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Modelling, Control and Stability Analysis of Flexible Rotating Beam's Impacts During Contact Scenario

Andrea Mattioni[1] and Yongxin Wu[1] and Yann Le Gorrec[1]

Abstract— **This paper considers the problem of a rotating flexible beam in collision with an external object. The flexible beam's colliding equations exhibit instant changes during impact times, therefore the model is cast in the class of switched infinite dimensional operator systems. The aim is to study the stability of the closed loop system with a PD control law, making use of the semigroup formalism together with the Lyapunov stability theory. To this end, we present a new stability result making use of multiple Lyapunov functions obtained as an adaptation of a theorem from finite dimensional hybrid systems theory. We show the port-Hamiltonian modelling procedure for a controlled rotating flexible beam in impact scenario, using distributed parameter equations to describe the beam's dynamic. Then, we compute the equilibrium position of the closed loop system and using the shifted variables with respect to the equilibrium position, we cast the system in the class of switched infinite dimensional operator systems. Finally we select the Lyapunov functions for the contact and non-contact phases and we show, through numerical simulations, that they respect the assumptions of the proposed stability theorem.**

I. INTRODUCTION

A lot of critical tasks in robotics involve the contact between the manipulator and an external object or the environment. In some cases, flexible manipulators are preferable to rigid ones due to their lightweight and because they can assure smooth contact force in impact scenario. This is why they can be encountered in many application fields ranging from spatial [1] to micro-manipulation applications [2]. The major challenge is to come up with a suitable model for control purposes that is enough accurate on taking into account the impact dynamics.

The main difficulty is that the distributed parameter nature of the flexible beam's system would require an infinite dimensional analysis. A finite dimensional analysis provides a good approximation of the flexible phenomena in case of unconstrained conditions, but it can bring misleading results in presence of impact, where a large bandwidth of frequencies will be excited. While there exist many studies on the control of flexible manipulators in impact scenario using finite dimensional models [3], [4], [5], very few have discussed the collision issue using infinite-dimensional models [6]. The dynamical model of a colliding flexible beam is expected to have instant changes in impact times. Therefore the model will combine behaviours that are typical of continuous-time dynamical systems with behaviours that are typical of discrete-time dynamical systems. This definition perfectly fits into the class of hybrid dynamical systems. The stability, as well as the control design theory, have been extensively studied for finite dimensional hybrid systems [7]. On the other side, some results have been established for infinite dimensional hybrid systems. In [8] are presented some general results on Lagrange, asymptotic and exponential stability (in all their variations) for the class of hybrid infinite dimensional systems, that do not require the determination of Lyapunov functions, as well as results that do involve Lyapunov functions. In particular, the result of Lagrange stability requires that the composition along solutions of all the different Lyapunov functions should be non-increasing in all the switching times. In [9] some conditions for obtaining exponential stability are given for a subclass of hybrid systems, namely switched operator systems. Other characterizations of exponentially stable switched operator equations can be founded in [10], [11].

In this preliminary work, we are interested in studying the Lagrange stability of a flexible beam in impact scenario in feedback with a simple PD controller. To do so, we recast the closed-loop system in the class of switched linear operator systems and we study its stability using an adaptation for infinite dimensional systems of the Lagrange stability result, that makes use of multiple Lyapunov functions, proposed in [12]. We decided to do not use the theorem proposed in [8] due to the difficulty in finding good Lyapunov functions for our specific application case.

The remainder of this paper is organized as follows. In the next section, we give some background on infinite dimensional switching linear systems; in section III we propose a model for the colliding flexible beam together with the design of a PD controller, then we show the equilibrium position computation together with the stability study; in section IV are given numerical simulations to validate the theoretical development. We conclude the paper with some final remarks and comments on future research.

II. PRELIMINARIES

In this section we provide the necessary background concerning dynamical systems determined by switching operator equations. Consider the general operator equation

$$\dot{x}(t) = f(x(t), m), \tag{1}$$

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[1]Andrea Mattioni, Yongxin Wu and Yann Le Gorrec are with AS2M department of the FEMTO-ST research lab University of Bourgogne Franche-Comté, 24 rue Savary, 25000 Besançon, France. andrea.mattioni@femto-st.fr yongxin.wu@femto-st.fr yann.le.gorrec@ens2m.fr

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where $x \in X$ is the *continuous state* and belongs to an appropriate Hilbert space, and $m \in M = \{1,2,\dots,N\}$ is the *discrete state*. The couple defined as the composition of the continuous and discrete state (x,m) is called *hybrid state*. The $i-th$ (in order of activation) discrete state m_i depends in general on the continuous state x and on the previous discrete state m_{i-1}, *i.e.* $m_i = \eta(x, m_{i-1})$ where $\eta : X \times M \to M$ is a discrete transition. If for each $x \in X$, only one $m \in M$ is possible, then the system is called a switching system, otherwise is an hybrid system. Here, we consider switched systems, then we partition the state space in N disjoint regions

$$\Omega_1 \dots \Omega_N \subset X$$

where $\Omega_i \bigcap \Omega_j = \emptyset,\ i \neq j$.

