Exponential Stabilisation of Port-Hamiltonian Boundary Control Systems via Energy-Shaping

Alessandro Macchelli, Yann Le Gorrec, and Héctor Ramírez

***Abstract*—This paper is concerned with the exponential stabilisation of a class of linear boundary control systems (BCS) in port-Hamiltonian form through energy-shaping. Starting from a first feedback loop that is in charge of modifying the Hamiltonian function of the plant, a second control loop that guarantees exponential convergence to the equilibrium is designed. In this way, a major limitation of standard energy-shaping plus damping injection control laws applied to linear port-Hamiltonian BCS, namely the fact that only asymptotic convergence is assured, has been removed.**

***Index Terms*—port-Hamiltonian systems, boundary control systems, exponential stability, passivity**

I. Introduction

Boundary control systems (BCS) [1], [2] are dynamical systems modelled by partial differential equations (PDEs) with input and output defined at the boundary of the spatial domain. The study of the existence of solution and stabilisation of undamped or weakly damped linear BCS via static or dynamic controllers has raised a major attention in the last decade [3]–[7] due to an increase use of boundary controlled flexible / wave-like structures in engineering applications (smart grids, traffic-flows, compliant structures, etc.). For the class of linear BCS in port-Hamiltonian form introduced in [8], powerful techniques that exploit the geometric structure of the system to study the well-posedness or to design in a constructive manner stabilising control laws have been presented in the last years, see e.g. [5], [8]–[12] and references therein.

Two control synthesis strategies have been proposed so far. The first one extends an analogous approach originally developed for lumped-parameter systems, [13]. The feedback law is designed to map the open-loop system into a *target* dynamic still in port-Hamiltonian form, but characterised by different energy function and internal dissipation. This control technique consists of two feedback loops: the first one implements the so-called energy-shaping, i.e. it is responsible for modifying the Hamiltonian function e.g. to move its minimum at the desired equilibrium configuration. The second one, instead, is designed to let the total energy to decrease until the "new" minimum is reached. From a physical point of view, the damping injection control law is "equivalent" to the interconnection of a linear dissipative element at the input / output port of the system. In other words, the control action is just the multiplication of the output of the BCS by a negative gain. For the class of BCS studied in this paper, if combined with energy-shaping, such technique assures only the asymptotic stability of the closed-loop system, [11, Theorem 5.3]. A different control design technique has been proposed in [10], where it is shown that the closed-loop system resulting from the power-conserving interconnection of a linear, port-Hamiltonian BCS and a linear control system is exponentially stable if such regulator is exponentially stable and strictly input passive. This result has been generalised to the general case in which the BCS is dissipative in [12].

The contribution of this paper is to show how to extend the energy-shaping and damping injection control synthesis to guarantee the exponential stability of the closed-loop system. For that purpose, we start from the feedback loop proposed in [11] and capable of modifying the shape of the Hamiltonian function as much as possible. Then, a second loop that assures exponential convergence towards the equilibrium is computed. This stabilising action can be seen as an extension of the damping injection control law. As a matter of fact, if compared to it, the novel stabilising law is characterised by the presence of two additional terms. The first one depends on the integral of the output, while the second one is related to the total dissipated energy in the BCS. To prove this result, it is shown that the complete control action, namely the energy-shaping law plus the "extended" damping injection contribution, can be generated by a linear, finite dimensional, control system in port-Hamiltonian form that meets the requirements for the exponential stability of the closed-loop system stated in [12].

This paper is organised as follows. Section II introduces the BCS in port-Hamiltonian form, while in Section III two results dealing with the control design for this class of systems are reported: in Section III-A the control by energy-shaping is illustrated, while in Section III-B, a characterisation of linear control systems that assure exponential stability is presented. Such results are the starting point to obtain the novel formulation of the damping injection loop capable to assure exponential stability that is presented in Section IV. Conclusions and final remarks are in Section V.

II. BCS in port-Hamiltonian form

We refer to the class of linear port-Hamiltonian systems on real Hilbert spaces described by the PDE, [8], [9]:

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z) \quad (1)$$

with $x \in X = L^2(a,b;\mathbb{R}^n)$, and $\mathcal{L} \in C^2(a,b;\mathbb{R}^{n\times n})$ a matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z) > 0$ for all

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$z \in [a, b]$. Since $\mathcal{L}$ is a coercive operator, X is then endowed with the inner product $\langle x_1 \mid x_2 \rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L} x_2 \rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1 \rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot \rangle$ denotes the natural L^2-inner product. X is also called the space of energy variables, and $(\mathcal{L}x)(t, z) = \mathcal{L}(z)x(t, z)$ denotes the co-energy variables. Moreover, P_1, P_0 and G_0 are $n \times n$ real matrices, with $P_1 = P_1^{\mathrm{T}}$ and invertible, $P_0 = -P_0^{\mathrm{T}}$, and $G_0 = G_0^{\mathrm{T}} \geq 0$. Finally, $0_{n\times m}$ denotes the zero $n \times m$ real matrix; if $n = m$, we compactly write 0_n. The same notation is adopted for the identity matrix I_n, and used when the dimension of such matrices is not immediate from the context.

For (1), we define the boundary variables f_∂, $e_\partial \in \mathbb{R}^n$ as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I_n & I_n \end{pmatrix}}_{=:R} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}. \tag{2}$$

Then, the characterisation of the (boundary) inputs and outputs for (1) in terms of f_∂ and e_∂ to have a BCS on X in the sense of the semigroup theory [2, Definition 3.3.2] has been addressed in the next proposition, a particular case of the framework introduced in [8].

Proposition 2.1: Denote by W a full rank $n \times 2n$ matrix, and define the input $u(t)$ as

$$u(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{3}$$

Given $\Sigma = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}$, if W satisfies $W \Sigma W^{\mathrm{T}} = 0_n$, then (1) with input (3) so that $u \in C^2(0, \infty; \mathbb{R}^n)$ and initial condition $x(0) \in C^2(a, b; \mathbb{R}^n)$ is a BCS on X in the sense of the semigroup theory, [2, Definition 3.3.2]. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, $\tilde{W} \Sigma \tilde{W}^{\mathrm{T}} = 0_n$, and $W \Sigma \tilde{W}^{\mathrm{T}} = I_n$, and define the output as

$$y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{4}$$

Then, we have that $\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2_{\mathcal{L}} \leq y^{\mathrm{T}}(t) u(t)$.

Proof: See [8, Theorems 4.4 and 5.3], but also [12, Theorem 2], where the more general case in which $\mathcal{L} : [a, b] \to \mathbb{R}^{n\times n}$ is bounded and Lipschitz continuous, and the initial condition is $(\mathcal{L}x)(0) \in H^1(a, b; \mathbb{R}^n)$[1] is discussed. ∎

Example 2.1: Let us consider the normalised wave equation with possible internal dissipation, [14]:

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1(t, z) \\ x_2(t, z) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & -g \end{pmatrix} \begin{pmatrix} x_1(t, z) \\ x_2(t, z) \end{pmatrix} \tag{5}$$

in which $z \in [0, \ell]$ is the spatial coordinate, and $x = (x_1, x_2) \in L^2(0, \ell; \mathbb{R}^2)$ the state variable. We assume that $g \geq 0$ and, for simplicity, that $\mathcal{L} = I$. Then, the Hamiltonian is $\mathcal{H}(x_1, x_2) = \frac{1}{2} \int_0^\ell (x_1^2 + x_2^2)\,\mathrm{d}z$. However, the same results hold when $\mathcal{L}$ is not unitary and depends on the spatial coordinate. Input and output are selected in accordance with Proposition 2.1 as

$$u = \begin{pmatrix} u_0 \\ u_\ell \end{pmatrix} = \begin{pmatrix} x_2(0) \\ x_1(\ell) \end{pmatrix} \quad y = \begin{pmatrix} y_0 \\ y_\ell \end{pmatrix} = \begin{pmatrix} x_1(0) \\ -x_2(\ell) \end{pmatrix}.$$

[1]Here, $H^1(a, b; \mathbb{R}^n)$ is the Sobolev space of order one.

