In-domain finite dimensional control of distributed parameter port-Hamiltonian systems via energy shaping *

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Abstract

In this paper we consider in-domain control of distributed parameter port-Hamiltonian systems defined on a one dimensional spatial domain. Through an early lumping approach we extend the control by interconnection and energy shaping approach to the use of distributed control over the spatial domain. With the established finite dimensional controller, the closed-loop performances can be modified over a given range of frequencies while guaranteeing the closed-loop stability of the infinite dimensional system. Two cases are investigated, the ideal case where the controller acts on the complete spatial domain (infinite dimensional distributed control), and the more realistic one where the control is piecewise homogeneous (finite rank distributed control). The proposed control strategies are illustrated through simulations on the stabilization of a vibrating Timoshenko beam.

1 Introduction

Finite dimensional port-Hamiltonian systems (PHSs) have firstly been introduced in the 90's in [11]. This modeling framework consists in representing the system dynamics by using the energy variables as state variables and by expressing the power exchanges within the system and with its environment through an intrinsic geometric structure, named Dirac structure. PHSs are particularly well adapted for the modeling and control of multiphysical and nonlinear systems [2]. PHSs have been extended to distributed parameter systems governed by partial differential equations (PDEs) in [10] and extensively studied in the one dimensional (1D) linear case in [8] with applications to different fields of research such as fluid dynamics, process control, vibroacoustics, etc.

The intrinsic passivity properties of PHSs pave the way for passivity based control (PBC) techniques. Many dedicated PBC designs have been proposed for finite dimensional PHSs (see for instance [16] for a general introduction). Among these control designs, the Control by Interconnection (CbI) methodology is based on the power preserving interconnection of the system plant and the controller, and consists in shaping the closed-loop energy function (closed-loop Hamiltonian) by an appropriate choice of the controller parameters. The closed-loop stability is guaranteed by Lyapunov arguments and damping injection [12]. The CbI method has been generalized to 1D boundary controlled infinite dimensional PHS in [9, 13]. A first result on the ideal in-domain control that allows to take advantage of the distributed nature of the control action with the help of structural invariants has been proposed in [15]. In this case the control action allows to modify the shape of the overall total energy while ensuring the closed-loop system stability.

In this paper we propose to apply and extend the in-domain control strategy [15] to the under-actuated case using an early lumping approach and the passive interconnection between the infinite dimensional plant and the finite dimensional controller. The early lumping approach consists firstly in approximating the PDE plant with a structure-preserving discretization method [4, 7] and then in designing the finite dimensional controller in order to shape the closed-loop energy function on the discretized plant. The stability of this infinite dimensional system using a controller designed for an approximation is proved with the Lyapunov arguments and LaSalle's invariant principle. This control strategy takes advantage of both the early lumping approach, leading to a directly implementable controller with guaranteed performances on the discretized model, *i.e.* on the infinite dimensional system over a given range of frequencies, and of the passivity of the controller, guaranteeing the asymptotic stability.

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Two different cases are investigated : the ideal *fully-actuated case* where the control input works independently at each point of the spatial domain *i.e.* on each element of the discretized model, and the more realistic *under-actuated case* where the input acts identically on sets of elements, providing less degrees of freedom (DOFs). This latter case is closer to the real implementation scenario as the control is usually carried out through actuator patches that act similarly over spatial intervals. It is shown how to change the closed-loop energetic properties of the discretized system in a perfect way when the system is fully-actuated and in an *optimal way* when the system is under-actuated. In this latter case a constructive method is proposed to guarantee the closed-loop performances and convergence rate with the optimization. Results are illustrated through simulations.

The paper is organized as follows: in Section 2 the port-Hamiltonian formulation of the flexible Timoshenko beam with dissipation and in-domain control, as well as its structure preserving discretization are given as an illustrative example. The aforementioned CbI method and energy shaping applied to the discretized PHS are detailed in Section 3. The closed-loop stability is then studied. Section 4 provides some simulation results with a comparison between the fully- and under-actuated scenarios. Section 5 ends with conclusions and perspectives.

2 Port-Hamiltonian formulation of a flexible Timoshenko beam with in-domain control

In this section, we recall the infinite dimensional PHS formulation of a flexible Timoshenko beam with dissipation defined on the 1D spatial domain $\zeta \in [0, L]$. We consider here clamped-free boundary conditions and distributed control over the domain. This leads to the following system of equations

(2.1a)
$$\underbrace{\frac{\partial}{\partial t}\begin{pmatrix} x_1(\zeta,t)\\ x_2(\zeta,t)\\ x_3(\zeta,t)\\ x_4(\zeta,t)\end{pmatrix}}_{x(\zeta,t)} = (\mathcal{J}-\mathcal{R})\begin{pmatrix} e_1(\zeta,t)\\ e_2(\zeta,t)\\ e_3(\zeta,t)\\ e_4(\zeta,t)\end{pmatrix} + \underbrace{\begin{pmatrix}0\\0\\0\\b\end{pmatrix}}_{B}\mathbf{u}_d(t),$$