Consider a family of linear operators $\mathbb{A} = \{\mathscr{A}_m,\ m \in M\}$ defined on a common domain $\mathscr{D}(\mathscr{A}_\alpha) = \mathscr{D}(\mathscr{A}_\beta)$ for $\alpha, \beta \in M$ and a family of functions $\mathbb{F} = \{f_m, m \in M\}$. The considered switched operator system is given by

$$\dot{x}(t) = \mathscr{A}_\eta x(t) + f_\eta(x(t)). \tag{2}$$

The continuous state evolution of (2) can be described as: starting at (x_0, m_0) at time t_0, the continuous trajectory evolves according to $\dot{x} = \mathscr{A}_{m_0} x + f_{m_0}(x)$. Let us assume that at time t_1, x reaches a value x_1 that triggers a discrete change from m_0 to m_1; then the process evolves according to $\dot{x} = \mathscr{A}_{m_1} x + f_{m_1}(x)$. Here, we consider hybrid systems with continuous state that does not change during switching and therefore the hybrid state (x, m_i) becomes (x, m_j). We define a *switching sequence* anchored to a certain initial state

$$\{S_n(x_0)\} = (m_0, t_0),\ (m_1, t_1), \dots, (m_n, t_n),\ \dots.$$

The switching sequence along (2) describes completely the trajectory of the system according to the following rule: (m_i, t_i) means that the system evolves according to $\dot{x}(t) = \mathscr{A}_{m_i} x + f_{m_i}(x)$ for $t_i \leq t \leq t_{i+1}$. We denote by $S(x_0)|_m$ the endpoints of times for which the system m is active. Finally, let $\mathscr{E}(T)$: $t_0,\ t_2,\ t_4, \dots$ denote the even sequence of T : $t_0,\ t_1,\ t_2, \dots$.

The solution of a dynamical system sometimes converges to an *equilibrium point*, of witch we propose the definition given in [8, page 1278].

Definition 2.1: An hybrid state (x_{eq}, m_{eq}) is said to be an *hybrid equilibrium* of (1) if the trajectory generated by the initial conditions (x_{eq}, m_{eq}) is such that $x_{S(x_{eq})}(t) = x_{eq}$ for all $t \geq 0$. □

The hybrid equilibrium points may be obtained by finding the states satisfying

$$\mathscr{A}_m x + f_m(x) = 0 \quad \forall m \in M. \tag{3}$$

All the continuous states satisfying (3) are not hybrid equilibria because they may be not possible hybrid states. For example one solution of (3) (x_{eq}, m_i) may not be possible in the sense that x_{eq} is not contained in the region of the state space that is associated with the discrete state m_i.

Without loss of generality the origin is assumed to be a continuous equilibrium of (1) for which stability is investigated.

Now, we can define a single candidate Lyapunov function V_m for a certain system's dynamic $\mathscr{A}_m x + f_m(x)$.

Definition 2.2: A continuous functional $V_m : X \to [0, \infty)$ such that $\forall x \in \Omega_m\ \ \alpha(||x||) \leq V_m(x) \leq \beta(||x||)$, where $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that $\alpha(0) = \beta(0) = 0$ and $\lim_{||x|| \to \infty} \alpha(||x||) = \lim_{||x|| \to \infty} \beta(||x||) = \infty$, is a Lyapunov functional for $\mathscr{A}_m x + f_m x$ and the trajectory $x(t)$ if:

- $V_m(x(t))$ is Dini differentiable [13];
- $\dot{V}_{m,+}(x_0) := \limsup_{t \to 0} \frac{V_m(x(t)) - V_m(x_0)}{t} \leq 0 \quad \forall x(0) = x_0 \in \Omega_m$. □

Since the Dini derivative is usually difficult to compute, we introduce in the next lemma an easy way to compute it. Note that for a functional V_m to be considered a Lyapunov functional for $\mathscr{A}_m x + f_m(x)$, it is necessary that $\dot{V}_{m,+}(x)$ is non positive only in the region Ω_m, but in principle $\dot{V}_{m,+}(x_0)$ can be computed in the whole state space X.

Lemma 2.1: If the functional V_m is Frchet differentiable, then for $x \in \Omega_\kappa \bigcap D(\mathscr{A}_\kappa)\ \ \kappa \in M$, $V_m(x(t))$ is differentiable for $t = 0$ and

$$\dot{V}_{m,t}(x) = \frac{dV_m(x(t))}{dt}|_{t=0} = dV_m(x) \mathscr{A}_\kappa x + f_\kappa(x) \quad \forall x \in \Omega_\kappa$$

where dV_m denotes the Frchet derivative of V_m.

Proof. Divide the state space in the different subspaces Ω_κ. Then, the time derivative equality in each Ω_κ is shown to hold as in Lemma 11.2.5 of [13]. □

In the previous lemma, we gave the formula for computing the time derivative of the Lyapunov function V_m in any subspace Ω_κ. At this point we are in position to state the bounded trajectory theorem for switched linear operator systems, that is an adaptation of theorem 2.3 in [12] for infinite dimensional systems.