III. Control of BCS in port-Hamiltonian form

A. Energy-shaping and damping injection

The aim of this section is to illustrate how to design a state-feedback control law in the form

$$u(t) = \beta(x(t, \cdot)) + u'(t) \tag{6}$$

that is able to map (1) into the target system

$$\begin{aligned} \frac{\partial x}{\partial t}(t, z) &= P_1 \frac{\partial}{\partial z} \frac{\delta \mathcal{H}_d}{\delta x}(x(t, z)) + (P_0 - G_0) \frac{\delta \mathcal{H}_d}{\delta x}(x(t, z)) \\ u'(t) &= W R \begin{pmatrix} \frac{\delta \mathcal{H}_d}{\delta x}(x(t, b)) \\ \frac{\delta \mathcal{H}_d}{\delta x}(x(t, a)) \end{pmatrix} \end{aligned} \tag{7}$$

in which $\mathcal{H}_d(x) = \frac{1}{2} \|x\|^2_{\mathcal{L}} + \mathcal{H}_a(x)$ is the "desired" Hamiltonian, being $\mathcal{H}_a(x)$ a functional to be determined later on. Here, $\frac{\delta \mathcal{H}}{\delta x}$ denotes the variational derivative of the functional $\mathcal{H}(x)$, see [15, Definition 4.1].

Proposition 3.1: Let us consider the BCS of Proposition 2.1, and introduce the matrix $\Psi(z) = \big(\psi_1(z), \ldots, \psi_n(z)\big)$, in which the functions $\psi_i \in C^\infty(a, b; \mathbb{R}^n)$, $i = 1, \ldots, n$, are independent solutions of

$$P_1 \frac{\mathrm{d}\psi_i}{\mathrm{d}z}(z) + (P_0 - G_0)\psi_i(z) = 0_{n\times 1}. \tag{8}$$

The feedback law (6) maps (1) into the target system (7) in which $\mathcal{H}_d(x) = \frac{1}{2} \|x\|^2_{\mathcal{L}} + \mathcal{H}_a(x)$, if $\mathcal{H}_a(x)$ is in the form $\mathcal{H}_a(x) = H_a(\xi(x))$ being $H_a(\xi)$ a real-valued function with

$$\xi(x(t, \cdot)) = \int_a^b \Psi^{\mathrm{T}}(z) x(t, z)\,\mathrm{d}z, \tag{9}$$

and if

$$\beta(x) = -W R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} \frac{\partial H_a}{\partial \xi}(\xi(x)). \tag{10}$$

In (6), u' is an auxiliary input, to be defined later.

Proof: This result is an equivalent reformulation of [11, Proposition 4.1 and Lemma 4.2]. ∎

Assumption 3.1: The following matrix is invertible:

$$G_\xi^{\mathrm{T}} = W R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}. \tag{11}$$

The functions $\psi_i(z)$ are instrumental for shaping the closed-loop energy function $\mathcal{H}_d(x)$ to have a minimum at the equilibrium configuration $x_\star(z) \in X$, solution of

$$P_1 \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}x_\star)(z) + (P_0 - G_0)(\mathcal{L}x_\star)(z) = 0_{n\times 1}.$$

More details on this point can be found in [11, Lemma 4.2]. Here, without loss of generality, it is assumed that $x_\star(z) = 0$. Moreover, to have again a linear system in closed-loop, $H_a(\xi)$ is selected to be quadratic:

$$H_a(\xi) = \frac{1}{2} \xi^{\mathrm{T}} Q_\xi \xi, \quad Q_\xi = Q_\xi^{\mathrm{T}} > 0 \tag{12}$$

so that, from (10), the energy-shaping control action becomes

$$\beta(x(t, \cdot)) = -W R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} Q_\xi \xi(x(t, \cdot)). \tag{13}$$

Note that, by acting on Q_ξ, different responses can be obtained: Q_ξ can be interpreted as the gain in a proportional regulator. Furthermore, from (9), we see that (13) is a state-feedback action since $\xi(\cdot)$ depends explicitly on $x(t,z)$.

Under the conditions of Proposition 3.1, when $u' = 0$, energy is not increasing along the trajectories of (7) since $\dot{\mathcal{H}}_d \leq 0$. If $\mathcal{H}_a$ is selected as $\mathcal{H}_a(x) = H_a(\xi(x))$, with $H_a(\xi)$ as in (12), this implies that with (6) only simple Lyapunov stability is guaranteed. On the other hand, convergence to the equilibrium can be obtained by damping injection, provided that a dual output to u' is defined. With (4)-(7) in mind, the "natural" choice is

$$\begin{aligned} y'(t) &= \tilde{W}R\begin{pmatrix} \frac{\delta\mathcal{H}_d}{\delta x}(x(t,b)) \\ \frac{\delta\mathcal{H}_d}{\delta x}(x(t,a)) \end{pmatrix} \\ &= y(t) + \tilde{W}R\begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} Q_\xi \xi(x(t,\cdot)). \end{aligned} \tag{14}$$

It turns out that $\dot{\mathcal{H}}_d \leq {y'}^{\mathrm{T}} u'$ and, to force the energy to decrease, we impose that

$$u'(t) = -K_D y'(t), \qquad K_D = K_D^{\mathrm{T}} \geq 0. \tag{15}$$

In [11, Theorem 5.3], it has been proved that the closed-loop system is asymptotically stable (or strongly stable) [3, Definition 3.1] if $K_D > 0$. This latter contribution is similar to a derivative action in a PD regulator.

Example 3.1: Let us consider the system introduced in Example 2.1. The idea is to design a feedback law $u = \beta(x_1,x_2)+u'$ based on energy-shaping plus damping injection that asymptotically stabilises the equilibrium $(x_1^\star(z), x_2^\star(z))$. Without loss of generality, it is assumed that $x_1^\star(z) = x_2^\star(z) = 0$. We start with the lossless case, i.e. with $g = 0$ in (5). From Proposition 3.1, the functions $\mathcal{H}_a(x_1,x_2)$ that can be employed in the energy-shaping procedure are in the form

$$\mathcal{H}_a(x_1,x_2) = H_a(\xi_1(x_1,x_2), \xi_2(x_1,x_2)) \tag{16}$$

with

$$\xi(x_1,x_2) = \begin{pmatrix} \xi_1(x_1) \\ \xi_2(x_2) \end{pmatrix} = \int_0^\ell \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix} \mathrm{d}z$$

as in (9), and where $H_a(\xi_1,\xi_2)$ can be freely chosen. If

$$H_a(\xi_1,\xi_2) = \frac{1}{2}Q_{\xi_1}\xi_1^2 + \frac{1}{2}Q_{\xi_2}\xi_2^2, \quad Q_{\xi_1}, Q_{\xi_2} > 0 \tag{17}$$