(2.1b)
$$\mathbf{y}_d(t) = \begin{pmatrix}0 & 0 & 0 & b^*\end{pmatrix}\begin{pmatrix} e_1(\zeta,t)\\ e_2(\zeta,t)\\ e_3(\zeta,t)\\ e_4(\zeta,t)\end{pmatrix},$$

with $x_1(\zeta,t) = \left(\frac{\partial\omega}{\partial\zeta}(\zeta,t) - \theta(\zeta,t)\right)$, $x_2(\zeta,t) = \rho A_r(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t)$, $x_3(\zeta,t) = \frac{\partial\theta}{\partial\zeta}(\zeta,t)$, $x_4(\zeta,t) = I\rho(\zeta)\frac{\partial\theta}{\partial t}(\zeta,t)$. $\omega(\zeta,t)$ is the longitudinal displacement over the spatial domain, $\theta(\zeta,t)$ denotes the bending angle. $\rho(\zeta)$, $A_r(\zeta)$, $I(\zeta)$, represent the mass density, the cross section area, the moment of inertia respectively. The interconnection operator $\mathcal{J}$ and the dissipation operator $\mathcal{R}$ are formulated as follows:

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{\partial}{\partial\zeta} & 0 & -1\\ \frac{\partial}{\partial\zeta} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\partial}{\partial\zeta}\\ 1 & 0 & \frac{\partial}{\partial\zeta} & 0\end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & R_t & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & R_r\end{pmatrix},$$

where $R_t > 0$ and $R_r > 0$ are related to the translational dissipation and the rotational dissipation, respectively. The constitutive relation between the effort variables and the energy variables yields:

(2.2)
$$\begin{pmatrix} e_1(\zeta,t)\\ e_2(\zeta,t)\\ e_3(\zeta,t)\\ e_4(\zeta,t)\end{pmatrix} = \underbrace{\begin{pmatrix} GA_r(\zeta) & 0 & 0 & 0\\ 0 & \frac{1}{\rho A_r(\zeta)} & 0 & 0\\ 0 & 0 & EI(\zeta) & 0\\ 0 & 0 & 0 & \frac{1}{I\rho(\zeta)}\end{pmatrix}}_{\mathcal{Q}>0} x(\zeta,t),$$

where $G(\zeta)$ and $E(\zeta)$ represent the shear modulus and the Young's modulus of the beam, respectively. The bounded input operator B maps the distributed input $\mathbf{u}_d$ (the total exterior bending moment density) into the state space. The power conjugated output $\mathbf{y}_d$ is the sum of the angular velocities over the domain. For the sake of compactness, the time parameter t is omitted hereafter. The Hamiltonian of the system, *i.e. the stored energy*, is defined as follows:

(2.3)
$$H(x(\zeta)) = \frac{1}{2}\int_0^L \left(GA_r(\zeta)x_1^2(\zeta) + EI(\zeta)x_3^2(\zeta) + \frac{1}{\rho A_r(\zeta)}x_2^2(\zeta) + \frac{1}{I\rho(\zeta)}x_4^2(\zeta)\right)d\zeta.$$

The boundary port variables [8] are defined as

(2.4)
$$f_\partial = \begin{pmatrix} e_2(0)\\ e_1(L)\\ e_4(0)\\ e_3(L)\end{pmatrix}, \quad \text{and} \quad e_\partial = \begin{pmatrix} -e_1(0)\\ e_2(L)\\ -e_3(0)\\ e_4(L)\end{pmatrix}.$$

Clamped-free boundary conditions are considered, i.e.

(2.5a)
$$u_b = W\begin{pmatrix} f_\partial\\ e_\partial\end{pmatrix} = \begin{pmatrix} e_2(0)\\ e_1(L)\\ e_4(0)\\ e_3(L)\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

(2.5b)
$$y_b = \tilde{W}\begin{pmatrix} f_\partial\\ e_\partial\end{pmatrix} = \begin{pmatrix} -e_1(0)\\ e_2(L)\\ -e_3(0)\\ e_4(L)\end{pmatrix},$$

with $W = \begin{pmatrix} I_{2,2} & 0_{2,2} \end{pmatrix}$, $\tilde{W} = \begin{pmatrix} 0_{2,2} & I_{2,2} \end{pmatrix}$, $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ being invertible leading to:

$$\begin{aligned}\frac{dH}{dt}(\zeta) &= y_b^T u_b + \mathbf{y}_d^T \mathbf{u}_d - \int_0^L e_2(\zeta)^T R_t e_2(\zeta) d\zeta \\ &\quad - \int_0^L e_4(\zeta)^T R_r e_4(\zeta) d\zeta \\ &= \mathbf{y}_d^T \mathbf{u}_d - \int_0^L \left(e_2(\zeta)^T R_t e_2(\zeta) + e_4(\zeta)^T R_r e_4(\zeta) \right) d\zeta.\end{aligned}$$