Theorem 2.2: Let assume that there exists a unique local mild solution of (2). If there exist Lyapunov functions V_m for every $\mathscr{A}_m x + f_m(x)$ that are non increasing in $\mathscr{E}(S(x_0)|_m)$ $\forall m \in M$, then (2) has a global bounded mild solution for every initial condition $x_0 \in X$.

Proof: [Proof (for N=2)] Since we know that there exists a local mild solution, we know that for any x_0 there exists a t_{max} such that (2) possess a mild solution on $[0, t_{max})$ and $t_{max} < \infty$ only if $||x(t, x_0)||$ diverges when $t \to t_{max}$.

Let $R > 0$ be arbitrary. Let $d_m(\alpha) = \min\{V_m(x_0) \mid ||x_0|| = \alpha\}$. Pick $r_m < R$ such that $\forall x_0 \in B(r_m) = \{x_0 \in X \mid ||x_0|| \leq r_m\}$, $V_m(x_0) < d_m(R)$ for all $m \in M$. Let $r = min(r_m)$. With this choice, if we select $||x_0|| \leq r$, the evolution of the trajectory with either vector field $A_m x + f_m(x)$ will be such that $||x(t)|| \leq R \quad \forall t \in [0, t_0]$.

Now pick $\rho_m < r$ such that $\forall x_0 \in B(\rho_m) = \{x_0 \in X \mid ||x_0|| \leq \rho_m\}$, $V_m(x_0) < d_m(r)$ for all $m \in M$. Set $\rho = \min(\rho_m)$. Thus, if we choose $||x_0||_X \leq \rho$ the trajectory's evolution with either vector field $\mathscr{A}_m x + f_m(x)$ will be such that $||x(t)|| \leq r \quad \forall t \in [0, t_0]$.

Therefore, select $||x_0||_X \leq \rho$ such that at the first transition at time t_1 we have that $||x(t_1)|| \leq r$ and at the second transition at time t_2 we have $||x(t_2)|| \leq R$. Then, because of the "switching-in" condition

$$V_m(x(t_2)) \leq V_m(x(t_0)) \leq r.$$

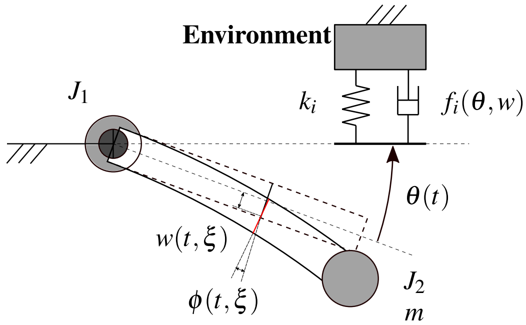

Fig. 1. Rotating flexible Timoshenko's beam in impact with the external environment.

This procedure can be repeated to the infinite to conclude

$$||x|| \leq R \quad \forall t \in \mathbb{R}^+.$$

Since the trajectory remains bounded the solution does not diverge and then $t_{max} = \infty$. ■

The non-increasing condition of V_m in $\mathscr{E}(S(x_0)|_m)$ concerns the value of each function V_m each time is "switched in". It means that the value of V_m at switching points should be smaller than that of the previous time it has become active or "switched in". We remark that to conclude about the existence of a globally bounded mild solution it is not necessary that the solution has a finite number of switching in a finite time interval (See *Zeno Behaviour* in Hybrid systems [7]).

III. CONTROLLED FLEXIBLE ROTATING BEAM IN IMPACT SCENARIO

A. Modelling and Control design

Consider a rotating flexible beam in absence of gravity as depicted in Figure 1. The rotor angle $\theta(t)$ is a real function of time, while $\xi \in [0,L]$ identifies the spatial coordinate of the beam. The deflection of the beam in the rotating frame is defined by $w(t,\xi)$, while $\phi(t,\xi)$ represents the relative rotation of the beam cross section. All the physical parameters are positive definite. J_1 and J_2 represent the rotary inertia of the hub to which the beam is connected and the end effector's rotary inertia, respectively. m is the end effector's mass. $E(\xi), I(\xi)$ are the Young's modulus and the moment of inertia of the beam's cross section, respectively. $\rho(\xi)$ and $I_\rho(\xi)$ are the density and the mass moment of inertia of the beam's cross section, respectively, and $G(\xi)$ is the Shear modulus.

According to [14], the compliant surface is considered as a mass-less system composed by a spring and a damper. In this paper we consider a constant spring coefficient k_i and a damper coefficient depending on the environment's deformation $f_i(\theta,w) = c_i(L\theta + w(t,L))$, with c_i constant. Note that the quantity $L\theta + w(t,L)$ corresponds to the distance of the end-effector from the external environment when is negative, and to the environment's deformation when it is positive. From now on we will not explicit the dependency from time and space of the variables when it is clear from the context. The kinetic energy H_k and the potential energy H_p, using Timoshenko's assumptions, write as