then $\mathcal{H}_d(x_1,x_2) = \mathcal{H}(x_1,x_2) + H_a(\xi_1(x_1), \xi_2(x_2))$ has a minimum in $(0,0)$. From (13) and (14), the energy-shaping and the damping injection contributions are

$$\begin{aligned} \beta(x_1,x_2) &= -\begin{pmatrix} Q_{\xi_2}\xi_2(x_2) \\ Q_{\xi_1}\xi_1(x_1) \end{pmatrix} \\ u'(x_1,x_2) &= -\begin{pmatrix} K_{D_1}(y_0 + Q_{\xi_1}\xi_1(x_1)) \\ K_{D_2}(y_\ell - Q_{\xi_2}\xi_2(x_2)) \end{pmatrix} \end{aligned} \tag{18}$$

respectively, where K_{D_1} and K_{D_2} are two positive gains. On the other hand, when $g > 0$ the energy function $\mathcal{H}_a(x_1,x_2)$ takes again the form (16), with

$$\xi(x_1,x_2) = \begin{pmatrix} \xi_1(x_1) \\ \xi_2(x_1,x_2) \end{pmatrix} = \int_0^\ell \begin{pmatrix} 1 & 0 \\ g(\ell - z) & 1 \end{pmatrix} \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix} \mathrm{d}z$$

and $H_a(\xi)$ that can be selected as in (17). The energy-shaping control $\beta(x_1,x_2)$ and the damping injection term $u'(x_1,x_2)$ are the same as in (18). For all $g \geq 0$, the control action $u(x_1,x_2) = \beta(x_1,x_2) + u'(x_1,x_2)$ leads to an asymptotically stable closed-loop system.

To conclude, as illustrated in the next section, it is possible to define linear control systems to be interconnected at the input / output port (u,y) of (1) that assures exponential stability. This property is exploited in Section IV to compute a different expression for u' so that the closed-loop system is exponentially stable when the control action is given as in (6), with $\beta(\cdot)$ obtained thanks to the energy-shaping procedure of Proposition 3.1 in which $\mathcal{H}_a(x) = H_a(\xi(x))$ and $H_a(\xi)$ is selected as in (12), i.e. equal to (13).

B. Exponential stabilisation of BCS in port-Hamiltonian form

Let us consider the linear control system in the port-Hamiltonian form, [16, Definition 6.1.2]:

$$\begin{cases} \dot{x}_C(t) = A_C x_C(t) + B_C u_C(t) \\ y_C(t) = C_C x_C(t) + D_C u_C(t) \end{cases} \tag{19}$$

with

$$\begin{aligned} A_C &= (J_C - R_C)\, Q_C & B_C &= G_C - P_C \\ C_C &= (G_C + P_C)^{\mathrm{T}} Q_C & D_C &= M_C + S_C \end{aligned} \tag{20}$$

that is interconnected to the BCS of Proposition 2.1 as

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \begin{pmatrix} u_C(t) \\ y_C(t) \end{pmatrix} + \begin{pmatrix} u'(t) \\ 0_n \end{pmatrix}. \tag{21}$$

In (20), we have that $x_C \in \mathbb{R}^{n_C}$ and $u_C, y_C \in \mathbb{R}^n$, while $J_C = -J_C^{\mathrm{T}}$, $M_C = -M_C^{\mathrm{T}}$, $R_C = R_C^{\mathrm{T}}$, $S_C = S_C^{\mathrm{T}}$, and

$$\begin{pmatrix} R_C & P_C \\ P_C^{\mathrm{T}} & S_C \end{pmatrix} \geq 0, \quad \text{with } Q_C = Q_C^{\mathrm{T}} > 0. \tag{22}$$

As illustrated in the next proposition, under certain conditions this controller guarantees the exponential stability [3, Definition 3.1] of the closed-loop system.

Proposition 3.2: Let us consider the power-conserving interconnection (21), with $u' = 0$, of the BCS of Proposition 2.1 and of the control system (19). Then, if the pair (A_C, B_C) is controllable, $R_C \geq 0$ is such that $A_C = (J_C - R_C)Q_C$ is Hurwitz, and if there exists $\delta_R, \delta_S > 0$ such that

$$\begin{pmatrix} (1-\delta_R)R_C & P_C \\ P_C^{\mathrm{T}} & S_C - \delta_S I_n \end{pmatrix} \geq 0_{2n} \tag{23}$$

then the closed-loop system is exponentially stable.

Proof: Since (19) is passive with storage function $\frac{1}{2}x_C^{\mathrm{T}} Q_C x_C$, in [12, Example 5.1] it is shown that the closed-loop system is exponentially stable if the pair (A_C, B_C) is controllable, $A_C = (J_C - R_C)Q_C$ is Hurwitz, and if there exist $\delta_x, \delta_u > 0$ such that

$$\begin{pmatrix} (1-2\delta_x)(Q_C A_C + A_C^{\mathrm{T}} Q_C) & Q_C B_C - C_C^{\mathrm{T}} \\ B_C^{\mathrm{T}} Q_C - C_C & 2\delta_u I_n - D_C - D_C^{\mathrm{T}} \end{pmatrix} \leq 0_{2n}. \tag{24}$$

Then, (23) easily follows from (20) and (24) by defining $\delta_R = 2\delta_x$ and $\delta_S = \delta_u$. ■

When (1) is lossless, i.e. $G_0 = 0$, the energy-shaping plus damping injection control law, i.e. (6) with u' given as in (15), can be generated by a linear dynamical system in the form (19). This is not surprising since in the lossless case the Hamiltonian of (1) can be shaped with a finite amount

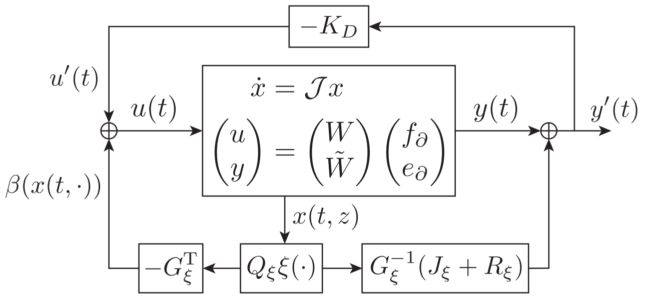

Fig. 1. The two loops that implement the energy-shaping plus damping injection control strategy. Here, $\mathcal{J}x = P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)(\mathcal{L}x)$ is the differential operator in (1), and $\beta(x) = -G_\xi^{\mathrm{T}} Q_\xi \xi(x)$ is the energy-shaping term; y' is computed as in (14), since (54) has been taken into account. Note the similarities with the scheme describing the control of port-Hamiltonian systems via canonical transformations, [18].

of energy, i.e. via energy-balancing. This fact is exploited in the so-called energy-Casimir method that shows that all the energy-balancing control laws can be generated by a properly initialised port-Hamiltonian system with a lower-bounded Hamiltonian, [13], [17]. However, the requirements for having exponential stability in closed-loop stated in Proposition 3.2 and, in particular, condition (23), are not met. This is coherent with [11, Theorem 5.3], where it has been shown that the control action (6) based on energy-shaping and damping injection (15) is only able to asymptotically stabilise the system, provided that $K_D > 0$. Finally, note that the existence of such linear control system, is not guaranteed when $G_0 \neq 0$ in (1) because of the so-called "dissipation obstacle", [13].

