_\pm \end{pmatrix},$$

with

$$\mathcal{L}_0 C_0 = C_0'' - \sin^2\alpha \left(-1 + g_1(C_+^{*2} + C_-^{*2})\right) C_0,$$

$$\mathcal{L}_\pm \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} C_+'' - \left(-1 + 2C_+^{*2} + g_3 C_-^{*2}\right) C_+ - C_+^{*2}\overline{C_+} - g_3 C_+^* C_-^* (C_- + \overline{C_-}) \\ C_-'' - \left(-1 + g_3 C_+^{*2} + 2C_-^{*2}\right) C_- - C_-^{*2}\overline{C_-} - g_3 C_+^* C_-^* (C_+ + \overline{C_+}) \end{pmatrix},$$

and we have the complex conjugated actions on the components $\overline{C_0}$ and $(\overline{C_+}, \overline{C_-})$. The operator $\mathcal{L}_\pm$ is precisely the one from [5, Lemma 7.3]. For any $g_3 > 1$, it is a Fredholm operator with index -1, has a trivial kernel, and the one-dimensional kernel of its L^2-adjoint is spanned by $(\mathrm{i}C_+^*, -\mathrm{i}C_-^*, -\mathrm{i}C_+^*, \mathrm{i}C_-^*)$. To complete the proof it remains to show that the operator $\mathcal{L}_0$ is invertible.

Taking as new variables

$$y = (\sin\alpha)x, \quad U_0 = \frac{1}{2}(C_0 + \overline{C_0}), \quad V_0 = \frac{1}{2i}(C_0 - \overline{C_0}), \quad \widetilde{C}_\pm^*(y) = C_\pm^*(x),$$

we obtain the matrix operator

$$\mathcal{M}_0 \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} U_0'' + U_0 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})U_0 \\ V_0'' + V_0 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})V_0 \end{pmatrix},$$

acting in

$$X_\eta^0 = \left\{(U_0, V_0) \in (L_\eta^2)^2 \; ; \; U_0(y) = U_0(-y),\ V_0(y) = -V_0(-y),\ y \in \mathbb{R}\right\}.$$

The invertibility of $\mathcal{L}_0$ is equivalent to the invertibility of the matrix operator $\mathcal{M}_0$. The action of $\mathcal{M}_0$ on the two components U_0 and V_0 being the same its invertibility in X_η^0 is equivalent to that of the scalar operator

$$L_0[g_1] = \partial_{yy} + 1 - g_1(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2})$$

acting in L_η^2. We show the invertibility of this operator in L^2, except for at most a sequence of values g_1 converging to 1, which by a standard perturbation argument implies its invertibility L_η^2 for sufficiently small η.

The function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ converging towards 1 as $x \to \pm\infty$, the operator $L_0[g_1]$ is a relatively compact perturbation of the asymptotic selfadjoint operator $\partial_{yy} + 1 - g_1$. Consequently, they have the same essential spectrum,

$$\sigma_{ess}(L_0[g_1]) = \sigma_{ess}(\partial_{yy} + 1 - g_1) = \sigma(\partial_{yy} + 1 - g_1) = (-\infty, 1 - g_1].$$

This implies that for any $g_1 > 1$ the operator $L_0[g_1]$ is Fredholm with index 0, so that its invertibility is equivalent to the property that its kernel is trivial.

We claim that if $L_0[g_1^*]$ has a nontrivial kernel for some $g_1^* > 1$, then $L_0[g_1]$ is invertible for any $g_1 = g_1^* + \gamma \neq g_1^*$ with sufficiently small γ. Indeed, consider an orthogonal basis $\{\xi_1^*, \dots, \xi_n^*\}$ of the finite dimensional kernel of $L_0[g_1^*]$, which is the spectral subspace associated with the eigenvalue 0, because 0 is an isolated eigenvalue and the operator is selfadjoint. For sufficiently small γ, the operator $L_0[g_1]$ has at most n eigenvalues close to 0, which are the continuation of the eigenvalue 0 of $L_0[g_1^*]$, and the spectral subspace associated to these eigenvalues has a basis $\{\xi_1(\gamma), \dots, \xi_n(\gamma)\}$ which is the smooth continuation of the basis above. These eigenvalues of $L_0[g_1]$ are the eigenvalues of the $n \times n$-matrix $M[\gamma]$ representing the action of $L_0[g_1]$ on the basis $\{\xi_1(\gamma), \dots, \xi_n(\gamma)\}$. A direct computation of this matrix shows that

$$M[\gamma] = M_1\gamma + O(\gamma^2), \quad M_1 = \left(\langle B\xi_i^*, \xi_j^*\rangle\right)_{1\leqslant i,j\leqslant n},$$

where

$$B = \frac{d}{dg_1} L_0[g_1]\big|_{g_1 = g_1^*} = -(\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}).$$

The function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ being continuous and positive with limits equal to 1 at $x = \pm\infty$, it is bounded from below by a positive constant c_*. This implies that B is a negative selfadjoint operator, so that the eigenvalues of M_1 are all negative. Consequently, 0 is not an eigenvalue of $M[\gamma]$ for $\gamma \neq 0$ sufficiently small, which implies that $L[g_1]$ is invertible, for g_1 close enough to g_1^*, $g_1 \neq g_1^*$, and proves the claim.