$$\begin{aligned} H_k &= \tfrac{1}{2}J_1\dot{\theta}^2 + \tfrac{1}{2}J_2(\dot{\theta}+\dot{\phi}(t,L))^2 + \tfrac{1}{2}m(L\dot{\theta}+\dot{w})^2 \\ &+\tfrac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}+\xi\dot{\theta}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t}+\dot{\theta}\right)^2\right] d\xi \\ H_p &= \tfrac{1}{2}\int_0^L \left[K\left(\frac{\partial w}{\partial \xi}-\phi\right)^2 + EI\left(\frac{\partial \phi}{\partial \xi}\right)^2\right] d\xi \\ &+\tfrac{1}{2}k_i\gamma(\theta,\dot{\theta},w(t,L),\dot{w}(t,L))(L\theta+w(t,L))^2 \end{aligned}$$

where $K(\xi) = kA(\xi)G(\xi)$, with k a positive parameter depending on the beam's cross section and $A(\xi)$ the cross sectional area. γ denotes the function $\gamma : \mathbb{R}^4 \to \{0,1\}$ defined as

$$\gamma = \begin{cases} 0 & \begin{array}{c} if \quad (L\theta + w(t,L) < 0) \\ or \\ [(L\theta + w(t,L) = 0)\ and\ (\dot{\theta}+\dot{\phi}(t,L) < 0)] \end{array} \\ 1 & \begin{array}{c} if \quad (L\theta + w(t,L) > 0) \\ or \\ [(L\theta + w(t,L) = 0)\ and\ (\dot{\theta}+\dot{\phi}(t,L) \geq 0)]. \end{array} \end{cases}$$

The Hamilton's principle is used to obtain the system's dynamical equations, considering $\delta W_{nc} = u(t)\delta\theta - f_i(\theta,w)\gamma(\theta,\dot{\theta},w(t,L),\dot{w}(t,L))(L\dot{\theta}+\dot{w}(t,L))\delta(L\theta+w(t,L))$ the virtual work of non-conservative forces, where $u(t)$ identifies the external torque, and the other term corresponds to the dissipation of the soft-impact model. The dissipation force results nonlinear because we considered a damping coefficient depending on the environment's deformation. The derived set of mixed partial and ordinal differential equations writes

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho\left(\frac{\partial w}{\partial t}+\xi\dot{\theta}\right)\right) = \frac{\partial}{\partial \xi}\left(K\left(\frac{\partial w}{\partial \xi}-\phi\right)\right) \\ \frac{\partial}{\partial t}\left(I_\rho\left(\frac{\partial \phi}{\partial t}+\dot{\theta}\right)\right) = \frac{\partial}{\partial \xi}\left(EI\frac{\partial \phi}{\partial \xi}\right) + K\left(\frac{\partial w}{\partial \xi}-\phi\right) \\ J_1\frac{d}{dt}\dot{\theta} = +EI(0)\frac{\partial \phi(t,0)}{\partial \xi} + u(t) \\ m\frac{d}{dt}(L\dot{\theta}+\dot{w}(t,L)) = -K(L)\left[\frac{\partial w}{\partial \xi}(L,t) - \phi(L,t)\right] \\ \qquad -k_i\gamma(\theta,\dot{\theta},w(t,L),\dot{w}(t,L))(L\theta+w(t,L)) \\ \qquad -f_i(\theta,w)\gamma(\theta,\dot{\theta},w(t,L),\dot{w}(t,L))(L\dot{\theta}+\dot{w}(t,L)) \\ J_2\frac{d}{dt}(\dot{\theta}+\dot{\phi}(t,L)) = -EI(L)\frac{\partial \phi}{\partial \xi}(t,L) \end{cases} \tag{4}$$

with boundary conditions

$$w(t,0) = 0 \quad \phi(t,0) = 0. \tag{5}$$

The energy states of the infinite dimensional system are defined by

$$\begin{array}{ll} \varepsilon_t = \frac{\partial w}{\partial \xi} - \phi & p_t = \rho\left(\frac{\partial w}{\partial t}+\xi\dot{\theta}\right) \\ \varepsilon_r = \frac{\partial \phi}{\partial \xi} & p_r = I_\rho\left(\frac{\partial \phi}{\partial t}+\dot{\theta}\right). \end{array} \tag{6}$$

The equations describing the infinite dimensional system can be written as a port-Hamiltonian (pH) system

$$\dot{x}_b = \mathscr{J}x_b = P_1\frac{\partial}{\partial \xi}(\mathscr{H}_b x_b) + P_0(\mathscr{H}_b x_b) \tag{7}$$

with $x_b = [p_t\ p_r\ \varepsilon_t\ \varepsilon_r]^T \in X_b \subset L_2([0,L],\mathbb{R}^4)$ representing the system's state. The matrices in equation (7) are defined as

$$P_1 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \quad P_0 = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{bmatrix},$$

$$\mathscr{H}_b(\xi)=\begin{bmatrix}\rho^{-1}(\xi) & 0 & 0 & 0\\ 0 & I_\rho^{-1}(\xi) & 0 & 0\\ 0 & 0 & K(\xi) & 0\\ 0 & 0 & 0 & EI(\xi)\end{bmatrix}.$$