IV. Exponential stability for energy-shaping control laws

The energy-shaping plus damping injection control law (6) consists of two main loops. The first one is a feedback action $\beta(\cdot)$ that is responsible for shaping the Hamiltonian function and can be regarded as the proportional action in a PD-like controller. The second one is designed to dissipate energy and let the trajectories to converge to the equilibrium as the derivative action in a PD regulator. In (15), u' implements the standard damping injection strategy, and this assures that the closed-loop system is asymptotically stable if $K_D = K_D^{\mathrm{T}} > 0$. The complete control scheme is represented in Fig. 1. The idea is to compute a different expression for u' that assures exponential convergence. To achieve this, we determine a dynamical system that meets the requirements of Proposition 3.2 and that is also able to generate the control action (6).

Proposition 4.1: Let us consider the BCS of Proposition 2.1, and the control action (6) where $u' \in C^2(0,\infty;\mathbb{R}^n)$ is arbitrary, and $\beta(x(t,\cdot))$ is obtained as in Proposition 3.1, with $\mathcal{H}_a(x) = H_a(\xi(x))$ and $H_a(\xi)$ defined in (12). Then, (6) can be equivalently generated by

$$\begin{cases} \dot{x}_\xi(t) = (J_\xi - R_\xi)\left[Q_\xi x_\xi(t) - G_\xi^{-\mathrm{T}} u'(t)\right] + G_\xi u_\xi(t) \\ \qquad\qquad - 2\displaystyle\int_a^b \Psi^{\mathrm{T}}(z) G_0 \mathcal{L}(z) x(t,z)\,\mathrm{d}z \\ y_\xi(t) = G_\xi^{\mathrm{T}} Q_\xi x_\xi(t) - u'(t) \end{cases} \tag{25}$$

with initial condition

$$x_\xi(0) = \int_a^b \Psi^{\mathrm{T}}(z) x(0,z)\,\mathrm{d}z, \tag{26}$$

being $x(0,z) \in L^2(a,b;\mathbb{R}^n)$ such that $(\mathcal{L}x)(0,z) \in H^1(a,b;\mathbb{R}^n)$, and where x_ξ, u_ξ, $y_\xi \in \mathbb{R}^n$, $J_\xi = -J_\xi^{\mathrm{T}}$ and $R_\xi = R_\xi^{\mathrm{T}} \geq 0$ are $n \times n$ matrices, and G_ξ has been defined in (11). System (25) is interconnected to (1) in feedback, i.e.:

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \begin{pmatrix} u_\xi(t) \\ y_\xi(t) \end{pmatrix}. \tag{27}$$

Proof: The starting point is Proposition A.1 reported in the Appendix, and in particular relation (47), in which J_ξ and R_ξ are defined in (48) and (49), respectively. From (11) and (13), we have that (6) is given by

$$u = -G_\xi^{\mathrm{T}} Q_\xi \xi(x) + u', \tag{28}$$

which implies that (47) can be re-written as

$$\begin{aligned} \dot{\xi} = (J_\xi - R_\xi) Q_\xi \xi(x) + G_\xi y - (J_\xi - R_\xi) G_\xi^{-\mathrm{T}} u' \\ - 2\int_a^b \Psi^{\mathrm{T}}(z) G_0 (\mathcal{L}x)(z)\,\mathrm{d}z. \end{aligned} \tag{29}$$

The result then follows once (28) and (29) are compared with (25) and the interconnection constraint (27) is taken into account. In fact, for all $u'(t)$ we have that $x_\xi(t) = \xi(x(t,\cdot))$ for all $t \geq 0$ if and only if the initial condition for (25) is selected as in (26). ■

The case in which (1) is lossless, i.e. when $G_0 = 0$ is treated in the next corollary.

Corollary 4.1: Let us consider the BCS of Proposition 2.1 with $G_0 = 0$, and the control action (6) where $u' \in C^2(0,\infty;\mathbb{R}^n)$ is arbitrary, and $\beta(x(t,\cdot))$ is obtained as in Proposition 3.1, with $\mathcal{H}_a(x) = H_a(\xi(x))$ and $H_a(\xi)$ defined as in (12). Then, (6) can be equivalently generated by

$$\begin{cases} \dot{x}_\xi(t) = J_\xi Q_\xi x_\xi(t) + G_\xi u_\xi(t) - J_\xi G_\xi^{-\mathrm{T}} u'(t) \\ y_\xi(t) = G_\xi^{\mathrm{T}} Q_\xi x_\xi(t) - u'(t) \end{cases} \tag{30}$$

with initial condition (26), being $x(0,z) \in L^2(a,b;\mathbb{R}^n)$ such that $(\mathcal{L}x)(0,z) \in H^1(a,b;\mathbb{R}^n)$, and where x_ξ, u_ξ, $y_\xi \in \mathbb{R}^n$, $J_\xi = -J_\xi^{\mathrm{T}}$ is a $n \times n$ matrix, and G_ξ has been defined in (11). System (30) is interconnected to (1) in feedback as in (27)

Proof: From Proposition A.1 reported in the Appendix, we have that since $G_0 = 0$, also $R_\xi = 0$. Then, the result immediately follows from (25). ■

In the general case, i.e. when $G_0 \neq 0$ in (1), the system that corresponds to the case in which u' is designed to introduce damping as in (15) is obtained in the next proposition.

Proposition 4.2: Under the conditions of Proposition 4.1, if $u'(t) = -K_D y'(t)$, with $K_D = K_D^{\mathrm{T}} \geq 0$ as in (15), then the control action can be generated by

$$\begin{cases} \dot{x}_\xi(t) = (J_\xi - R_\xi - \tilde{R}_\xi) Q_\xi x_\xi(t) + (G_\xi - P_\xi) u_\xi(t) \\ \qquad\qquad - 2\displaystyle\int_a^b \Psi^{\mathrm{T}}(z) G_0 \mathcal{L}(z) x(t,z)\,\mathrm{d}z \\ y_\xi(t) = (G_\xi + P_\xi)^{\mathrm{T}} Q_\xi x_\xi(t) + K_D u_\xi(t) \end{cases} \tag{31}$$

with initial condition (26), and where

$$
\begin{aligned}
P_\xi &= \left[G_\xi^{-1}\left(J_\xi + R_\xi\right)\right]^{\mathrm{T}} K_D, \\
\tilde{R}_\xi = \tilde{R}_\xi^{\mathrm{T}} &= P_\xi \left[G_\xi^{-1}\left(J_\xi + R_\xi\right)\right] \geq 0.
\end{aligned} \tag{32}
$$

Such system is interconnected to (1) as in (27).

