Control design for linear port-Hamiltonian boundary control systems. An overview

A. Macchelli, Y. Le Gorrec, H. Ramírez, H. Zwart and F. Califano

Abstract In this paper, we provide an overview of some control synthesis methodologies for boundary control systems (BCS) in port-Hamiltonian form. At first, it is shown how to design a state-feedback control action able to shape the energy function to move its minimum at the desired equilibrium, and how to achieve asymptotic stability via damping injection. Secondly, general conditions that a linear regulator has to satisfy to have a well-posed and exponentially stable closed-loop system are presented. This second methodology is illustrated with reference to two specific stabilisation scenarios, namely when the plant is in impedance or in scattering form. It is also shown how these techniques can be employed in the analysis of more general systems described by coupled PDEs and ODEs. As an example, the repetitive control scheme is studied, and conditions to have asymptotic tracking of generic periodic reference signals are presented.

Key words: distributed parameter systems, port-Hamiltonian systems, control design, passivity-based control, stability

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1 Introduction

Port-Hamiltonian systems [16] have been introduced about twenty-five years ago to describe lumped parameter physical systems in an unified manner [6]. The generalisation to the infinite dimensional scenario led to the definition of distributed port-Hamiltonian systems [26], introduced about fifth-teen years ago. Most of the current research on stabilisation techniques deal with the development of boundary controllers, see e.g. [1, 11–15, 20, 22, 23]. In this paper, we illustrate the basic control design techniques for a particular class of linear, infinite dimensional port-Hamiltonian systems with one-dimensional domain, and boundary actuation and sensing. As proved in [10], such systems are boundary control systems (BCS) in the sense of the semigroup theory [4], and have been studied in detail in [9].

The simplest way of designing such boundary controllers is to add some dissipation at the boundary (damping injection), and use the total energy as Lyapunov function to prove asymptotic / exponential stability of the zero equilibrium state. A more sophisticated approach consists in adding a further step (energy-shaping), in which the closed-loop energy function is shaped to shift its equilibrium; stability is assured by the passivity of the closed-loop system. Two possible implementations of such technique are presented here. In the first one, the energy-shaping task is accomplished by generating a set of invariants (Casimir functions) that relate the state of the BCS to the state of the dynamical controller, [14, 15, 19, 22–24], and the shape of the closed-loop energy function is changed by acting on the Hamiltonian of the controller itself. The main drawback is that it is not possible to deal with equilibria that require an infinite amount of supplied energy in steady state (dissipation obstacle). This limitations is solved by the second approach. The idea is to mimic the energy-Casimir method without requiring the existence of invariants, and going through dynamic extension / reduction. The control action is selected among all the possible state-feedback laws able to shape the closed-loop Hamiltonian function e.g. to have an isolated minimum at the equilibrium. In this way, simple stability is obtained and, to have asymptotic stability, it is necessary to add damping. The result is that the final system is asymptotically stable, [13].

A second general approach for control design consists in determining the conditions that a control system has to meet so that the related closed-loop system is asymptotically / exponentially stable. The technique presented in this paper is an extension of [28] or, more precisely, of [20]. Differently, here the BCS is no longer required to be passive and the stability result can be applied to all the possible parametrisation of the input-output mapping presented in [10]. The resulting BCS turns out to be dissipative, and the control design follows two main steps. In the first one, conditions on the controller structure are obtained so that the system of coupled PDEs and ODEs associated with the closed-loop dynamics is a well-posed BCS. Then, in a second step, dissipation is added to let the closed-loop energy (storage) function decrease exponentially. This fact implies the exponential stability of the equilibrium, [12]. The potentialities of the approach are illustrated in case the port-Hamiltonian BCS is in impedance or in scattering form. In both cases, sufficient conditions that the finite dimensional controller has to satisfy to have an exponentially stable closed-loop system are provided. The proposed methodology can be also applied for the analysis of dynamical systems resulting from the interconnection of sub-systems modelled by means of PDEs and ODEs. To illustrate this feature, the stability analysis of the repetitive control [8] in the linear case is presented.

2 Distributed port-Hamiltonian systems

In this paper, we refer to the class of linear distributed port-Hamiltonian systems on real Hilbert spaces studied in [9, 10, 20, 27], i.e. to systems described by the PDE

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z) \tag{1}$$

with $x \in X := L^2(a,b;\mathbb{R}^n)$, and $z \in [a,\ b]$. Moreover, it is assumed that $P_1 = P_1^{\mathrm{T}}$ is invertible, $P_0 = -P_0^{\mathrm{T}}$, $G_0 = G_0^{\mathrm{T}} \geq 0$, and $\mathcal{L}(\cdot)$ is a bounded and Lipschitz continuous matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z)$ and $\mathcal{L}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a,\ b]$. For the sake of clarity, $(\mathcal{L}x)\,(t,z) := \mathcal{L}(z)x(t,z)$. We say that the symmetric matrix M is positive definite, in short $M > 0$, if all its eigenvalues are positive, and positive semi-definite, in short $M \geq 0$, if its eigenvalues are non-negative. The state space X is endowed with the inner product $\langle x_1 \mid x_2\rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2\rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1\rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot\rangle$ denotes the natural L^2-inner product. The selection of this space for the state variable is motivated by the fact that $\|\cdot\|^2_{\mathcal{L}}$ is strongly linked to the energy function of (1). As a consequence, X is also called the space of energy variables, and $\mathcal{L}x$ denote the co-energy variables.