The state space X_b is equipped with the L_2 inner product $\langle x_b,x_b\rangle_{X_b}=\langle x_b,\mathscr{H}_b x_b\rangle_{L_2}$, such to express the energy related to the flexible part of the system as $H_b=\frac{1}{2}\langle x_b,x_b\rangle_{X_b}$. The boundary variables are defined as [15]

$$\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}P_1 & -P_1\\ I & I\end{bmatrix}\begin{bmatrix}(\mathscr{H}_b x_b)(t,0)\\ (\mathscr{H}_b x_b)(t,L)\end{bmatrix}.$$

Then, define the boundary input and output operators as

$$\begin{aligned}\mathscr{B}_1(\mathscr{H}_b x_b)&=W_2\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=\begin{bmatrix}I_\rho^{-1}p_r(t,0)\\ \rho^{-1}p_t(t,L)\\ I_\rho^{-1}p_r(t,L)\end{bmatrix}=u_{b,1}\\ \mathscr{B}_2(\mathscr{H}_b x_b)&=W_1\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=\rho^{-1}p_t(t,0)=u_{b,2}\\ \mathscr{C}_1(\mathscr{H}_b x_b)&=\tilde{W}_2\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=\begin{bmatrix}-EI\varepsilon_r(t,0)\\ K\varepsilon_t(t,L)\\ EI\varepsilon_r(t,L)\end{bmatrix}=y_{b,1}\\ \mathscr{C}_2(\mathscr{H}_b x_b)&=\tilde{W}_1\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=-K\varepsilon_t(t,0)=y_{b,2}\end{aligned}\qquad(8)$$

where $W=\begin{bmatrix}W_1\\ W_2\end{bmatrix}$ and $\tilde{W}=\begin{bmatrix}\tilde{W}_1\\ \tilde{W}_2\end{bmatrix}$ are appropriate matrices, and are such that $\begin{bmatrix}W\\ \tilde{W}\end{bmatrix}$ is non-singular. The total boundary input-output operators are defined as

$$\begin{aligned}\mathscr{B}(\mathscr{H}_b x_b)&=\begin{bmatrix}\mathscr{B}_1(\mathscr{H}_b x_b)\\ \mathscr{B}_2(\mathscr{H}_b x_b)\end{bmatrix}=\begin{bmatrix}W_1\\ W_2\end{bmatrix}\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=u_b\\ \mathscr{C}(\mathscr{H}_b x_b)&=\begin{bmatrix}\mathscr{C}_1(\mathscr{H}_b x_b)\\ \mathscr{C}_2(\mathscr{H}_b x_b)\end{bmatrix}=\begin{bmatrix}\tilde{W}_1\\ \tilde{W}_2\end{bmatrix}\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix}=y_b.\end{aligned}\qquad(9)$$

Denote the restoring torques and forces with u_r and the hub's and end-effector's velocities with y_r

$$u_r=\begin{bmatrix}EI\frac{\partial\phi}{\partial\xi}(t,0)\\ -K\left[\frac{\partial w}{\partial\xi}(t,L)-\phi(t,L)\right]\\ -EI\frac{\partial\phi}{\partial\xi}(t,L)\end{bmatrix}\quad y_r=\begin{bmatrix}\dot\theta\\ L\dot\theta+\dot w(t,L)\\ \dot\theta+\dot\phi(t,L)\end{bmatrix}.$$

The states related to the finite dimensional part are defined as

$$\begin{aligned}&p_1=J_1\dot\theta && q_1=\theta\\ &p_2=m(L\dot\theta+\dot w(t,L)) && q_2=L\theta+w(t,L)\\ &p_3=J_2(\dot\theta+\dot\phi(t,L))\end{aligned}\qquad(10)$$

and the related equations write as

$$\begin{cases}\dot p=-u_r+f(p,q)+gu\\ \dot q=\begin{bmatrix}\frac{1}{J_1}p_1 & \frac{1}{m}p_2\end{bmatrix}^T\\ y_r(t)=\begin{bmatrix}\frac{1}{J_1}p_1 & \frac{1}{m}p_2 & \frac{1}{J_2}p_3\end{bmatrix}^T\end{cases}\qquad(11)$$

where $p=[p_1\ p_2\ p_3]^T$, $q=[q_1\ q_2]^T$ and the matrices and the nonlinear vector are defined as

$$g=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad f(p_2,q_2)=\begin{bmatrix}0\\ -k_i\gamma(q_2,p_2)q_2-\frac{c_i}{m}\gamma(q_2,p_2)q_2p_2\\ 0\end{bmatrix}.$$

with γ function in the new variable defined as

$$\gamma=\begin{cases}0 & if\quad (q_2<0)\ or\ [(q_2=0)\ and\ (p_2<0)]\\ 1 & if\quad (q_2>0)\ or\ [(q_2=0)\ and\ (p_2\geq 0)].\end{cases}$$