Proof: From (14) and (54), we have that

$$
y' = y + G_\xi^{-1}(J_\xi + R_\xi)Q_\xi \xi(x),
$$

which combined with (28) leads to

$$
-u = G_\xi^{\mathrm{T}} Q_\xi \xi(x) + K_D y + P_\xi^{\mathrm{T}} Q_\xi \xi(x),
$$

because from (15) $u' = -K_D y'$, and where (32) is taken into account. From (29) we have that

$$
\begin{aligned}
\dot{\xi} = (J_\xi - R_\xi)Q_\xi \xi(x) + (G_\xi - P_\xi)\, y - \tilde{R}_\xi Q_\xi \xi(x) \\
- 2\int_a^b \Psi^{\mathrm{T}}(z) G_0 \mathcal{L}(z) x(\cdot, z)\, \mathrm{d}z,
\end{aligned}
$$

and the result follows as in the proof of Proposition 4.1. ■

Corollary 4.2: Under the conditions of Corollary 4.1, let us consider the BCS of Proposition 2.1 with $G_0 = 0$. If $u'(t) = -K_D y'(t)$, with $K_D = K_D^{\mathrm{T}} \geq 0$, the control action can be generated by the following system with initial condition (26)

$$
\begin{cases}
\dot{x}_\xi(t) = (J_\xi - \tilde{R}_\xi)Q_\xi x_\xi(t) + (G_\xi - P_\xi)u_\xi(t) \\
y_\xi(t) = (G_\xi + P_\xi)^{\mathrm{T}} Q_\xi x_\xi(t) + K_D u_\xi(t)
\end{cases} \tag{33}
$$

that is interconnected to (1) as in (27), and where

$$
P_\xi = -J_\xi G_\xi^{-\mathrm{T}} K_D \quad \tilde{R}_\xi = \tilde{R}_\xi^{\mathrm{T}} = P_\xi G_\xi^{-1} J_\xi \geq 0. \tag{34}
$$

Proof: This result is immediate from Proposition 4.2 since $G_0 = 0$ implies that $R_\xi = 0$. ■

Remark 4.1: As discussed in Proposition 2.1, (1) is a BCS in the sense of [2, Definition 3.3.2] if $u \in C^2(0,\infty;\mathbb{R}^n)$. Since $u(t)$ is given by (6), with $\beta(x(t,\cdot))$ equal to (13) and $u'(t)$ to (15), we see that $u(t)$ results from the sum of a state-feedback term that depends on $\xi(x)$, defined in (9), and an output-feedback contribution that depends on $y(t)$. Since we have assumed that $\mathcal{L} \in C^2(a,b;\mathbb{R}^{n\times n})$ and that in Proposition 2.1 $x(0) \in C^2(a,b;\mathbb{R}^n)$, we know that $u(t)$ evolves in the prescribed space. Less restrictive conditions are also possible, but this problem is not investigated here.

With Corollary 4.2 in mind, it is immediate to check that (33) is in the port-Hamiltonian form (19)-(20). In fact, if $\bar{P} = G_\xi^{-1} J_\xi$, then from (34) we have that $\tilde{R}_\xi = \bar{P}^{\mathrm{T}} K_D \bar{P}$, $P_\xi = \bar{P}^{\mathrm{T}} K_D$, and condition (22) holds true since

$$
\begin{aligned}
\begin{pmatrix} \tilde{R}_\xi & P_\xi \\ P_\xi^{\mathrm{T}} & K_D \end{pmatrix} = \begin{pmatrix} \bar{P} & 0_n \\ 0_n & I_n \end{pmatrix}^{\mathrm{T}} \\
\times \begin{pmatrix} K_D & K_D \\ K_D & K_D \end{pmatrix} \begin{pmatrix} \bar{P} & 0_n \\ 0_n & I_n \end{pmatrix} \geq 0_{2n}
\end{aligned} \tag{35}
$$

for all $K_D \geq 0$. However, the requirements for having an exponentially stable closed-loop system stated in Proposition 3.2 and, in particular, condition (23), are not met. This is coherent with [11, Theorem 5.3], where it has been shown that the control action (6) based on energy-shaping plus damping injection is only able to asymptotically stabilise the system, provided that $K_D > 0$.

By slightly modifying (31), a linear control system in port-Hamiltonian form that meets the hypotheses of Proposition 3.2, thus assuring exponential stability in closed-loop, is defined. This property is instrumental to compute a different expression for u' in (6) that guarantees exponential convergence to the equilibrium, under the condition that $\beta(\cdot)$ is still obtained thanks to the energy-shaping procedure of Proposition 3.1. This result is presented in the next proposition.

Proposition 4.3: Let us consider the BCS of Proposition 2.1 and the stabilising law (6) with $\beta(x(t,\cdot))$ obtained thanks to the energy-shaping methodology discussed in Proposition 3.1. Given $Q_\xi = Q_\xi^{\mathrm{T}} > 0$ and $K_D = K_D^{\mathrm{T}} \geq 0$ such that the pair

$$
\Big(\big[J_\xi - (1+\kappa)(R_\xi + \tilde{R}_\xi)\big]Q_\xi, G_\xi - P_\xi\Big) \tag{36}
$$

is controllable, the matrix $\big[J_\xi - (1+\kappa)(R_\xi + \tilde{R}_\xi)\big]Q_\xi$ is Hurwitz for some $\kappa > 0$, where $\bar{P} = G_\xi^{-1}(J_\xi + R_\xi)$, with $J_\xi = -J_\xi^{\mathrm{T}}$ and $R_\xi = R_\xi^{\mathrm{T}} \geq 0$ a couple of $n \times n$ matrices, and G_ξ, P_ξ, and $\tilde{R}_\xi$ have been defined in (11), and (32), and for any $K_I = K_I^{\mathrm{T}} > 0$, if

$$
\begin{aligned}
u'(t) = -K_D y'(t) - K_I y(t) - K_G \int_0^t e^{-K_R(t-\tau)} \\
\times \bigg\{ \int_a^b \big[2\Psi^{\mathrm{T}}(z) G_0 \mathcal{L}(z) - K_R \Psi^{\mathrm{T}}(z)\big] x(\tau, z)\, \mathrm{d}z \\
+ \bar{P} K_I y(\tau) \bigg\}\, \mathrm{d}\tau
\end{aligned} \tag{37}
$$

with $K_G = (G_\xi^{\mathrm{T}} + K_D \bar{P})Q_\xi$ and $K_R = \kappa(R_\xi + \bar{P}^{\mathrm{T}} K_D \bar{P})Q_\xi$, then the closed-loop system is exponentially stable.

Proof: With an eye on (31), let us consider the port-Hamiltonian control system

$$
\begin{cases}
\dot{\bar{x}}_\xi = \big[J_\xi - (1+\kappa)(R_\xi + \tilde{R}_\xi)\big]Q_\xi \bar{x}_\xi + (G_\xi - P_\xi)u_\xi \\
y_\xi = (G_\xi + P_\xi)^{\mathrm{T}} Q_\xi \bar{x}_\xi + (K_D + K_I)u_\xi
\end{cases} \tag{38}
$$

with $\bar{x}_\xi \in \mathbb{R}^n$, and P_ξ defined in (32). Since $\bar{P} = G_\xi^{-1}(J_\xi + R_\xi)$, we can compactly write that $P_\xi = \bar{P}^{\mathrm{T}} K_D$ and that $\tilde{R}_\xi = \bar{P}^{\mathrm{T}} K_D \bar{P}$. Now, we can check that (23) holds true under the condition that $\kappa > 0$ and $K_I = K_I^{\mathrm{T}} > 0$. As a consequence, by following Proposition 3.2, we can say that the closed-loop system resulting from the power-conserving interconnection (27) of the BCS of Proposition 2.1 and of the linear system (38) is exponentially stable if K_D and κ are selected in such a way that the pair (36) is controllable, and the matrix $\big[J_\xi - (1+\kappa)(R_\xi + \tilde{R}_\xi)\big]Q_\xi$ is Hurwitz. With Proposition 4.1 in mind, the idea is to determine u' such that (25) is "equivalent" to (38), i.e. both generate the same control action. To achieve this, at first note that since from (27) $u_\xi = y$ and $y_\xi = -u$, the output equation in (38) gives that