In this paper, we consider an early lumping approach. Hence, the first step in the design procedure is to spatially discretize (2.1). For a sake of simplicity we consider here the homogeneous case, *i.e.* the physical parameters ρ, A_r, I, G and E do not depend on ζ. Since the PBC is considered, the discretization has to preserve the structure and the passivity of the system. To this aim we use the mixed finite element method proposed in [4]. The discretization of the Timoshenko beam with dissipation (2.1) in n elements with boundary conditions (2.5) leads to the finite dimensional system:

$$\frac{d}{dt} \underbrace{\begin{pmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \\ x_{4d} \end{pmatrix}}_{x_d} = (J_n - R_n) \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \\ Q_3 x_{3d} \\ Q_4 x_{4d} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_d \end{pmatrix} \mathbf{u}_d, \tag{2.6a}$$

$$\mathbf{y}_d = b_d^T Q_4 x_{4d}, \tag{2.6b}$$

where $x_{id} = \begin{pmatrix} x_i^1 & \cdots & x_i^n \end{pmatrix}^T$ for $i \in \{1, 2, 3, 4\}$,

$$J_n = \begin{pmatrix} 0 & J_i & 0 & -S^T \\ -J_i^T & 0 & 0 & 0 \\ 0 & 0 & 0 & J_i \\ S & 0 & -J_i^T & 0 \end{pmatrix}, S = \operatorname{diag}(L_{ab})_{n \times n},$$

$$J_i = \begin{pmatrix} \frac{1}{\beta} & & & \\ -\frac{1}{\beta^2} & \frac{1}{\beta} & & \\ \vdots & \ddots & \ddots & \\ (-1)^{n-1} \frac{(\beta')^{n-2}}{\beta^n} & \cdots & -\frac{1}{\beta^2} & \frac{1}{\beta} \end{pmatrix}_{n \times n},$$

$$R_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R_{td} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{rd} \end{pmatrix}, \quad L_{ab} = \frac{L}{n},$$

$Q_1 = \operatorname{diag}\left(\frac{GA_r}{L_{ab}}\right) \in \mathbb{R}^{n \times n}$, $Q_2 = \operatorname{diag}\left(\frac{1}{\rho A_r L_{ab}}\right) \in \mathbb{R}^{n \times n}$, $Q_3 = \operatorname{diag}\left(\frac{EI}{L_{ab}}\right) \in \mathbb{R}^{n \times n}$, $Q_4 = \operatorname{diag}\left(\frac{1}{I\rho L_{ab}}\right) \in \mathbb{R}^{n \times n}$, $R_{td} = \operatorname{diag}(R_t L_{ab}) \in \mathbb{R}^{n \times n}$, and $R_{rd} = \operatorname{diag}(R_r L_{ab}) \in \mathbb{R}^{n \times n}$. β denotes the effort mapping parameter and $\beta' = 1 - \beta$ [4]. They are chosen in this case equal to $\frac{1}{2}$ in order to get a centered scheme.

The Hamiltonian of the discretized model (2.6) is given as follows:

$$\begin{aligned}H_d(x_d) = \frac{1}{2} &\left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} + x_{3d}^T Q_3 x_{3d} \right. \\ &\left. + x_{4d}^T Q_4 x_{4d} \right).\end{aligned} \tag{2.7}$$

It is important to notice that the input matrix b_d depends on the considered case, whether the system is fully- or under-actuated. These two cases are presented in Subsection 3.1 and 3.2, respectively.

3 Control by interconnection and energy shaping

In this section, we extend the CbI method to the spatially distributed input and output case. As for finite dimensional systems [12, 16], the CbI consists in interconnecting the port-Hamiltonian plant with a port-Hamiltonian controller in a power persevering way. The main difference here is that the control and controller action are distributed in space. This allows to shape the distributed energy function all over the system by choosing an appropriate parametrization of the controller and the use of structural invariants *i.e.* Casimir functions [2]. Therefore, this control allows not only to change the desired equilibrium but also to modify the closed-loop dynamic performances of the distributed parameter system. The controller is formulated as a PHS in the form of:

$$\dot{x}_c = (J_c - R_c) Q_c x_c + B_c u_c, \tag{3.8a}$$

$$y_c = B_c^T Q_c x_c + D_c u_c, \tag{3.8b}$$

where $x_c \in \mathbb{R}^{m \times 1}$ is the state of controller, $J_c = -J_c^T \in \mathbb{R}^{m \times m}$, $\mathbb{R}^{m \times m} \ni R_c = R_c^T \geq 0$, $\mathbb{R}^{m \times m} \ni Q_c \geq 0$, $B_c \in \mathbb{R}^{m \times m}$, $u_c \in \mathbb{R}^{m \times 1}$ and $y_c \in \mathbb{R}^{m \times 1}$. $\mathbb{R}^{m \times m} \ni D_c > 0$ works as the damping injection. m denotes the dimension of input ($m \leq n$). Without considering external signals, the passive interconnection [12] between the plant (2.6) and the controller (3.8) is:

$$\mathbf{u}_d = -y_c, \quad u_c = \mathbf{y}_d, \tag{3.9}$$

leading to a new PHS in closed-loop:

(3.10)

$$\dot{x}_{cl} = \begin{pmatrix} 0 & J_i & 0 & -S^T & 0 \\ -J_i^T & -R_{td} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_i & 0 \\ S & 0 & -J_i^T & -\tilde{R}_{rd} & -b_d B_c^T \\ 0 & 0 & 0 & B_c b_d^T & J_c - R_c \end{pmatrix} e_{cl},$$

where $x_{cl} = \begin{pmatrix} x_{1d}^T & x_{2d}^T & x_{3d}^T & x_{4d}^T & x_c^T \end{pmatrix}^T$, $e_{cl} = Q_{cl} x_{cl}$, $Q_{cl} = \operatorname{diag}\begin{pmatrix} Q_1, & Q_2, & Q_3, & Q_4, & Q_c \end{pmatrix}$, and $\tilde{R}_{rd} = R_{rd} + b_d D_c b_d^T$.

Due to the power preserving interconnection relation (3.9) the passivity is preserved in closed-loop. The following Proposition 1 characterizes how the closed-loop energy can be shaped by using structural invariants.

PROPOSITION 1. *Choosing* $J_c = 0$, $R_c = 0$ *the closed-loop system* (3.10) *admits the Casimir function* $C(x_{3d}, x_c)$ *defined by:*

$$C(x_{3d}, x_c) = B_c b_d^T J_i^{-1} x_{3d} - x_c, \tag{3.11}$$

as structural invariant, i.e. $\dot{C}(x_{3d}, x_c) = 0$ *along the closed-loop trajectories. If the initial conditions of* $x_{3d}(0)$ *and* $x_c(0)$ *satisfy* $C(x_{3d}(0), x_c(0)) = 0$, *the control law* (3.9) *is equivalent to the state feedback:*

$$\mathbf{u}_d = -B_c^T Q_c B_c b_d^T J_i^{-1} x_{3d} - D_c \mathbf{y}_d, \tag{3.12a}$$

$$\mathbf{y}_d = b_d^T Q_4 x_{4d}. \tag{3.12b}$$

Therefore, the closed-loop system (3.10) *becomes:*

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \\ \dot{x}_{3d} \\ \dot{x}_{4d} \end{pmatrix} = \begin{pmatrix} 0 & J_i & 0 & -S^T \\ -J_i^T & -R_{td} & 0 & 0 \\ 0 & 0 & 0 & J_i \\ S & 0 & -J_i^T & -\tilde{R}_{rd} \end{pmatrix} \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \\ \tilde{Q}_3 x_{3d} \\ Q_4 x_{4d} \end{pmatrix}, \tag{3.13}$$

where $\tilde{Q}_3 = Q_3 + J_i^{-T} b_d B_c^T Q_c B_c b_d^T J_i^{-1}$ *is the new closed-loop energy matrix associated to* x_{3d}.

Proof. Given the Casimir function in its general form:

$$C(x_d, x_c) = F(x_d) - x_c, \tag{3.14}$$

from $\dot{C}(x_d, x_c) = 0$ and (3.10), one gets:

$$\frac{dC}{dt} = \frac{\partial^T C}{\partial x_{cl}} \frac{\partial x_{cl}}{\partial t} = \begin{pmatrix} \frac{\partial^T F}{\partial x_d}, & -I \end{pmatrix} (J_{cl} - R_{cl}) e_{cl} = 0. \tag{3.15}$$

As Casimir functions should not depend on the trajectories of the closed-loop system *i.e.* on the Hamiltonian, (3.15) gives rise to the following matching equations:

$$\frac{\partial^T F}{\partial x_{2d}} \left(-J_i^T\right) + \frac{\partial^T F}{\partial x_{4d}} S = 0, \tag{3.16}$$

$$\frac{\partial^T F}{\partial x_{1d}} J_i + \frac{\partial^T F}{\partial x_{2d}} \left(-R_{td}\right) = 0, \tag{3.17}$$

$$\frac{\partial^T F}{\partial x_{4d}} \left(-J_i^T\right) = 0, \tag{3.18}$$

$$\frac{\partial^T F}{\partial x_{1d}} \left(-S^T\right) + \frac{\partial^T F}{\partial x_{3d}} J_i - \frac{\partial^T F}{\partial x_{4d}} \tilde{R}_{rd} - B_c b_d^T = 0, \tag{3.19}$$

$$\frac{\partial F}{\partial x_{4d}}^T \left(-b_d B_c^T\right) - (J_c - R_c) = 0. \tag{3.20}$$