Let $H^1(a,b;\mathbb{R}^n)$ denote the Sobolev space of order one. The PDE (1) can be compactly written as $\dot{x} = \mathcal{J}x$, where

$$\mathcal{J}x := P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x \tag{2}$$

is a linear operator with domain $D(\mathcal{J}) = \left\{\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)\right\}$. To have a port-Hamiltonian system, such PDE is completed by the set of boundary port variables $f_\partial,\ e_\partial \in \mathbb{R}^n$ that are a linear combination of the restriction of the co-energy variables $\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)$ to the boundary and are defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R}\begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}. \tag{3}$$

Theorem 1 *Let W be a full rank $n \times 2n$ real matrix, and define the input mapping $\mathcal{B} : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$ and the input $u(t)$ as*

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: \mathcal{B}x(t). \tag{4}$$

The operator $\bar{\mathcal{J}}x := P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ *with domain*

$$D(\bar{\mathcal{J}}) = \left\{ \mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \;\middle|\; \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \operatorname{Ker} W \right\} \tag{5}$$

generates a contraction semigroup on X if and only if

$$W \Sigma W^{\mathrm{T}} \geq 0, \qquad \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6}$$

and the system (1) with input (4) is a boundary control system on X, [4, Theorem 3.3.3], provided that $u \in C^2(0,\infty;\mathbb{R}^n)$. *Note that (5) is equivalent to require that* $u = \mathcal{B}x = 0$. *Moreover, let* $\tilde{W}$ *be a full rank* $n \times 2n$ *matrix such that* $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ *is invertible, and let* P_W *be given by*

$$P_W = \begin{pmatrix} W\Sigma W^{\mathrm{T}} & W\Sigma \tilde{W}^{\mathrm{T}} \\ \tilde{W}\Sigma W^{\mathrm{T}} & \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} \end{pmatrix}^{-1} = \begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-\mathrm{T}} \Sigma \begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-1}. \tag{7}$$

Define the output as

$$y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} =: Cx(t) \tag{8}$$

with $C : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$. *Then, for* $(\mathcal{L}x)(0) \in H^1(a,b;\mathbb{R}^n)$, *the following energy-balance equation is satisfied:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{9}$$

Proof See [10, Theorem 4.1]. □

The energy-balance relation (9) shows that (1) with input-output mapping defined by (4) and (8) is a dissipative system, [2], with storage function $H(x) := \frac{1}{2}\|x\|^2_{\mathcal{L}}$, and supply rate

$$s(u,y) = \frac{1}{2}\begin{pmatrix} u \\ y \end{pmatrix}^{\mathrm{T}} P_W \begin{pmatrix} u \\ y \end{pmatrix} =: \frac{1}{2}\begin{pmatrix} u \\ y \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U & S \\ S^{\mathrm{T}} & Y \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \tag{10}$$

where $U = U^{\mathrm{T}}$, and $Y = Y^{\mathrm{T}}$.

Remark 1 The input-output mapping of system (1) is in impedance form if W and $\tilde{W}$ in (4) and (8), respectively, are chosen such that $W\Sigma W^{\mathrm{T}} = \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$ and $\tilde{W}\Sigma W^{\mathrm{T}} = I$, which leads to a supply rate (10) equal to $s(u,y) = y^{\mathrm{T}}u$. Differently, (1) is in scattering form if W and $\tilde{W}$ are such that $W\Sigma W^{\mathrm{T}} = -\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = I$ and $\tilde{W}\Sigma W^{\mathrm{T}} = 0$, which leads to a supply rate (10) equal to $s(u,y) = \frac{1}{2}u^{\mathrm{T}}u - \frac{1}{2}y^{\mathrm{T}}y$.

Before presenting the design methodologies, a preliminary problem is to understand when the linear system resulting from the feedback interconnection of (1) with a linear control system is well-posed, i.e. it is BCS. In this respect, let us consider

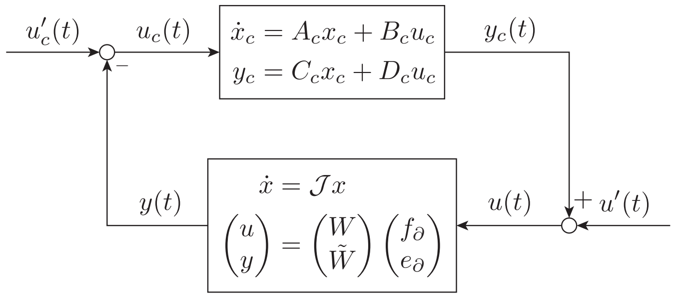

Fig. 1 Distributed port-Hamiltonian system (1) with boundary controller (11).

the following linear control system

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \\ y_c(t) = C_c x_c(t) + D_c u_c(t) \end{cases} \tag{11}$$

where $x_c \in \mathbb{R}^{n_c}$ and $u_c, y_c \in \mathbb{R}^n$. It is assumed that A_c has eigenvalues with non-positive real part, and that the pair (A_c, B_c) is controllable. System (11) is interconnected to the boundary of (1) in standard feedback interconnection through the input $u(t)$ and $y(t)$ defined in (4) and (8), respectively, as shown in Fig. 1. This means that

$$\begin{pmatrix} u_c(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y_c(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} u_c'(t) \\ u'(t) \end{pmatrix} \tag{12}$$

where $u_c', u' \in \mathbb{R}^n$ are auxiliary signals. Finally, it is assumed that there exists a symmetric, positive definite $n_c \times n_c$ real matrix Q_c such that (11) is dissipative with storage function $H_c(x_c) := \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$ and supply rate

$$s_c(u_c, y_c) = \frac{1}{2} \begin{pmatrix} u_c \\ y_c \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \begin{pmatrix} u_c \\ y_c \end{pmatrix} \tag{13}$$

with $U_c = U_c^{\mathrm{T}}$, and $Y_c = Y_c^{\mathrm{T}}$.