$$
K_D\left[\bar{P} Q_\xi \bar{x}_\xi + y\right] = -u - G_\xi^{\mathrm{T}} Q_\xi \bar{x}_\xi - K_I y,
$$

since $P_\xi = \bar{P}^{\mathrm{T}} K_D$. Moreover, the $\bar{x}_\xi$ dynamic is

$$
\begin{aligned}
\dot{\bar{x}}_\xi = \Big[J_\xi - (1+\kappa)R_\xi - \kappa \tilde{R}_\xi\Big] Q_\xi \bar{x}_\xi + G_\xi y \\
- \bar{P}^{\mathrm{T}} K_D \left[\bar{P} Q_\xi \bar{x}_\xi + y\right]
\end{aligned}
$$

which combined with the previous relation leads to

$$\begin{aligned}\dot{\bar{x}}_\xi &= \left[J_\xi - (1+\kappa)R_\xi - \kappa\tilde{R}_\xi\right] Q_\xi \bar{x}_\xi + G_\xi y \\ &\quad + \bar{P}^{\mathrm{T}}\left(u + G_\xi^{\mathrm{T}} Q_\xi \bar{x}_\xi + K_I y\right) \\ &= -K_R \bar{x}_\xi + G_\xi y + \bar{P}^{\mathrm{T}} u + \bar{P}^{\mathrm{T}} K_I y\end{aligned} \tag{39}$$

since $\bar{P}^{\mathrm{T}} G_\xi^{\mathrm{T}} = -J_\xi + R_\xi$. For all $u' \in C^2(0,\infty;\mathbb{R}^n)$, we can also verify that the dynamic of x_ξ presented in (25) can be written in terms of the input and output u and y of (1) as

$$\dot{x}_\xi = G_\xi y + \bar{P}^{\mathrm{T}} u - 2\int_a^b \Psi^{\mathrm{T}}(z) G_0(\mathcal{L}x)(z)\,\mathrm{d}z$$

which, if combined with (39), leads to

$$\dot{\bar{x}}_\xi = -K_R\bar{x}_\xi + \dot{x}_\xi + 2\int_a^b \Psi^{\mathrm{T}}(z) G_0(\mathcal{L}x)(z)\,\mathrm{d}z + \bar{P}^{\mathrm{T}} K_I y.$$

By integrating this differential equation in time, and under the condition that $\bar{x}_\xi(0) = x_\xi(0) = \int_a^b \Psi^{\mathrm{T}} x(0,z)\,\mathrm{d}z$, we get that

$$\begin{aligned}\bar{x}_\xi(t) &= x_\xi(t) - \int_0^t e^{-K_R(t-\tau)} K_R x_\xi(\tau)\,\mathrm{d}\tau \\ &\quad + \int_0^t e^{-K_R(t-\tau)} \bar{P}^{\mathrm{T}} K_I y(\tau)\,\mathrm{d}\tau \\ &\quad + 2\int_0^t e^{-K_R(t-\tau)} \int_a^b \Psi^{\mathrm{T}}(z) G_0(\mathcal{L}x)(\tau,z)\,\mathrm{d}z\,\mathrm{d}\tau.\end{aligned}$$

After taking into account the output equation of (38) and the expression of $\bar{x}_\xi$, since $y_\xi = -u$ and $u_\xi = y$, the control action that such system generates is

$$\begin{aligned}-u(t) &= K_G x_\xi(t) + (K_D + K_I) y(t) \\ &\quad + K_G \int_0^t e^{-K_R(t-\tau)} \bar{P}^{\mathrm{T}} K_I y(\tau)\,\mathrm{d}\tau \\ &\quad + K_G \int_0^t e^{-K_R(t-\tau)} \cdot \Big[-K_R x_\xi(\tau) \\ &\qquad + 2\int_a^b \Psi^{\mathrm{T}}(z) G_0(\mathcal{L}x)(\tau,z)\,\mathrm{d}z \Big]\mathrm{d}\tau.\end{aligned} \tag{40}$$

As discussed in Proposition 4.1, the condition on $x_\xi(0)$ assures that $x_\xi(t) = \xi(x(t,\cdot))$ for $t \geq 0$. Consequently, since $K_G = (G_\xi^{\mathrm{T}} + K_D\bar{P})Q_\xi$, (40) is in fact equal to

$$\begin{aligned}-u(t) &= G_\xi^{\mathrm{T}} Q_\xi \xi(t) + K_D\left[y(t) + \bar{P} Q_\xi \xi(t)\right] + K_I y(t) \\ &\quad + K_G \int_0^t e^{-K_R(t-\tau)} \bar{P}^{\mathrm{T}} K_I y(\tau)\,\mathrm{d}\tau \\ &\quad + K_G \int_0^t e^{-K_R(t-\tau)} \cdot \Big[-K_R \xi(\tau) \\ &\qquad + 2\int_a^b \Psi^{\mathrm{T}}(z) G_0(\mathcal{L}x)(\tau,z)\,\mathrm{d}z \Big]\mathrm{d}\tau\end{aligned}$$

where, to keep the expression compact, $\xi(t) \equiv \xi(x(t,\cdot))$. From (13), the energy-shaping contribution is $\beta(x) = -G_\xi^{\mathrm{T}} Q_\xi \xi(x)$, while from (14) and (54) we have that the dual output to u' is $y' = y + \bar{P} Q_\xi \xi(x)$. Then, from a comparison between the latter expression for u and (6), we get that the exponentially stabilising feedback law u' in (6) is equal to (37), because $\xi(x(t,\cdot))$ is given by (9). ■

When (1) is lossless, i.e. when $G_0 = 0$, the expression for u' to be used in (6) takes a simpler form than (37). The result is reported in the next corollary.

Corollary 4.3: Let us consider the BCS of Proposition 2.1 with $G_0 = 0$, and the control law (6), with $\beta(x(t,\cdot))$ obtained in Proposition 3.1. Given $Q_\xi = Q_\xi^{\mathrm{T}} > 0$ and $K_D = K_D^{\mathrm{T}} \geq 0$ such that the pair

$$\Big(\big[J_\xi - (1+\kappa)\tilde{R}_\xi\big] Q_\xi, G_\xi - P_\xi\Big) \tag{41}$$

is controllable, the matrix $\big[J_\xi - (1+\kappa)\tilde{R}_\xi\big] Q_\xi$ is Hurwitz for some $\kappa > 0$, with $\tilde{R}_\xi$ and P_ξ defined in (34), and for any $K_I = K_I^{\mathrm{T}} > 0$, if

$$\begin{aligned}u'(t) &= -K_D y'(t) - K_I y(t) + K_G \int_0^t e^{-K_R(t-\tau)} \\ &\quad \times \left[K_R \int_a^b \Psi^{\mathrm{T}}(z) x(\tau,z)\,\mathrm{d}z - \bar{P} K_I y(\tau)\right]\mathrm{d}\tau\end{aligned} \tag{42}$$

in which $\bar{P} = G_\xi^{-1} J_\xi$, $K_G = (G_\xi^{\mathrm{T}} + K_D\bar{P})Q_\xi$ and $K_R = \kappa\bar{P}^{\mathrm{T}} K_D \bar{P} Q_\xi$, then the closed-loop system is exponentially stable.