Solving (3.18), since the matrix J_i is full rank, one gets $\partial F/\partial x_{4d} = 0$. Substituting (3.18) into (3.16), one obtains $\partial F/\partial x_{2d} = 0$. Solving (3.17), one has $\partial F/\partial x_{1d} = 0$. Hence, it indicates that x_c does not hinge on x_{1d}, x_{2d}, nor on x_{4d}, and that the controller allows to modify the angular potential energy only. Substituting again (3.18) to (3.20), because J_c is skew-symmetric and R_c is symmetric and positive semi-definite, one gets $J_c = 0$ and $R_c = 0$. Solving (3.19), one gets (3.11) as structural invariant as soon as the initial condition x_c has been chosen properly. Taking the initial conditions $x_{3d}(0)$ and $x_c(0)$ such that $C(x_{3d}(0), x_c(0)) = 0$, (3.11) becomes $B_c b_d^T J_i^{-1} x_{3d} - x_c = 0$, linking the state of the controller with the state of the plant. Replacing the state variable of the controller x_c in (3.10) by $B_c b_d^T J_i^{-1} x_{3d}$, the control law (3.9) becomes a state feedback formulated in (3.12). Thus the closed-loop system (3.10) becomes (3.13) which concludes the proof. □

From Proposition 1, the closed-loop Hamiltonian function reads:

$$H_{cl}(x_d) = \frac{1}{2} \Big(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} + x_{3d}^T \tilde{Q}_3 x_{3d} + x_{4d}^T Q_4 x_{4d} \Big), \tag{3.21}$$

and satisfies

$$\frac{dH_{cl}}{dt} = -x_{2d}^T Q_2 R_{td} Q_2 x_{2d} - x_{4d}^T Q_4 \tilde{R}_{rd} Q_4 x_{4d} \leq 0. \tag{3.22}$$

From a physical point of view, (3.21) implies that with the dynamic controller (3.8) equivalent to the state feedback (3.12), it is possible to change, at least partially, the distributed Young's modulus of the beam, thereby achieving desired performances in closed-loop. The available DOFs for this energy shaping depend on the range of B_c *i.e.* the number of distributed independent actuators. For a given problem *i.e.* for a given number of independent control variables, the objective is to find the matrices B_c and Q_c such that the *distance* (considered here in the Frobenius norm[1]) between the closed-loop angular potential energy matrix $\tilde{Q}_3$ and the desired one $\tilde{Q}_{3d}$ is minimal, *i.e.*

$$\min_{B_c^T Q_c B_c} \left\| J_i^{-T} b_d B_c^T Q_c B_c b_d^T J_i^{-1} + Q_3 - \tilde{Q}_{3d} \right\|_F. \tag{3.23}$$

This problem can be formalized by the optimization Problem 1.

[1]For a matrix $\mathbb{R}^{m\times n} \ni A = [a_{ij}]$, the Frobenius norm is defined as $\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2\right)^{1/2}$, see Definition 6.4 of [14].

Problem 1. *The angular potential energy of the closed-loop system* (3.13) *is shaped in an optimal way if and only if* $X = B_c^T Q_c B_c \in SR_0^{m\times m}$ *minimizes the criterion*

$$f(X) = \left\| AXA^T - Q_m \right\|_F , \tag{3.24}$$

where $A = J_i^{-T} b_d \in \mathbb{R}^{n\times m}$*, and* $Q_m = \tilde{Q}_{3d} - Q_3 \in D_0^{n\times n}$*.* $SR_0^{m\times m}$ *represents the set of symmetric positive semi-definite matrices, and* $D_0^{n\times n}$ *stands for the set of diagonal positive semi-definite matrices.*

To solve Problem 1, we consider two different cases: the *ideal* fully-actuated case ($m = n$) and the under-actuated case ($m < n$).

3.1 Fully-actuated case. We first consider the ideal case where each discretized beam element is actuated by an independent input (bending moment), *i.e.* $\mathbf{u}_d \in \mathbb{R}^n$, as illustrated in Fig.1. In this case, the input

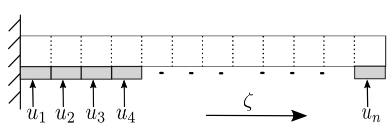


Figure 1: Clamped-free Timoshenko beam with fully-actuated case.

matrix $b_d \in \mathbb{R}^{n\times n}$ is the identity, and the power conjugated output $\mathbf{y}_d$ contains all the angular velocities of the discretized system. Therefore, the optimization Problem 1 admits an exact solution that is given in Proposition 2.

Proposition 2. *In the fully-actuated case, i.e.* $m = n$ *the optimization Problem 1 has an exact analytic solution* $\hat{X} = J_i^T Q_m J_i$ *leading to* $f(X) = 0$*, where the controller matrices* B_c *and* Q_c *can be chosen as:*

$$B_c = J_i, \qquad Q_c = Q_m. \tag{3.25}$$

Proof. The matrix A is full rank, therefore, (3.24) admits a minimum $f(X) = 0$ when:

$$\hat{X} = A^{-1} Q_m A^{-T} = b_d^{-1} J_i^T Q_m J_i b_d^{-T}. \tag{3.26}$$

As b_d being identity, and $\hat{X} = B_c^T Q_c B_c$, one can choose B_c and Q_c as in (3.25) to satisfy (3.26). $\square$

Remark 1. *The choice* $B_c = J_i$ *can be regarded as the finite dimensional approximation of the spatial derivation* $\frac{\partial}{\partial \zeta}$*. This choice has also been used in the late lumping control design approach [15].*

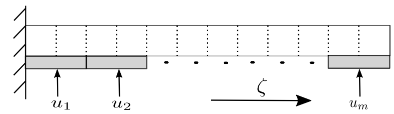


Figure 2: Under-actuated case with $k = 2$.