The closed-loop system resulting from the interconnection of (1) and (11) through the set of relations (12) can be compactly written as

$$\begin{cases} \dot{\xi}(t) = \mathcal{J}_{cl}\xi(t) + B_{cl} u_c'(t) \\ D_c u_c'(t) + u'(t) = \left(\mathcal{B} + D_c \mathcal{C} \;\; -C_c\right) \xi(t) =: \mathcal{B}' \xi(t) \end{cases} \tag{14}$$

where the operators $\mathcal{B}$ and $\mathcal{C}$ are defined in (4) and (8), respectively, $\xi = (x, x_c) \in X_{cl} := X \times \mathbb{R}^{n_c}$ is the state variable, $\mathcal{J}_{cl} : D(\mathcal{J}_{cl}) \subset X_{cl} \to X_{cl}$ and $B_{cl} : \mathbb{R}^n \to X_{cl}$ are the linear operators

$$\mathcal{J}_{cl}\xi := \begin{pmatrix} \mathcal{J} & 0 \\ -B_c \mathcal{C} & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} \qquad\qquad B_{cl} v := \begin{pmatrix} 0 \\ B_c v \end{pmatrix} \tag{15}$$

with

$$D(\mathcal{J}_{cl}) := D(\mathcal{J}) \times \mathbb{R}^{n_c} \tag{16}$$

being $\mathcal{J}$ the operator introduced in (2). Moreover, the state space X_{cl} is endowed with the inner product $\langle \xi_1 \mid \xi_2 \rangle_{X_{cl}} = \langle x_1 \mid x_2 \rangle_{\mathcal{L}} + x_{c,1}^{\mathrm{T}} Q_c x_{c,2}$. Some fundamental properties associated to the coupled PDEs and ODEs that describe the closed-loop dynamics are discussed in the next proposition.

Proposition 1 *Let us consider the closed-loop system resulting from the feedback interconnection (12) of (1) and (11), which results in (14). If*

$$\begin{pmatrix} Y & -S^{\mathrm{T}} \\ -S & U \end{pmatrix} + \begin{pmatrix} U_c & S_c \\ S_c^{\mathrm{T}} & Y_c \end{pmatrix} \leq 0, \tag{17}$$

the operator $\bar{\mathcal{J}}_{cl}$ defined as $\bar{\mathcal{J}}_{cl}\xi := \begin{pmatrix} \mathcal{J} & 0 \\ -B_c C & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$ *with domain*

$$D(\bar{\mathcal{J}}_{cl}) = \left\{ \begin{pmatrix} x \\ x_c \end{pmatrix} \in X_{cl} \mid x \in D(\mathcal{J}), \text{ and } \mathcal{B}' \begin{pmatrix} x \\ x_c \end{pmatrix} = 0 \right\} \tag{18}$$

and $\mathcal{B}'$ defined in (14) generates a contraction semigroup on X_{cl}. Moreover, (14) with $\mathcal{J}_{cl}$ and B_{cl} defined by (15) and (16) is a BCS on X_{cl} if $u_c', u' \in C^2(0, \infty; \mathbb{R}^n)$.

Proof This result is an extension of [12, Proposition 7]. □

Remark 2 If system (1) is in impedance form, see Remark 1, the control system (11) meets the condition of the previous proposition, for example, if it is passive, i.e. dissipative with respect to the supply rate $s_c(u_c, y_c) = y_c^{\mathrm{T}} u_c$. This result has been proved in [11, 28]. On the other hand, if (1) is in scattering form, the control system (11) can be selected such that it is dissipative with respect to the supply rate $s_c(u_c, y_c) = \frac{1}{2}\gamma^2 \|u_c\|^2 - \frac{1}{2}\|y_c\|^2$, with $|\gamma| \leq 1$. In other words, (11) should have a L^2-gain lower than γ, with $|\gamma| \leq 1$, [25].

3 Energy-shaping design by interconnection and state-feedback

Proposition 1 shows when the system resulting from the feedback interconnection (12) of (1) and (11) is well-posed. It is also easy to check that such system is dissipative with storage function $H_{cl}(x, x_c) := H(x) + H_c(x_c)$, and the idea is to use H_{cl} as Lyapunov function. The control design procedure starts by guaranteeing that H_{cl} has a minimum at the desired equilibrium with a proper choice of H_c. As in the finite dimensional case [18], if it is possible to find invariants of the form $C(x, x_c) = x_c - F(x)$ that do not depend on the Hamiltonian of the system, then on every invariant manifold $x_c - F(x) = \kappa$, with $\kappa \in \mathbb{R}$ a constant which depends on the initial condition, the closed-loop Hamiltonian may be written as $H_{cl}(x) = H(x) + H_c(F(x) + \kappa)$. Hence, the equilibrium now depends on the choice of H_c and, on the invariant manifold, H_{cl} is function of the state of (1) only. These invariants are called Casimir functions, and their definition is reported below, [5,15]. For simplicity and with Remarks 1 and 2 in mind, we assume that (1) and (11) are passive, and that the latter one has a port-Hamiltonian structure, i.e.:

$$A_c = (J_c - R_c)Q_c \quad B_c = G_c - P_c \quad C_c = (G_c + P_c)^{\mathrm{T}} Q_c \quad D_c = M_c + S_c \tag{19}$$

where $J_c = -J_c^{\mathrm{T}}$, $M_c = -M_c^{\mathrm{T}}$, $R_c = R_c^{\mathrm{T}}$ and $S_c = S_c^{\mathrm{T}}$, and such that

$$\begin{pmatrix} R_c & P_c \\ P_c^{\mathrm{T}} & S_c \end{pmatrix} \geq 0. \tag{20}$$

Note that (11) with (19) and (20) is passive and, once interconnected to (1), leads to a well-posed closed-loop system in the sense of Proposition 1.