Proof: The result is a trivial consequence of Proposition 4.3, in which we impose that $G_0 = 0$ in (37). Differently, we can start from Corollary 4.2 and system (33) that realises the energy-shaping plus damping injection control action and leads to an asymptotically stable closed-loop system. As pointed out at the end of Section III-B, such system does not meets the requirements of exponential stability stated in Proposition 3.2, but it is easy to check that the following system does:

$$\begin{cases}\dot{\bar{x}}_\xi = \big[J_\xi - (1+\kappa)\tilde{R}_\xi\big] Q_\xi \bar{x}_\xi + (G_\xi - P_\xi) u_\xi \\ y_\xi = (G_\xi + P_\xi)^{\mathrm{T}} Q_\xi \bar{x}_\xi + (K_D + K_I) u_\xi\end{cases} \tag{43}$$

if $\kappa > 0$, $K_I = K_I^{\mathrm{T}} > 0$, the pair (41) is controllable, and $\big[J_\xi - (1+\kappa)\tilde{R}_\xi\big] Q_\xi$ is Hurwitz. Then, the result is proved in the same way as Proposition 4.3. ■

Example 4.1: For the BCS (5) introduced in Example 2.1, a different design for u' is obtained so that, if combined with the energy-shaping control action $\beta(x_1,x_2)$ obtained in Example 3.1, exponential stability is achieved. As before, we start with the lossless case, i.e. by assuming that $g = 0$ in (5). The first step consists in determining the dynamical system (33). From a direct computation we obtain that

$$J_\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad R_\xi = 0 \qquad G_\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{44}$$

Note that G_ξ is invertible, as required in Assumption 3.1. As discussed in Corollary 4.2, (33) is able to generate the energy-shaping plus damping injection control action once properly initialised. In particular, if $x_\xi = (x_{\xi_1}, x_{\xi_2})$ denotes the state variable of (33), we have that $(x_{\xi_1}(0), x_{\xi_2}(0)) = \int_0^\ell (x_1(0,z), x_2(0,z))\,\mathrm{d}z$. Moreover, it is easy to obtain that $\tilde{R}_\xi = \mathrm{diag}(K_{D_1}, K_{D_2})$, and that $P_\xi = \mathrm{diag}(K_{D_1}, -K_{D_2})$. To have exponential stability, it is necessary to modify the dissipation and feedthrough terms of (33) by adding $-\kappa\tilde{R}_\xi Q_\xi x_\xi$ and $K_I u_\xi$ to the state and output equations, respectively, to obtain (43). Here, $\kappa > 0$ and $K_I = \mathrm{diag}(K_{I_1}, K_{I_2}) > 0$.

Moreover, at least one constant among K_{D_1} and K_{D_2} has to be strictly positive so that $\big[J_\xi - (1+\kappa)\tilde{R}_\xi\big]Q_\xi$ is Hurwitz, and it is possible to check that the controllability condition (41) holds true. If all these conditions are met, then the control action u' that leads to an exponentially stable closed-loop system is given by (42), where $\bar{P} = \operatorname{diag}(1,-1)$, $K_R = \kappa\operatorname{diag}(K_{D_1}Q_{\xi_1}, K_{D_2}Q_{\xi_2})$, and

$$K_G = \begin{pmatrix} K_{D_1}Q_{\xi_1} & Q_{\xi_2} \\ Q_{\xi_1} & -K_{D_2}Q_{\xi_2} \end{pmatrix}$$

The result is that

$$u'(x_1,x_2) = -\begin{pmatrix} (K_{D_1}+K_{I_1})y_0 + K_{D_1}Q_{\xi_1}\xi_1(x_1) \\ (K_{D_2}+K_{I_2})y_\ell - K_{D_2}Q_{\xi_2}\xi_2(x_2) \end{pmatrix}$$
$$+K_G\begin{pmatrix} \int_0^t \mathrm{e}^{-\kappa K_{D_1}Q_{\xi_1}(t-\tau)}\left[\kappa K_{D_1}Q_{\xi_1}\xi_1(x_1) - K_{I_1}y_0\right]\mathrm{d}\tau \\ \int_0^t \mathrm{e}^{-\kappa K_{D_2}Q_{\xi_2}(t-\tau)}\left[\kappa K_{D_2}Q_{\xi_2}\xi_2(x_2) + K_{I_2}y_\ell\right]\mathrm{d}\tau \end{pmatrix}.$$

A comparison with u' in (18) shows that exponential stability is obtained with the addition of a "small" integral action. As expected, the energy-shaping control action $\beta(x_1,x_2)$ remains the same as in (18).

When $g>0$, to determine u' so that exponential stability in closed-loop is assured, the same steps of the lossless case have to be followed: it is necessary to compute the terms J_ξ, R_ξ and G_ξ that appear in (47). In this respect, J_ξ and G_ξ are the same as in (44), but now $R_\xi = \operatorname{diag}(0, g\ell)$. Then, given $\kappa>0$ and with the same choices for K_D and K_I, the expression for u' that assures exponential convergence of the trajectories follows from (37) in Proposition 4.3, in which

$$\tilde{R}_\xi = \begin{pmatrix} K_{D_1} & DLK_{D_1} \\ g\ell K_{D_1} & (g\ell)^2K_{D_1}+K_{D_2} \end{pmatrix}$$
$$P_\xi = \underbrace{\begin{pmatrix} 1 & g\ell \\ 0 & -1 \end{pmatrix}}_{\equiv\bar{P}}\begin{pmatrix} K_{D_1} & 0 \\ 0 & K_{D_2} \end{pmatrix} = \begin{pmatrix} K_{D_1} & g\ell K_{D_2} \\ 0 & -K_{D_2} \end{pmatrix}$$

and where $\bar{P}$ appears in Proposition 4.3. Note that also the controllability condition (36) holds true. Note that, as requested in Proposition 4.3, the matrix $[J_\xi - (1+\kappa)(R_\xi + \tilde{R}_\xi)]Q_\xi$ is Hurwitz for all $K_D \geq 0$. Then, it is possible to select $K_D = 0$, and this implies that $P_\xi = 0$. With this choice, the gains that appear in (37) take the simpler expression

$$K_G = \begin{pmatrix} 0 & Q_{\xi_2} \\ Q_{\xi_1} & 0 \end{pmatrix} \qquad K_R = \kappa\begin{pmatrix} 0 & 0 \\ 0 & g\ell Q_{\xi_2} \end{pmatrix}.$$

Such simplification is a consequence of the fact that internal dissipation makes the damping injection term based on the "new" output $y'(t)$ not strictly necessary for exponential stability. In this case, we obtain that $u' = (u'_0, u'_\ell)$, where

$$u'_0(x_1,x_2) = -K_{I_1}y_0 - Q_{\xi_1}\int_0^t \left[K_{I_1}y_0 + g\ell K_{I_2}y_\ell\right]\mathrm{d}\tau$$
$$u'_\ell(x_1,x_2) = -K_{I_2}y_\ell - Q_{\xi_2}\int_0^t \mathrm{e}^{-\kappa g\ell Q_{\xi_2}(t-\tau)}$$
$$\times\left[2\int_0^\ell gx_2\,\mathrm{d}z - \kappa g\ell Q_{\xi_2}\xi_2(x_1,x_2) - K_{I_2}y_\ell\right]\mathrm{d}\tau.$$

Remark 4.2: This design methodology can be applied to all the distributed parameter systems in the form (1) and for which the energy-shaping loop is obtained as in Proposition 3.1. In this respect, the unique requirement is to have actuation at both sides of the spatial domain, as stated in Proposition 2.1. With minor modifications, it is possible to treat the case in which actuation is only at one side of the domain, provided that the other one is interconnected to a passive system: as a matter of fact, the whole system is passive.