3.2 Under-actuated case. We consider now the more realistic case depicted in Fig.2 where the control is delivered by the use of patches, *i.e.* the same control is applied to a set of elements. Let k be the number of elements under the same actuator. $m = \frac{n}{k}$ is then the number of independent inputs distributed all over the spatial domain. This is the case when the beam is discretized in n elements and actuated by m uniform piezoelectric patches, each patch acting homogeneously over k elements. The plant system can still be formulated in the form of (2.6) with:

$$b_d = I_m \otimes \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{n\times m}, \tag{3.27}$$

where $\otimes$ denotes the Kronecker product. The input vector $\mathbf{u}_d \in \mathbb{R}^m$ contains m independent (and distributed) bending moments that can be used for control design.

Unlike the fully-actuated case, the under-actuated case contains less DOFs. As suggested in [17], $f(X)$ is convex and the minimization of $f(X)$ is equivalent to the minimization of $f^2(X)$. This kind of optimization problem has been studied in [3] and [6] under the symmetric/Hermitian or positive semi-definite constraint. The solution of the optimization Problem 1 is given in Proposition 3.

Proposition 3. $f^2(X)$ *has a unique minimum given for* $\hat{X} = V\Sigma_0^{-1} U_1^T Q_m U_1 \Sigma_0^{-1} V^T$*, with* V*,* Σ_0 *and* U_1 *the matrices of the singular value decomposition (SVD) of the matrix* A*, i.e.*

$$A = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_0 \\ 0 \end{pmatrix} V^T, \tag{3.28}$$

where $U \in \mathbb{R}^{n\times n}$ *and* $V \in \mathbb{R}^{m\times m}$ *are unitary matrices,* $U_1 \in \mathbb{R}^{n\times m}$*,* $U_2 \in \mathbb{R}^{n\times(n-m)}$ *and* $\Sigma_0 \in D_0^{m\times m}$ *is the diagonal matrix of singular values of* A*.*

Proof. Calculating $f^2(X)$, one obtains:

(3.29)
$$\begin{aligned} f^2(X) &= \left\| AXA^T - Q_m \right\|_F^2 \\ &= tr\left(Q_m Q_m + AXA^T AXA^T - 2AXA^T Q_m \right), \end{aligned}$$

where $tr(\cdot)$ denotes the trace of a matrix.

Substituting the SVD of A (3.28) into (3.29) and after some computations, one gets the equivalent optimization of Problem 1 as:

(3.30)

$$\min_{X\in SR_0^{m\times m}} f^2(X) = \min_{X\in SR_0^{m\times m}} \left(\left\| \Sigma_0 V^T X V \Sigma_0^T - T_1 \right\|_F^2 + 2\left\| U_1^T Q_m U_2 \right\|_F^2 + \left\| U_2^T Q_m U_2 \right\|_F^2 \right),$$

where $T_1 = U_1^T Q_m U_1$. Since the last two terms are given, the minimization problem (3.30) becomes:

$$\min_{\bar{X}\in SR_0^{m\times m}} \left\| \bar{X} - T_1 \right\|_F^2, \text{ with } \bar{X} = \Sigma_0 V^T X V \Sigma_0^T. \quad (3.31)$$

It has been proven in [5] that, using the Frobenius norm, the nearest positive semi-definite matrix to an arbitrary matrix $\mathcal{A}$ is unique and is given by $(\mathcal{B}+\mathcal{H})/2$ with $\mathcal{B} = \left(\mathcal{A}+\mathcal{A}^T\right)/2$ and $\mathcal{H}$ the symmetric polar factor of $\mathcal{B}$. In our case, $T_1 \in SR_0^{m\times m}$ hence (3.31) admits a unique solution $\hat{\bar{X}} = T_1$. Finally we get the solution of (3.30) as:

$$\hat{X} = V\Sigma_0^{-1}\hat{\bar{X}}\Sigma_0^{-1}V^T = V\Sigma_0^{-1}T_1\Sigma_0^{-1}V^T \quad (3.32)$$

$\square$

Remark 2. *The choice of controller matrices B_c and Q_c is not unique, as long as their product $B_c^T Q_c B_c$ satisfies the condition* (3.32)*.*

3.3 Closed-loop stability. In this subsection we consider the closed-loop stability of the infinite dimensional system (2.1) controlled by the finite dimensional controller (3.8) derived from the early lumping approach. The closed-loop is formulated as:

$$\dot{\mathcal{X}} = \underbrace{\begin{pmatrix} (\mathcal{J}-\mathcal{R})\mathcal{Q} & -BB_c^T Q_c \\ B_c B^* \mathcal{Q} & 0 \end{pmatrix}}_{\mathcal{J}_{cl}} \mathcal{X}, \quad (3.33)$$

with $\mathcal{X} = \begin{pmatrix} x \\ x_c \end{pmatrix} \in L_2\left([0,L],\mathbb{R}^4\right)\times\mathbb{R}^m$. We make the following assumptions:

Assumption 1. *The operator $\mathcal{A}_{cl}$ defined by $\mathcal{A}_{cl}\mathcal{X} = \mathcal{J}_{cl}\mathcal{X}$ generates a contraction semigroup on $L_2\left([0,L],\mathbb{R}^4\right)\times\mathbb{R}^m$.*

Assumption 2. *The closed-loop resolvent set is compact.*

From these assumptions and the fact that the open loop system is exponentially stable and the controller passive, one can derive the following theorem using Lyapunov argument and evoking LaSalle's invariant principle.