Definition 1 Consider the BCS of Proposition 1, and assume that $u' = u_c' = 0$ in (12) and that (11) is such that (19) and (20) hold. A function $C : X \times \mathbb{R}^{n_c} \to \mathbb{R}$ is a Casimir function if $\dot{C}(x(t), x_c(t)) = 0$ along the solutions for every possible choice of $\mathcal{L}(\cdot)$ and Q_c.

Proposition 2 *Under the conditions of Definition 1, the functional*

$$C(x(t), x_c(t)) := \Gamma^{\mathrm{T}} x_c(t) + \int_a^b \Psi^{\mathrm{T}}(z) x(t,z)\,\mathrm{d}z \tag{21}$$

with $\Gamma \in \mathbb{R}^{n_c}$ *and* $\Psi \in H^1(a,b;\mathbb{R}^n)$*, is a Casimir function for the closed-loop system if and only if*

$$\begin{aligned} P_1 \frac{\mathrm{d}\Psi}{\mathrm{d}z}(z) + (P_0 + G_0)\Psi(z) &= 0 \\ (J_c + R_c)\Gamma + (G_c + P_c)\tilde{W} R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} &= 0 \\ (G_c + P_c)^{\mathrm{T}}\Gamma + [W + (M_c - S_c)\tilde{W}] R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} &= 0 \end{aligned} \tag{22}$$

From the first condition in (22) we can show that it is always possible to find n independent Casimir functions; so, in (11), we can assume that $n_c = n$. Now, let $\hat{\Gamma} := (\Gamma_1, \ldots, \Gamma_n)$ and $\hat{\Psi} := (\Psi_1, \ldots, \Psi_n)$ be the $n \times n$ matrices build from the elements that appear in each Casimir function (21). If $\hat{\Gamma}$ is invertible, under the conditions of Proposition 2, we have that

$$x_c(t) = -\hat{\Gamma}^{-1} \int_a^b \hat{\Psi}^{\mathrm{T}}(z) x(t,z)\,\mathrm{d}z + \kappa \tag{23}$$

with $\kappa \in \mathbb{R}^n$ a constant that depends only on the initial condition of (1) and (11). As a consequence, the controller Hamiltonian H_c is in fact a function of the state variables of the plant, and may be chosen to obtain a desired shape for the closed-loop energy.

In the linear case, we have that $H_c(x_c) := \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$, with $Q_c = Q_c^{\mathrm{T}} > 0$. Note that it is also possible to project the state of the closed-loop system (x, x_c), on that state space of the plant to write the control action in state feedback form, i.e.:

$$\begin{aligned} u(t) &= C_c x_c(t) - D_c y(t) + u'(t) \\ &= (G_c + P_c)^{\mathrm{T}} Q_c x_c(t) - (M_c + S_c) y(t) + u'(t) \end{aligned} \tag{24}$$

where the output equation of (11) has been taken into account. In a state-feedback realisation of (24), the expression of $x_c(t)$ is given by (23), in which κ can be conveniently chosen equal to 0. This step is usually named reduction step because it reduces the dynamic contribution of the controller to a static one using a state feedback approach. It is possible to verify that the final closed-loop dynamic is given by the following BCS:

$$\begin{aligned} \frac{\partial x}{\partial t}(t,z) &= P_1 \frac{\partial}{\partial z} \frac{\delta H_{cl}}{\delta x}(x(t,z)) + (P_0 - G_0) \frac{\delta H_{cl}}{\delta x}(x(t,z)) \\ u'(t) &= W' R \begin{pmatrix} \frac{\delta H_{cl}}{\delta x}(x(t,b)) \\ \frac{\delta H_{cl}}{\delta x}(x(t,a)) \end{pmatrix} \end{aligned} \tag{25}$$

in which δ is the variational derivative [17], $H_{cl}(x) := \frac{1}{2} \|x\|^2_{\mathcal{L}} + \frac{1}{2} x_c^{\mathrm{T}}(x) Q_c x_c(x)$ and W' is a $n \times 2n$ full rank matrix which satisfies the hypothesis of Theorem 1, i.e. $W' \Sigma W'^{\mathrm{T}} \geq 0$. Independently from the way in which the control action is implemented (dynamic extension or state-feedback), the closed-loop system has the same structure of the plant (1), i.e. the same matrices P_1, P_0 and G_0, but a different (shaped) Hamiltonian, namely H_{cl}.