V. Conclusions and Future Work

In this paper, a control synthesis methodology for linear port-Hamiltonian BCS has been presented. The state-feedback action consists of two main loops. The first one is responsible for shaping the Hamiltonian function, while the second one for adding dissipative effects and assuring convergence to the equilibrium. In [11], it has been shown that, if such second control action is based on damping injection, only asymptotic stability can be obtained. In this paper, instead, exponential stability is guaranteed by properly modifying this latter loop with the addition of two novel terms, one proportional to the integral of the natural output of the system, while the other one to the total dissipated power in the BCS. Differently from [11], exponential stability after energy-shaping is now obtained for a large class of linear BCS in port-Hamiltonian form. Future work deals with the extension of the design methodology to the case in which the open-loop system is passive but nonlinear, or it is unstable. For this latter case, some preliminary results can be found in [19], where a linear wave equation, i.e. (5) with $g=0$, subject to anti-damping at its free end and with control at the opposite one is studied.

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Appendix

To get to the main contribution of this note, it is necessary to determine how $\xi(x(t,\cdot))$ defined in (9) varies as a function of the state variable $x(t,\cdot)$ of (1), and of the input and output $u(t)$ and $y(t)$ introduced in Proposition 2.1. This result is illustrated in the next proposition. Before, a preliminary lemma which is instrumental for its proof is stated.

Lemma A.1: The matrices W and $\tilde{W}$ introduced in Proposition 2.1 are such that:

$$\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-1} = \Sigma \begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix} \Sigma \tag{45}$$

$$\tilde{W}^{\mathrm{T}} W = \frac{1}{2}\Sigma - \tilde{J}_\xi \tag{46}$$

for some $2n \times 2n$ skew-symmetric matrix $\tilde{J}_\xi = -\tilde{J}_\xi^{\mathrm{T}}$.

Proof: From the hypotheses of Proposition 2.1, we have that $\tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0_n$, and $W\Sigma\tilde{W}^{\mathrm{T}} = I_n$, which means that

$$\begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \Sigma \begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix} = \Sigma.$$

Then, (45) is immediate since $\Sigma\Sigma = I_{2n}$. As far as (46) is concerned, see [11, proof of Proposition 3.2]. ■

Proposition A.1: Under the same conditions of Propositions 2.1 and 3.1, given $\xi(x(t,\cdot))$ defined in (9), for all $u \in C^2(0,\infty;\mathbb{R}^n)$ we have that

$$\begin{aligned} \dot{\xi}(x(t,\cdot)) = &-2\int_a^b \Psi^{\mathrm{T}}(z) G_0 \mathcal{L}(z) x(t,z)\,\mathrm{d}z \\ &+ G_\xi y(t) - (J_\xi - R_\xi) G_\xi^{-\mathrm{T}} u(t) \end{aligned} \tag{47}$$

with

$$J_\xi = \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}} \tilde{J}_\xi R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} \tag{48}$$

and

$$\begin{aligned} R_\xi &= \frac{1}{2}\begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P_1 & 0_n \\ 0_n & -P_1 \end{pmatrix} \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} \\ &= \int_a^b \Psi^{\mathrm{T}}(z) G_0 \Psi(z)\,\mathrm{d}z, \end{aligned} \tag{49}$$

where G_ξ has been defined in (11), and $\tilde{J}_\xi = -\tilde{J}_\xi^{\mathrm{T}}$ follows from (46). Note that J_ξ and R_ξ are $n \times n$ matrices, and that $J_\xi = -J_\xi^{\mathrm{T}}$ and $R_\xi = R_\xi^{\mathrm{T}} \geq 0$.

Proof: With (1) in mind and by using integration by parts, the time derivative of (9) is

$$\begin{aligned} \dot{\xi} = &\int_a^b \left[-\frac{\mathrm{d}^{\mathrm{T}}\Psi}{\mathrm{d}z}(z) P_1 + \Psi^{\mathrm{T}}(z)(P_0 - G_0) \right] (\mathcal{L}x)(z)\,\mathrm{d}z \\ &+ \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P_1 & 0_n \\ 0_n & -P_1 \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}. \end{aligned} \tag{50}$$

Relation (8) implies that

$$0_n = -\frac{\mathrm{d}^{\mathrm{T}}\Psi}{\mathrm{d}z} P_1 + \Psi^{\mathrm{T}}(P_0 - G_0) + 2\Psi^{\mathrm{T}} G_0,$$

which combined with (50) leads to

$$\begin{aligned} \dot{\xi} = &-2\int_a^b \Psi^{\mathrm{T}}(z) G_0 (\mathcal{L}x)(z)\,\mathrm{d}z \\ &+ \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P_1 & 0_n \\ 0_n & -P_1 \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}. \end{aligned} \tag{51}$$

Since we have to prove that (47) holds true, it is necessary to focus on the second term in (51) only. From the definition of R reported in (2), it is possible to check that

$$R^{\mathrm{T}}\Sigma R = \begin{pmatrix} P_1 & 0_n \\ 0_n & -P_1 \end{pmatrix}. \tag{52}$$

Moreover, from (2), (3), (4) and (45) we have that

$$R\begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} = \begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-1} \begin{pmatrix} u \\ y \end{pmatrix} = \Sigma\left(W^{\mathrm{T}} y + \tilde{W}^{\mathrm{T}} u \right).$$

Then, because of (52), the second term in (51) is equal to

$$G_\xi y + \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}} \tilde{W}^{\mathrm{T}} u, \tag{53}$$

where (11) has been taken into account, and $\Sigma\Sigma = I_{2n}$. To conclude the proof, it is now necessary to manipulate the last term in (53). At first, with (11) in mind, note that

$$\begin{aligned} \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}} \tilde{W}^{\mathrm{T}} G_\xi^{\mathrm{T}} &= \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}} \left(\frac{1}{2}\Sigma - \tilde{J}_\xi \right) R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} \\ &= -J_\xi + R_\xi, \end{aligned}$$

where (46), (48), the first equality in (49), and (52) have been considered. Since G_ξ is invertible, we have that

$$\begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}} \tilde{W}^{\mathrm{T}} = -(J_\xi - R_\xi) G_\xi^{-\mathrm{T}}. \tag{54}$$

The final step consists in proving the second equality in (49). From (8), we get the matrix differential equation $0_n = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{2}\Psi^{\mathrm{T}} P_1 \Psi\right) + \Psi^{\mathrm{T}}(P_0 - G_0)\Psi$, which after integration on $[a,\, b]$ implies that

$$R_\xi = -\underbrace{\int_a^b \Psi^{\mathrm{T}}(z) P_0 \Psi(z)\,\mathrm{d}z}_{=0_n} + \int_a^b \Psi^{\mathrm{T}}(z) G_0 \Psi(z)\,\mathrm{d}z$$

and proves the second equality in (49), since P_0 is skew-symmetric, and so $\int_a^b \Psi^{\mathrm{T}} P_0 \Psi\,\mathrm{d}z$ is, while R_ξ and $\int_a^b \Psi^{\mathrm{T}} G_0 \Psi\,\mathrm{d}z$ are both symmetric because $G_0 = G_0^{\mathrm{T}} \geq 0$. ■