Theorem 3.1. *For any $\mathcal{X}(0) \in L_2\left([0,L],\mathbb{R}^4\right)\times\mathbb{R}^m$ the unique solution of* (3.33) *converges asymptotically to zero, and the closed-loop system is globally asymptotically stable.*

The proof of this Theorem 3.1 is omitted for the sake of brevity.

4 Numerical simulations

We consider here the Timoshenko beam example with length $L = 2$ m, width $b = 0.5$m, thickness $h = 0.2$m, Young's modulus $E = 90$MPa, density $\rho = 1.633 \times 10^3$kg/m^3, Poisson ratio $v = 0.3$, translational dissipation $R_t = 10^{-3}$ and rotational dissipation $R_r = 10^{-6}$ in a clamped-free scenario. The initial conditions are set to a spatial distribution $x_3(\zeta,0) \sim 0.1\mathcal{N}(1.5, 0.113)$ for the angular strain and to zero for other state variables. The beam is discretized into 50 elements.

We consider a time step of 5×10^{-4}s and mid-point time discretization method[2] for simulations. The open loop evolution of the angular strain x_{3d} is given in Fig.3.

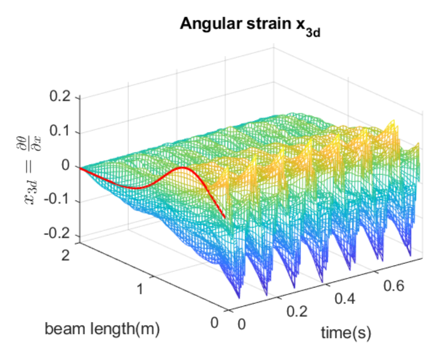


Figure 3: Open loop evolution of the angular strain.

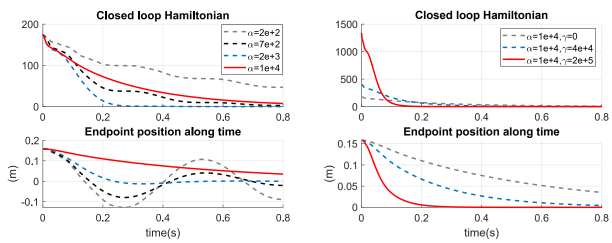
Next we investigate the numerical simulations of the closed-loop system considering both fully-actuated and under-actuated cases.

4.1 Fully-actuated case. Following Proposition 1 and 2 we choose $B_c = J_i$ in order to guarantee the existence of structural invariants, and the initial conditions of the controller such that $C = 0$. In this case (3.11) becomes: $x_c = x_{3d}$. The control law is given by:

$$\mathbf{u}_d = -\mathbf{y}_c = -B_c^T Q_c x_c - D_c \mathbf{y}_d. \quad (4.34)$$

[2]Implicit midpoint rule is known to be a structure-preserving time integrator for PHSs [1]. It is a particular case in the family of symplectic collocation methods for time integration which is investigated in [7].

In a first instance we consider the pure damping injection case, *i.e.* varying D_c with $Q_c = 0$. In Fig.4(a) we can see that this degree of freedom allows to damp the vibrations of the beam to the detriment of the time response. Next we fix $D_c = \operatorname{diag}\left(10^4 L_{ab}\right)$ corresponding to the slightly over-damped case in order to illustrate the effect of the energy shaping on the achievable performances. We can see in Fig.4(b) that we can speed up significantly the closed-loop system via energy shaping, without introducing any overshoot. A good dynamic performance is achieved when $Q_c = \operatorname{diag}\left(\frac{2\times 10^5}{L_{ab}}\right)$, which relates to an *equivalent Young's modulus* of $\tilde{E} = 690$MPa.

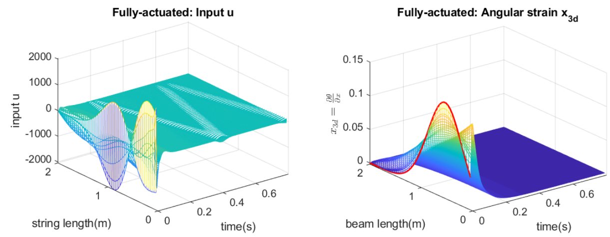


(a) (b)

Figure 4: Closed-loop Hamiltonian function and endpoint position in the fully-actuated case with (a) pure damping injection and with (b) energy shaping plus damping injection, with α denoting the damping injection coefficient and γ representing the energy shaping coefficient.