Even if this approach allows to shape the Hamiltonian function of (1) into H_{cl} and to obtain the closed-loop system (25), the presence of dissipation imposes strong constraints on the applicability of the method itself. In particular, (22) implies that

$$G_0 \hat{\Psi}(z) = 0 \qquad \qquad \begin{pmatrix} R_c & P_c \\ P_c^{\mathrm{T}} & S_c \end{pmatrix} \begin{pmatrix} \hat{\Gamma} \\ \tilde{W} R \begin{pmatrix} \hat{\Psi}(b) \\ \hat{\Psi}(a) \end{pmatrix} \end{pmatrix} = 0 \tag{26}$$

Such conditions, and the first one in particular, show that it is not possible to shape the closed-loop Hamiltonian in the coordinates in which dissipation is present. This limitation is called "dissipation obstacle," and is related to the fact that the (passive) control system (11) has just a finite amount of energy at disposal to drive the state of the plant (1) towards the desired equilibrium. To overcome this limitation, the idea is to start from the feedback law (23)-(24) derived in the context of the immersion / reduction scheme and to directly design a state feedback law that shapes the closed-loop Hamiltonian function, but without relying on the dynamic extension and on the Casimir functions. Asymptotic stability is then guaranteed by damping injection via the auxiliary input $u'(t)$.

Proposition 3 *Consider the system (1) with boundary input u defined in (4). The feedback law $u(t) = \beta(x(t)) + u'(t)$ in which u' is an auxiliary input, maps (1) into*

$$
\begin{aligned}
\frac{\partial x}{\partial t}(t,z) &= P_1 \frac{\partial}{\partial z}\frac{\delta H_d}{\delta x}(x(t,z)) + (P_0 - G_0)\frac{\delta H_d}{\delta x}(x(t,z)) \\
u'(t) &= WR\begin{pmatrix} \frac{\delta H_d}{\delta x}(x(t,b)) \\ \frac{\delta H_d}{\delta x}(x(t,a)) \end{pmatrix}
\end{aligned}
\tag{27}
$$

with $H_d(x) := \frac{1}{2}\|x\|^2_{\mathcal{L}} + H_a(\xi(x))$ and H_a an arbitrary C^1 function, if

$$
\beta(x) = -WR\begin{pmatrix}\Phi(b)\\ \Phi(a)\end{pmatrix}\frac{\partial H_a}{\partial \xi}(\xi(x)) \tag{28}
$$

in which $\xi(x(t,\cdot)) := \int_a^b \Phi^{\mathrm{T}}(z)x(t,z)\,\mathrm{d}z$, being $\Phi(z) := (\Phi_1(z),\ldots,\Phi_n(z))$ and each $\Phi_i \in H^1(a,b;\mathbb{R}^n)$ independent solution of

$$
P_1\frac{\mathrm{d}\Phi_i}{\mathrm{d}z}(z) + (P_0 - G_0)\Phi_i(z) = 0 \tag{29}
$$

Proof This result is a reformulation of [13, Proposition 4.1 and Lemma 4.2]. □

From the previous proposition, it is not clear how to select $H_a(\xi)$ and then $\beta(x)$ so that the energy function of (1) is properly shaped, for example to move the minimum at the desired equilibrium configuration $\mathcal{L}x^\star \in H^1(a,b;\mathbb{R}^n)$. Note that, due to the definition of $\Phi(z)$ in Proposition 3, there exists a unique $\phi \in \mathbb{R}^n$ such that $(\mathcal{L}x^\star)(z) = \Phi(z)\phi$. A possible choice for $H_a(\xi)$ is then

$$
H_a(\xi) := \frac{1}{2}(\xi - \xi^\star)^{\mathrm{T}}Q_a(\xi - \xi^\star) - \phi^{\mathrm{T}}\xi + H_a^\star \tag{30}
$$

where $\xi^\star := \xi(x^\star)$, $H_a^\star$ is a constant selected so that $H_a(\xi) > 0$ for all $\xi \neq 0$, and $Q_a = Q_a^{\mathrm{T}} > 0$. With this choice, we have that

$$
\frac{\delta H_d}{\delta x}(x(t,z)) = \mathcal{L}(z)x(t,z) + \Phi(z)Q_a\left[\xi(x(t,z)) - \xi_\star\right] - \Phi(z)\phi \tag{31}
$$

that is equal to 0 when $x(t,z) = x^\star(z)$ because of the definition of ϕ. Such critical "point" is isolated, and is a minimum for H_d. From (30) and thanks to (28), the corresponding energy-shaping control law $\beta(x)$ can be obtained. It is worth noticing that the same procedure can be applied for the selection of H_c once a set of invariants has been computed by following Proposition 2. In fact, for any $\hat{\Psi}(z)$ in (23), from (22) and (26), it is possible to check that $P_1\frac{\mathrm{d}\hat{\Psi}}{\mathrm{d}z}(z) + (P_0 - G_0)\hat{\Psi}(z) = 0$, which is the same condition that $\Phi(z)$ introduced in Proposition 3 has to meet. This means that H_c can be selected equal to H_a, and so $H_{cl} \equiv H_d$. In other words and similarly to the finite-dimensional case [18], the energy-shaping methodology based on Proposition 2 is a particular case of the procedure illustrated in Proposition 3. The difference is that in the latter case, it is possible to shape the Hamiltonian in the coordinates that have pervasive dissipation. Differently, not all the equilibria of (1) can be stabilised with the first approach.