Use the original boundary conditions (5) together with the state variables definition (6) to derive the interconnection relation between the infinite dimensional and the finite dimensional parts of the system

$$u_{b,1}=y_r\quad u_r=-y_{b,1},$$

while the remaining boundary condition of (7) is equal to zero, i.e. $u_{b,2}=0$. We can now define the input control torque as a simple PD controller

$$u(t)=-k(\theta(t)-\theta^o)-c\dot\theta(t)\qquad(12)$$

and defining the new error state $\tilde q_1=\theta-\theta^o$, we can write the closed loop equations in the following semi-linear operator form

$$\begin{aligned}\dot x&=\begin{bmatrix}P_1\frac{\partial}{\partial\xi}(\mathscr{H}x_b)+P_0(\mathscr{H}x_b)\\ +EI\varepsilon_r(t,0)-k\tilde q_1-\frac{c}{J_1}p_1\\ +K\varepsilon_t(t,L)-k_iq_2\\ +EI\varepsilon_r(t,L)\\ \frac{1}{J_1}p_1\\ \frac{1}{m}p_2\end{bmatrix}+\begin{bmatrix}0\\0\\ k_i\gamma(-x)q_2\\ -\frac{c_i}{m}\gamma(x)q_2p_2\\ 0\\0\\0\end{bmatrix}\\ &=\mathscr{A}x+f(x)\end{aligned}\qquad(13)$$

where $x=[x_b\ p_1\ p_2\ p_3\ \tilde q_1\ q_2]^T\in X=L_2([0,L],\mathbb{R}^4)\times\mathbb{R}^5$ is the state of the system and the domain of the linear operator $\mathscr{A}$ is defined as

$$\begin{aligned}D(\mathscr{A})=\ &\{x\in X|x_b\in H^1([0,L],\mathscr{R}^4),\ \mathscr{B}_2x=0,\\ &\mathscr{B}_1x=[p_1/J_1\ \ p_2/m\ \ p_3/J_2]^T\}.\end{aligned}$$

The inner product in the state space is defined for $x_1,x_2\in X$ as

$$\begin{aligned}\langle x_1,x_2\rangle_X=\ &\langle x_1,\mathscr{H}x_2\rangle_{L_2}+\frac{1}{J_1}p_{1,1}p_{1,2}+\frac{1}{m}p_{2,1}p_{2,2}\\ &+\frac{1}{J_2}p_{3,1}p_{3,2}+k\tilde q_{1,1}\tilde q_{1,2}+k_iq_{2,1}q_{2,2}\end{aligned}$$

with associated norm $||x||_X=\sqrt{\langle x,x\rangle_X}$.

B. Model definition as a switched system

The aim of this paper is to study the asymptotic behaviour of a flexible beam in collision with the external environment, therefore we assume $\theta^o>0$ in (12). We first notice that the equilibrium position for the non-contact equation (*i.e.*(13) with $\gamma(x)=0$) corresponds to a state in the contact region, hence it is not a possible hybrid state. Assuming $\gamma(x)=1$ in (13), the equilibrium point is such that all the momenta are null ($p_t^*=p_r^*=p_1^*=p_2^*=p_3^*=0$), the shear and angular deformations are

$$\varepsilon_t^*(\xi)=-\frac{k_iA}{K}Bq_1^o\quad \varepsilon_r^*(\xi)=\frac{k_iA}{EI}B(\xi-L)q_1^o.\qquad(14)$$

with $A(\xi)=L(1+k_i(\frac{L}{K(\xi))^{-1}+L^3(3EI(\xi)})^{-1}$, $B(\xi)=\frac{k}{k_iLA(\xi)+k}$ and the finite dimensional states equilibria write as

$$q_1^*=Bq_1^o,\quad q_2^*=ABq_1^o.$$

Next, we define a new set of shifted variables with respect to the founded equilibrium states:

$$\begin{aligned}&\varepsilon_t'(t,\xi)=\varepsilon_t(t,\xi)-\varepsilon_t^*(\xi) && \varepsilon_r'(t,\xi)=\varepsilon_r(t,\xi)-\varepsilon_r^*(\xi)\\ &p_t'(t,\xi)=p_t(t,\xi) && p_r'(t,\xi)=p_r(t,\xi)\\ &q_1'(t)=q_1(t)-q_1^* && q_2'(t)=q_2(t)-q_2^*\end{aligned}\qquad(15)$$

and $p'_1(t) = p_1(t)$, $p'_2(t) = p_2(t)$, $p'_3(t) = p_3(t)$. Then, defining $x' = [\varepsilon'_t\ \varepsilon'_r\ p'_t\ p'_r\ p'_1\ p'_2\ p'_3\ q'_1\ q'_2]^T$, equation (13) can be rewritten as (2), with discrete transition function $\eta = \gamma(x)$, $M = \{0,1\}$, non-contact operator

$$\mathscr{A}_0 x' = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathscr{H} x'_b) + P_0(\mathscr{H} x'_b) \\ EI\varepsilon'_r(t,0) - kq'_1 - \frac{c}{J_1} p'_1 \\ -K\varepsilon'_t(t,L) + Ak_i B q_1^o \\ -EI\varepsilon'_r(t,L) \\ \frac{1}{J_1} p'_1 \\ \frac{1}{m} p'_2 \end{bmatrix} \tag{16}$$