The evolution of the distributed input and of the angular strain along time are given in Fig.5(a) and (b), respectively. We can see in Fig.5(a) that the control remains smooth. Fig.5(b) shows that the closed-loop stabilization time is about 0.2s which is much faster than 0.8s resulting from the pure damping injection case.

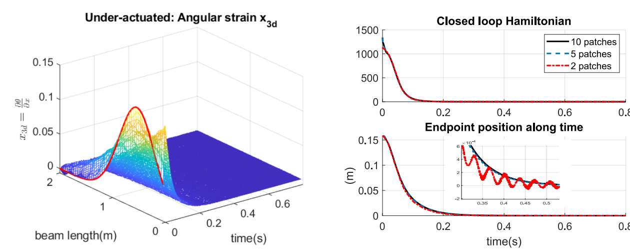
4.2 Under-actuated case. We now consider that the control is achieved using m patches as depicted in Fig.2. The aim of the control design is to modify as far as possible the closed-loop Young's modulus $\tilde{E}$ of the beam. $J_m \in \mathbb{R}^{m\times m}$ stems from the discretization of $\partial/\partial\zeta$, B_c is chosen to be $B_c = J_m$. According to (3.32), $Q_c = J_m^{-T} V \Sigma_0^{-1} U_1^T Q_m U_1 \Sigma_0^{-1} V^T J_m^{-1}$. D_c is chosen such that the time derivative of the Hamiltonian (3.22) behaves similarly than in the fully-actuated case, *i.e.* in order to satisfy $\min_{D_c\in\mathbb{R}^{m\times m}} \left\| b_d D_c b_d^T - \operatorname{diag}\left(\alpha L_{ab}\right)\right\|_F$, where b_d is given in (3.27). This optimization problem is similar to Problem 1, and the optimal D_c is given by $\hat{D}_c = \operatorname{diag}\left(\frac{\alpha L_{ab}}{k}\right)$.



(a) (b)

Figure 5: Evolution of the closed-loop input signal and angular strain in the energy shaping and damping injection case with full actuation, $D_c = \operatorname{diag}\left(10^4 L_{ab}\right)$, $Q_c = \operatorname{diag}\left(\frac{2\times 10^5}{L_{ab}}\right)$.

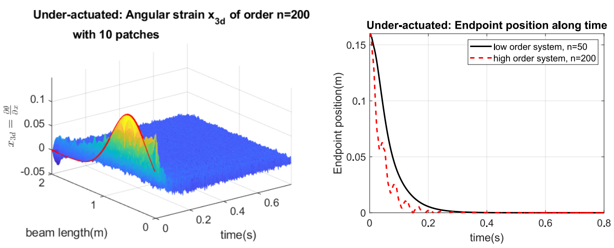
We first consider the case with 10 patches, *i.e.* $m = 10$, $n = 50$ and $k = 5$. In this case the angular strain evolution is quite similar to that obtained in the fully-actuated case as depicted in Fig.6(a). This indicates that if the controller matrices B_c, Q_c and D_c are adequately selected, the achievable performances in the under-actuated case can be optimized in order to be close to the ones obtained in the fully-actuated case. When the number of patches is reduced to 5 and to 2, *i.e.* $n = 50$, $k = 10$ and $n = 50$, $k = 25$ respectively, these performances are slightly deteriorated in the high frequencies as shown in Fig. 6(b).



(a) (b)

Figure 6: Closed-loop evolution of the angular strain for $m = 10$ (a), Hamiltonian function and endpoint position (b) in the under-actuated case for $m = 10$, $m = 5$ and $m = 2$.

In order to illustrate the effect of the neglected dynamics on the achievable performances we implement the controller designed considering 10 patches on the discretized system where $n = 50$ to a more precise model of the beam derived using $n = 200$. In Fig. 7 we can see that, due to the damping injection and the associated closed-loop bandwidth, the neglected dynamics do not impact significantly the closed-loop response of the system to the considered initial condition.



(a) (b)

Figure 7: Closed-loop evolution of the angular strain of the high order system (a), and comparison of the endpoint position of the low order and high order systems using the same controller (b).

5 Conclusion and future work

In this paper, we consider the in-domain control of infinite dimensional port-Hamiltonian systems using an early lumping approach. We extend the CbI method to the use of controllers distributed in space. The distributed structural invariants are used to modify the closed-loop angular potential energy of the system, *e.g.* the Young's modulus, in the considered Timoshenko beam example. Two different cases are investigated: the ideal case where the system is fully-actuated and the under-actuated case where the control action is achieved using piecewise homogeneous inputs. In this latter the controller is derived by optimization. Simulations of both fully-actuated and under-actuated cases show that the damping injection renders the system asymptotically stable, while the energy shaping improves the dynamic performances of the closed-loop system. Comparisons of the two cases also indicate that with an appropriate choice of the controller parameters, one can achieve similar performances for the under- and fully-actuated cases in a given frequency range. Future works aim at extending the approach to the use of observers and at generalizing the proposed control design procedure to classes of nonlinear infinite dimensional PHS.

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