In any case, once the closed-loop Hamiltonian has been defined, and these considerations are valid either for H_{cl} in (25) and H_d in (26), convergence of the trajectories towards the new minimum of the energy function is obtained by introducing dissipation via the auxiliary input $u'(t)$. With an eye on the energy-shaping procedure presented in Proposition 3 and on the closed-loop dynamic (26), by Theorem 1, a natural choice for the output dual to $u'(t)$ is

$$y'(t) = \tilde{W} R \begin{pmatrix} \frac{\delta H_d}{\delta x}(x(t,b)) \\ \frac{\delta H_d}{\delta x}(x(t,a)) \end{pmatrix} \tag{32}$$

for which it is immediate to check that $\frac{\mathrm{d}}{\mathrm{d}t} H_d(x(t)) \leq y'^{\mathrm{T}}(t) u'(t)$. A simple way to introduce dissipation is by imposing that, [18]:

$$u'(t) = -\Xi y'(t), \qquad \Xi = \Xi^{\mathrm{T}} \geq 0. \tag{33}$$

Proposition 4 *Consider the linear BCS of Theorem 1, and the equilibrium state $x^\star(z)$. Then, the control action $u(t) = \beta(x(t,\cdot)) + u'(t)$ in which β is defined as in (28) with the choice (30) for H_a, and u' as in (33) with $\Xi > 0$, makes $x^\star(z)$ asymptotically stable.*

Proof See [13, Theorem 4.5]. □

4 Exponential stabilisation of port-Hamiltonian linear BCS

In the previous section, we have shown that via energy-shaping and damping injection it is possible to asymptotically stabilise an equilibrium configuration for the BCS of Theorem 1. With an eye on the feedback scheme reported in Fig. 1, the aim is now to show how it is possible to choose the linear control system (11) so that the closed-loop system is not only well-posed in the sense of Proposition 1, but also exponentially stable. The result generalises what has been presented in [20] for port-Hamiltonian BCS in impedance form, under the further requirement that (11) is a port-Hamiltonian system.

Let us assume that the linear control system (11) is such that Proposition 1 holds. The main requirement is that the following LMI holds true:

$$\begin{pmatrix} Q_c A_c + A_c^{\mathrm{T}} Q_c & Q_c B_c \\ B_c^{\mathrm{T}} Q_c & 0 \end{pmatrix} - \begin{pmatrix} C_c^{\mathrm{T}} Y_c C_c & C_c^{\mathrm{T}} Y_c D_c \\ D_c^{\mathrm{T}} Y_c C_c & U_c + D_c^{\mathrm{T}} Y_c D_c \end{pmatrix} - \\ - \begin{pmatrix} 0 & C_c^{\mathrm{T}} S_c \\ S_c^{\mathrm{T}} C_c & D_c^{\mathrm{T}} S_c + S_c^{\mathrm{T}} D_c \end{pmatrix} \leq - \begin{pmatrix} -\delta_x (Q_c A_c + A_c^{\mathrm{T}} Q_c) & 0 \\ 0 & \delta_u I \end{pmatrix} \tag{34}$$

with δ_x and δ_u two positive constants. When $\delta_x = \delta_u = 0$, (34) states that (11) is dissipative with storage function $H_c(x_c) := \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$, with $Q_c = Q_c^{\mathrm{T}} > 0$, and supply rate (13). From a physical point of view, δ_x is related to the presence of internal damping in the control system responsible for attenuating the lower frequencies in the plant dynamics. Differently, δ_u assures that the higher frequencies are damped.

Proposition 5 *Under the same conditions of Proposition 1, assume that the control system (11) is such that A_c has all the eigenvalues with negative real part, the pair (A_c, B_c) is controllable, and (33) holds with $\delta_x > 0$ and $\delta_u > 0$. Then, the closed-loop system (14) with $u_c'(t) = u'(t) = 0$ is exponentially stable.*

Proof See [12, Proposition 12]. □

The previous result is now used to study the stability of systems whose dynamics are given in terms of coupled PDEs and ODEs. With Remark 1 in mind, we start with a standard regulation problem in which the distributed port-Hamiltonian system (1) is in impedance or in scattering form. Then, we focus on a different and apparently unrelated topic, i.e. repetitive control of linear systems, [8]. The goal is to determine the class of linear systems for which this control technique can be applied. Now, let us first assume that (1) is in impedance form, which implies that (1) is passive, i.e. dissipative with storage function given by the total energy $\frac{1}{2}\|x\|^2_{\mathcal{L}}$, and supply rate $s(u,y) = y^{\mathrm{T}}u$, where input $u(t)$ and output $y(t)$ are given in (4) and (8), respectively. From (34) in Proposition 5, the control system (11) leads to an exponentially stable closed-loop system if there exists $Q_c = Q_c^{\mathrm{T}} > 0$, $\delta_x > 0$ and $\delta_u > 0$ such that

$$\begin{pmatrix} (1-\delta_x)(Q_c A_c + A_c^{\mathrm{T}} Q_c) & Q_c B_c \\ B_c^{\mathrm{T}} Q_c & 0 \end{pmatrix} - \begin{pmatrix} 0 & C_c^{\mathrm{T}} \\ C_c & D_c + D_c^{\mathrm{T}} - \delta_u I \end{pmatrix} \leq 0 \tag{35}$$

This implies that $D_c + D_c^{\mathrm{T}} \geq \delta_u I > 0$, and that $\dot{H}_c(x_c(t)) \leq y_c^{\mathrm{T}}(t) u_c(t) - \delta_u \|u_c(t)\|^2$, where $H_c(x_c) = \frac{1}{2} x_c^{\mathrm{T}} Q_c x_c$ is the storage function of (11). This relation implies that the control system has to be input strictly passive, [25].