contact semi-linear operator,

$$\mathscr{A}_1 x' + f_1(x') = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathscr{H} x'_b) + P_0(\mathscr{H} x'_b) \\ EI\varepsilon'_r(t,0) - kq'_1 - \frac{c}{J_1} p'_1 \\ -K\varepsilon'_t(t,L) - k_i q'_2 \\ -EI\varepsilon'_r(t,L) \\ \frac{1}{J_1} p'_1 \\ \frac{1}{m} p'_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{c_i}{m}(q'_2 + q_2^*) p'_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

domains defined as

$$\begin{aligned} D(\mathscr{A}_0) = D(\mathscr{A}_1) = \{ x' \in X | x'_b \in H^1([0,L], \mathbb{R}^4), \\ \mathscr{B}_2 x' = 0,\ \mathscr{B}_1 x' = [p'_1/J_1\ \ p'_2/m\ \ p'_3/J_2]^T \}, \end{aligned} \tag{18}$$

and non-contact and contact regions defined, respectively, as

$$\begin{aligned} \Omega_0 = \{x' \in X | q'_2 < -q_2^*\} \\ \Omega_1 = \{x' \in X | q'_2 \geq -q_2^*\}. \end{aligned} \tag{19}$$

C. Stability Analysis

We now use Theorem 2.2 to study the stability of the switched system defined by operators (16)-(17) and state space partition (19), in case the control law sets the equilibrium position in the contact region, *i.e.* $q_1^o > 0$.

Proposition 3.1: Assume that there exists a mild solution of the switched system (2) with operators (16)-(17) and domains (18). Then, the solution $x'(t)$ is bounded for every initial condition $x'_0 \in X$.

Sketch of the proof. Let's consider the following Lyapunov functions for the non-contact and contact operators

$$\begin{aligned} V_0 = \ & \frac{1}{2} \int_0^L \left(K(\varepsilon'_t + \varepsilon_t^*)^2 + EI(\varepsilon'_r + \varepsilon_r^*)^2 + \frac{1}{\rho} p_t'^2 + \frac{1}{I_\rho} p_r'^2 \right) d\xi \\ & + \frac{1}{2J_1} p_1'^2 + \frac{1}{2J_2} p_2'^2 + \frac{1}{2m} p_3'^2 + \frac{1}{2} k(q'_1 - (1-B) q_1^o)^2, \end{aligned} \tag{20}$$

$$\begin{aligned} V_1 = \ & \frac{1}{2} \langle x'_b, \mathscr{H} x'_b \rangle_{L_2} + \frac{1}{2J_1} p_1'^2 + \frac{1}{2J_2} p_2'^2 + \frac{1}{2m} p_3'^2 \\ & + \frac{1}{2} k q_1'^2 + \frac{1}{2} k_i q_2'^2. \end{aligned} \tag{21}$$

We can see that both functions are positive definite in X, and in particular $V_0 > 0$ in Ω_0 and $V_1 \geq 0$ in Ω_1. It is possible to show that both Lyapunov functions are non-increasing in the respective region of the state space

$$\begin{aligned} \dot{V}_0(x') &= dV_{nc}(x') \mathscr{A}_{nc} x' = -\frac{c}{J_1^2} p_1'^2 \quad \forall x' \in \Omega_0 \\ \dot{V}_1(x') &= dV_c(x') \mathscr{A}_c x' = -\frac{c}{J_1^2} p_1'^2 - \frac{c_i}{m^2}(q'_2 + q_2^*) p_2'^2 \ \ \forall x' \in \Omega_1 \end{aligned}$$

and that they are non increasing in $\mathscr{E}(S(x_0)|_0)$ and $\mathscr{E}(S(x_0)|_1)$, respectively. By means of Theorem 2.2, we can conclude that the solutions are bounded for every initial condition $x'_0 \in X$. □

In the next section we will show, through the use of numerical simulations, the behaviour along solutions of the selected Lyapunov functions.

IV. NUMERICAL SIMULATIONS

To perform numerical simulations, we derived a finite dimensional approximation of equation (13) using the discretization procedure described in [16]. In particular, the approximated model has been obtained splitting the spatial domain in 150 elements. Simulations were made in the Simulink® environment using the "ode23t" time integration algorithm, and the set of parameters used for simulation are listed in Table I. The controller parameters are set as $k = 10$, $c = 3$ and $\theta^o = 1$, while the impact's model parameters are set equal to $k_i = 1000$ and $c_i = 30$. In accordance with section III-B, it is possible to compute the equilibrium configuration of the system: $q_1^* = 0.0424\ rad$, $q_2^* = 0.0096\ m$, $\varepsilon_t^*(\xi) = -1.3981 \times 10^{-8}$ and $\varepsilon_r(\xi) = 0.0985(\xi - L)$.

To perform numerical simulations, the beam's states as well as the finite dimensional momentum states are initialized to zero initial conditions $x_b(0,\xi) = 0$, $p_1(0) = p_2(0) = p_3(0)$. The initial hub's angle has been initialized to $\theta(0) = q_1(0) = -1\ rad$, accordingly with the load's initial position $q_2(0) = L\theta(0) + w(0,L) = -1\ m$.