As a consequence of this property, the set of values $g_1 > 1$ for which the operator $L_0[g_1]$ is not invertible is countable and has no accumulation point in $(1, \infty)$. In addition, the function $\widetilde{C}_+^{*2} + \widetilde{C}_-^{*2}$ being bounded from below by $c_* > 0$, it is straightforward to check that the operator $L_0[g_1]$ is negative, and in particular invertible, for any $g_1 > 1/c_*$. We conclude that the set of values $g_1 > 1$ for which $L_0[g_1]$ is not invertible is at most a sequence converging to 1, which completes the proof of the lemma. ■

A Coefficients of the cubic normal form

The formulas for the coefficients b_3 and b_5 found in [5] remain the same, and we compute in the same way the coefficients a_1 and a_3. We obtain

$$\begin{aligned}
a_1\langle \mathbf{\Psi}_0, \mathbf{\Psi}_0^* \rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{200000}, \overline{\boldsymbol{\zeta}_0}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{110000}, \boldsymbol{\zeta}_0), \mathbf{\Psi}_0^* \rangle, \\
a_3\langle \mathbf{\Psi}_0, \mathbf{\Psi}_0^* \rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001100}, \boldsymbol{\zeta}_0) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{101000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{100100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_0^* \rangle, \\
b_3\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{002000}, \overline{\boldsymbol{\zeta}_+}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001100}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^* \rangle, \\
b_5\langle \mathbf{\Psi}_+, \mathbf{\Psi}_+^* \rangle &= \langle 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001010}, \overline{\boldsymbol{\zeta}_-}) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{001001}, \boldsymbol{\zeta}_-) + 2\mathcal{B}_{\mu_c}(\mathbf{\Phi}_{000011}, \boldsymbol{\zeta}_+), \mathbf{\Psi}_+^* \rangle.
\end{aligned}$$

where $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_\pm$ are the eigenvectors of $\mathcal{L}_{\mu_c}$ from Section 2.3, $\mathbf{\Psi}_0^*$ and $\mathbf{\Psi}_+^*$ are eigenvectors of the adjoint operator $\mathcal{L}_{\mu_c}^*$ associated to the eigenvalues $-ik_c$ and $-ik_x$, respectively, and the vectors $\mathbf{\Phi}_{pqrstu}$ satisfy

$$\begin{aligned}
&(\mathcal{L}_{\mu_c} - 2ik_c)\mathbf{\Phi}_{200000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_0), \qquad \mathcal{L}_{\mu_c}\mathbf{\Phi}_{110000} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \overline{\boldsymbol{\zeta}_0}), \\
&\mathcal{L}_{\mu_c}\mathbf{\Phi}_{001100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_+}), \qquad (\mathcal{L}_{\mu_c} - i(k_c + k_x))\mathbf{\Phi}_{101000} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_+), \\
&(\mathcal{L}_{\mu_c} - i(k_c - k_x))\mathbf{\Phi}_{100100} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_0, \overline{\boldsymbol{\zeta}_+}), \qquad (\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{002000} = -\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_+), \\
&(\mathcal{L}_{\mu_c} - 2ik_x)\mathbf{\Phi}_{001010} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \boldsymbol{\zeta}_-), \qquad \mathcal{L}_{\mu_c}\mathbf{\Phi}_{001001} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_+, \overline{\boldsymbol{\zeta}_-}), \\
&\mathcal{L}_{\mu_c}\mathbf{\Phi}_{000011} = -2\mathcal{B}_{\mu_c}(\boldsymbol{\zeta}_-, \overline{\boldsymbol{\zeta}_-}).
\end{aligned}$$