Corollary 1 *Under the same conditions of Proposition 5, let us consider the port-Hamiltonian system (1), now assumed in impedance form. Moreover, let us denote by $H_c(s) = C_c(sI - A_c)^{-1} B_c + D_c$ the transfer matrix of (11). The closed-loop system (14) is exponentially stable if the linear system with transfer matrix $H(s-\epsilon)$ is strictly input passive for some $\epsilon > 0$.*

In [20], the same result of Corollary 1 has been proved in case the control system (20) is in a specific port-Hamiltonian form, i.e. if (19) holds with $P_c = 0$ and $M_c = 0$. It is easy to see that if $A_c = (J_c - R_c)Q_c$ is Hurwitz, $R_c \neq 0$, and $D_c + D_c^{\mathrm{T}} > 0$ then (35) holds, and the closed-loop system is exponentially stable. Note that (35) holds even for $\delta_x = 0$. In fact, from (19) we have that $C_c = G_c^{\mathrm{T}} Q_c = B_c^{\mathrm{T}} Q_c$, and $Q_c A_c + A_c^{\mathrm{T}} Q_c = -2 Q_c R_c Q_c \leq 0$ because J_c is skew-symmetric and $R_c \neq 0$. It is worth mentioning that an extension to the case in which the control system is nonlinear has been presented in [21].

Analogous considerations can be drawn if (1) is in scattering form, i.e. when input and output are selected in such a way that the distributed parameter system is dissipative with storage function $\frac{1}{2}\|x\|^2_{\mathcal{L}}$ and supply rate $s(u,y) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$.

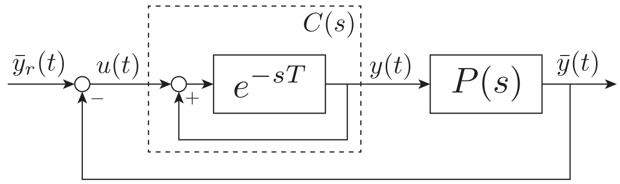

Fig. 2 Basic structure of (continuous-time) repetitive control, [8].

From (35) in Proposition 5, the regulator (11) exponentially stabilises the BCS (1) if there exists $Q_c = Q_c^{\mathrm{T}} > 0$, and $\delta_x > 0$ and $\delta_u > 0$ such that

$$\begin{pmatrix} (1-\delta_x)(Q_c A_c + A_c^{\mathrm{T}} Q_c) & Q_c B_c \\ B_c^{\mathrm{T}} Q_c & 0 \end{pmatrix} + \begin{pmatrix} C_c^{\mathrm{T}} C_c & C_c^{\mathrm{T}} D_c \\ D_c^{\mathrm{T}} C_c & D_c^{\mathrm{T}} D_c - (\gamma^2 - \delta_u) I \end{pmatrix} \leq 0 \qquad (36)$$

for some γ such that $|\gamma| \leq 1$. The LMI (36) implies that $D_c^{\mathrm{T}} D_c - (\gamma^2 - \delta_u) I \leq 0$, i.e. that $D_c^{\mathrm{T}} D_c - I < 0$, which means that the feedthrough gain has to be lower than 1 or, equivalently, that the dissipation inequality $\dot{H}_c(x_c(t)) \leq \frac{1}{2}\gamma^2 \|u_c(t)\|^2 - \frac{1}{2}\|y_c(t)\|^2$ holds true with $|\gamma| < 1$.

Corollary 2 *Under the same conditions of Proposition 5, let us consider the port-Hamiltonian system (1), now assumed in scattering form. Moreover, let us denote by* $H_c(s) = C_c(sI - A_c)^{-1} B_c + D_c$ *the transfer matrix of (11). The closed-loop system (14) is exponentially stable if, for some* $\epsilon > 0$*, the linear system with transfer matrix* $H(s-\epsilon)$ *has* L^2*-gain* $\gamma < 1$.

The final contribution is now to show how the previous methodological results can be applied on different control problems, provided that the closed-loop system is described by a set of coupled PDEs and ODEs. The focus is on repetitive control, [8], a simple technique to let a dynamical system to track and / or reject periodic exogenous signals with a known time period T. Its effectiveness relies on the Internal Model Principle [7], and the main properties depend on a particular element reported in Fig. 2 and denoted by $C(s)$. Such dynamical system, called *repetitive compensator*, is a pure time delay T surrounded by a positive feedback loop that represents, from an Internal Model Principle point of view, a generator of any periodic signal whose period equals the amount of time in the delay.

The repetitive compensator can be described by means of a delay PDE. When $u(t) = 0$, the particular structure of the compensator causes the initial condition associated to the delay equation, i.e. an arbitrary function defined on $[0, T]$, to be periodically transported along the domain to generate the periodic signal $y(t)$. The pair (u, y) defines the input-output mapping of the system, with u and y that depend on the boundary conditions of the PDE. The repetitive compensator admits an interpretation in terms of a BCS in port-Hamiltonian form in the sense of Theorem 1 if we select for (1) $P_1 = -I$, $P_0 = G_0 = 0$, and $\mathcal{L}(z) = I$, with $z \in [0, T]$, and if $W = \sqrt{2}\begin{pmatrix} I & 0 \end{pmatrix}$ and $\tilde{W} = \frac{\sqrt{2}}{2}\begin{pmatrix} -I & I \end{pmatrix}$. Moreover, it is easy to check that it obeys to the following energy-balance relation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|x(t)\|_2^2 = y^{\mathrm{T}}(t)u(t) + \frac{1}{2}u^{\mathrm{T}}(t)u(t) = \frac{1}{2}\begin{pmatrix}u(t)\\y(t)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}I & I\\I & 0\end{pmatrix}\begin{pmatrix}u(t)\\y(t)\end{pmatrix}$$

An immediate consequence is that it is possible to treat repetitive control within the port-Hamiltonian framework or, more precisely, to rely on the stability tools discussed in this section to determine under which conditions on the plant dynamics the closed-loop system depicted in Fig. 2 is exponentially stable.