Fig. 2 shows the evolution in time of the hub's angle and of the load position. It is important to note that the contact occurs when $q_2(t) \geq 0$, and in fact when it dynamically reaches this value, the q_2 variable is rejected back because of the spring force of the impact model. It is possible to appreciate that both angles asymptotically stabilize to the computed equilibrium positions. Fig. 3 shows firstly the Lyapunov functions (20)-(21) behaviour along solutions in the entire simulation time interval without distinguish between the active or non active time intervals, and secondly their behaviour during the respective activation time intervals. It is possible to appreciate that both the selected Lyapunov functions are non-increasing in their activation phases, and that the "Switching in" conditions are met. The time scale difference between the first and second image in Fig. 3 comes

TABLE I

SIMULATION PARAMETERS

Name	Variable	Value
Beam's Length	L	$1\ m$
Beam's Width	L_w	$0.1\ m$
Beam's Thickness	L_t	$0.02\ m$
Density	ρ	$8000\ \frac{kg}{m^3}$
Young's modulus	E	$2 \times 10^9\ \frac{N}{m^2}$
Bulk's modulus	K	$6.85 \times 10^8\ \frac{N}{m^2}$
Hub's inertia	J_1	$1\ kg \cdot m^2$
Load's mass	m	$1\ kg$
Load's inertia	J_2	$1\ kg \cdot m^2$

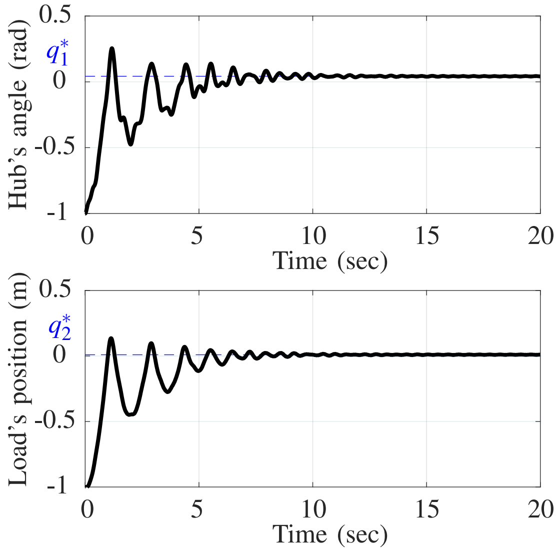

Fig. 2. Hub's angle evolution along time $q_1(t)$ and Load's position evolution along time $q_2(t)$.

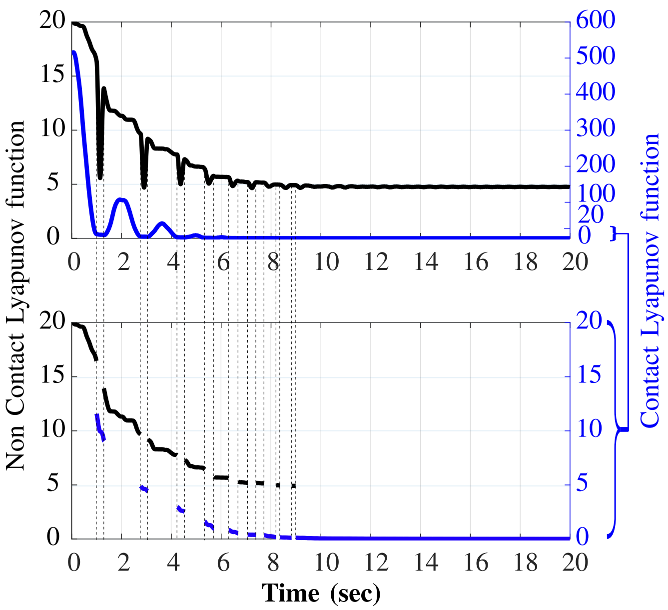

Fig. 3. Contact V_1 and non-contact V_0 Lyapunov functions behaviour along time.

from the big values of $||q_2'||$ and thus of the term $\frac{1}{2}k_i q_2'^2$ in the contact Lyapunov function (21) during the non-contact time periods.

V. CONCLUSIONS

In this preliminary work a general framework for switched infinite dimensional linear systems, together with a theorem concerning Lagrange stability have been presented. The proposed result makes use of multiple Lyapunov functions, each one associated to one of the operators defining the system. The theorem assures Lagrange stability if the Lyapunov functions have non-increasing time derivative in the subspace on which they are active and they respect the so called "switching-in" condition. Then, the modelling procedure together with the equilibrium computation of a controlled rotating flexible beam in impact scenario has been detailed using the port-Hamiltonian framework. The obtained free motion and contact scenario operators have been written such to be cast in the defined framework for switched infinite dimensional systems. Next, Lyapunov functions for the free and the contact case fulfilling the Lagrange stability theorem's assumptions have been proposed. Finally, with the help of numerical simulations, their non increasing behaviour in the respective active region together with the "switching-in" condition fulfilment have been shown. The future work will focus on the design of control laws capable of increasing performances in term of vibration suppression in case of impact scenario.

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