A direct computation (see also [5, Appendix A.1]) gives the formulas for the eigenvectors

$$\boldsymbol{\zeta}_0(y,z) = \begin{pmatrix} \frac{i}{k_c}DV \\ 0 \\ V \\ -\frac{1}{\mu_c k_c^2}D^3V \\ 0 \\ \frac{ik_c}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2 - k_c^2)^2V \\ \frac{i}{\mu_c k_c}(D^2 - k_c^2)^2V \end{pmatrix}, \quad \boldsymbol{\zeta}_\pm(y,z) = e^{\pm ik_y y}\begin{pmatrix} \frac{i\sin\alpha}{k_c}DV \\ \pm\frac{i\cos\alpha}{k_c}DV \\ V \\ -\frac{1}{\mu_c k_c^2}(D^2 - k_c^2\cos^2\alpha)DV \\ \mp\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ \frac{1}{\mu_c k_c^2}(D^2 - k_c^2)^2V \\ \frac{i\sin\alpha}{\mu_c k_c}(D^2 - k_c^2)^2V \end{pmatrix},$$

and

$$\mathbf{\Psi}_0^*(y,z) = \begin{pmatrix} -\frac{1}{\mu_c k_c^2}\left(D^3V - \langle D^3V\rangle\right) \\ 0 \\ \frac{ik_c}{\mu_c}V \\ -\frac{i}{k_c}DV \\ 0 \\ -V \\ -ik_c\phi \\ \phi \end{pmatrix}, \quad \mathbf{\Psi}_+^*(y,z) = e^{ik_y y}\begin{pmatrix} -\frac{1}{\mu_c k_c^2}(D^2 - k_c^2\cos^2\alpha)DV \\ -\frac{\sin\alpha\cos\alpha}{\mu_c}DV \\ \frac{ik_c\sin\alpha}{\mu_c}V \\ -\frac{i\sin\alpha}{k_c}DV \\ -\frac{i\cos\alpha}{k_c}DV \\ -V \\ -ik_c(\sin\alpha)\phi \\ \phi \end{pmatrix}.$$

In these formulas, V is a real-valued solution of the boundary value problem

$$
\begin{aligned}
&(D^2 - k_c^2)^3 V + \mu_c^2 k_c^2 V = 0, \\
&V = DV = (D^2 - k_c^2)^2 V = 0 \text{ in } z = 0, \\
&V = D^2 V = D^4 V = 0 \text{ in } z = 1,
\end{aligned}
$$

ϕ is the unique solution of the boundary value problem

$$
(D^2 - k_c^2)\phi = V, \quad \phi = 0 \text{ in } z = 0, 1,
$$

and

$$
\langle D^3 V \rangle = \int_{\Omega_{per}} D^3 V(z)\, dy\, dz.
$$

After very long computations we obtain that

$$
g_1 = \frac{a_3}{a_1} = \frac{b_{51}(\frac{1}{2}(1 + \sin\alpha)) + b_{51}(\frac{1}{2}(1 - \sin\alpha)) + b_{51}(0)}{\frac{1}{2} b_{51}(1) + b_{51}(0)}, \tag{A.1}
$$

$$
g_3 = \frac{b_5}{b_3} = \frac{b_{51}(\sin^2\alpha) + b_{51}(\cos^2\alpha) + b_{51}(0)}{\frac{1}{2} b_{51}(1) + b_{51}(0)}, \tag{A.2}
$$

in which

$$
b_{51}(\Theta) = A_{51}(\Theta) + B_{51}(\Theta)\mathcal{P}^{-1} + C_{51}(\Theta)\mathcal{P}^{-2},
$$

with

$$
\begin{aligned}
A_{51}(\Theta) &= 2\mu_c^3 \langle (D^2 - 4k_c^2\Theta)^2 V_1, R_1 \rangle, \\
B_{51}(\Theta) &= 4\mu_c^3 \Theta \left(\langle V_1, R_2 \rangle + \langle V_2, R_1 \rangle \right), \\
C_{51}(\Theta) &= -\frac{2\mu_c \Theta}{k_c^2} \langle (D^2 - 4k_c^2\Theta) V_2, R_2 \rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
&R_1 = V D\phi + (1 - 2\Theta)\phi DV, \\
&R_2 = \left(D^2 - 4k_c^2(1 - \Theta)\right)(V DV) - 4\Theta(DV)(D^2 V),
\end{aligned}
$$

and V_1, V_2 are the unique solutions of the boundary value problems

$$
\begin{aligned}
&(D^2 - 4k_c^2\Theta)^3 V_1 + 4k_c^2 \mu_c^2 \Theta\, V_1 = R_1, \\
&V_1 = DV_1 = (D^2 - 4k_c^2\Theta)^2 V_1 = 0 \text{ in } z = 0, \\
&V_1 = D^2 V_1 = D^4 V_1 = 0 \text{ in } z = 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&(D^2 - 4k_c^2\Theta)^3 V_2 + 4k_c^2 \mu_c^2 \Theta\, V_2 = R_2, \\
&V_2 = D^2 V_2 = (D^2 - 4k_c^2\Theta) DV_2 = 0 \text{ in } z = 0, \\
&V_2 = D^2 V_2 = D^4 V_2 = 0 \text{ in } z = 1,
\end{aligned}
$$

respectively.

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