Since the repetitive compensator is a BCS, the scheme of Fig. 2 can be equivalently represented as in Fig. 1 with $\mathcal{J} = -\frac{\partial}{\partial z}$, $\bar{y}_r(t) \equiv u'(t)$ the periodic reference signal, $u'_c(t) = 0$, and under the hypothesis that the plant is the linear system (11). Note that, in this case, the distributed parameter system is the controller (repetitive compensator) responsible for stabilising on a periodic trajectory the finite dimensional plant. From Proposition 1, existence of solution in closed-loop is guaranteed if (11) is dissipative with respect to the quadratic storage function $H_c(x_c) = \frac{1}{2}x_c^{\mathrm{T}}Q_c x_c$, and to the supply rate (13), in which $U_c = 0$, $S_c = I$, and $Y_c = -\sigma I$, with $\sigma \geq 1$. Then, Proposition 5 is instrumental to characterise the class of linear systems (11) for which the closed-loop system is exponentially stable. Then, the Internal Model Principle assures that the tracking error goes to zero in case of periodic reference signals $\bar{y}_r(t)$, [8].

Proposition 6 *The repetitive control scheme of Fig. 2 is well-posed and exponentially stable if the plant $P(s)$ takes the from (11), and it is such that A_c is Hurwitz, the pair (A_c, B_c) is controllable, and*

$$\begin{pmatrix}Q_c A_c + A_c^{\mathrm{T}}Q_c & Q_c B_c\\ B_c^{\mathrm{T}}Q_c & 0\end{pmatrix} - \begin{pmatrix}-\sigma C_c^{\mathrm{T}}C_c & C_c^{\mathrm{T}}\left(I-\sigma D_c\right)\\ \left(I-\sigma D_c^{\mathrm{T}}\right)C_c & D_c^{\mathrm{T}} + D_c - \sigma D_c^{\mathrm{T}}D_c\end{pmatrix} \leq \\ \leq -\begin{pmatrix}-\delta_x\left(Q_c A_c + A_c^{\mathrm{T}}Q_c\right) & 0\\ 0 & \delta_u I\end{pmatrix} \tag{37}$$

holds for a $Q_c = Q_c^{\mathrm{T}} > 0$ $\sigma \geq 1$, $\delta_x > 0$ and $\delta_u > 0$.

Proof The result follows from Prop. 5, in which the supply rate of (11) is given as in (13), with $U_c = 0$, $S_c = I$, and $Y_c = -\sigma I$. □

The property summarised in the previous proposition is consistent with the classical stability conditions of repetitive control. In fact, a necessary condition for (37) to hold true is that $D_c^{\mathrm{T}} + D_c - \sigma D_c^{\mathrm{T}}D_c \geq \delta_u I$, for all $\sigma \geq 1$, and $\delta_u > 0$. If for simplicity $D_c = \gamma I$, we have that $\sigma\gamma^2 - 2\gamma < 0$, i.e. that $0 < \gamma < \frac{2}{\sigma}$. So, it is necessary that (11) is strictly proper, and that the feedthrough gain γ is positive and lower than 2, which corresponds to $\sigma = 1$. Note that, since now the stability condition is given in time-domain and based on energy-considerations, it can be extended also to deal with nonlinear systems. A first attempt in this direction has been illustrated in [3]. Moreover, from Proposition 1, since $\sigma \geq 1$, it can be deduced that, to obtain a closed-loop system whose evolution is described in terms of a contraction C_0-semigroup, (11) has to be ν-output strictly passive [25], with $\nu \geq \frac{1}{2}$. Then, (37) forces (11) to have a non-null feed-through term, and low frequency dissipation to guarantee exponential stability. With Corollaries 1 and 2 in mind, this latter condition is equivalent to require that the linear system with transfer matrix $H(s-\epsilon)$ is ν-output strictly passive with $\nu \geq \frac{1}{2}$, where $H(s)$ is the transfer matrix of (11).

5 Conclusions and future works

The goal of the paper is to present in an unified manner some of the basic control design techniques developed so far for linear BCS in port-Hamiltonian form characterised by a 1D spatial domain. The first synthesis methodology is based on state-feedback and capable to shape the energy function to move its minimum at the desired equilibrium. In this case, asymptotic stability is obtained via damping injection. The second technique, instead, provides some general conditions in terms of an LMI that a linear regulator has to satisfy to obtain a closed-loop system that is well-posed and exponentially stable. This methodology is quite general and powerful, and it has been illustrated with reference to two stabilisation scenarios, i.e. when the plant is in impedance or in scattering form. Moreover, because of its generality, it can be employed in the analysis of systems described by coupled PDEs and ODEs. As an example, the repetitive control scheme is studied.

Future researches are mainly focused to the extension of such results to BCS in port-Hamiltonian form in which the spatial domain is 2D or 3D. Another stimulating research topic deals with nonlinear BCS. In this respect, some preliminary results have been discussed in [3,21], but some efforts are still required to develop a general theory that is applicable to a large class of systems.

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