A Lyapunov Approach to Robust Regulation of Distributed Port–Hamiltonian Systems

Lassi Paunonen, Yann Le Gorrec and Héctor Ramírez

***Abstract*—This paper studies robust output tracking and disturbance rejection for boundary controlled infinite-dimensional port–Hamiltonian systems including second order models such as the Euler–Bernoulli beam. The control design is achieved using the internal model principle and the stability analysis using a Lyapunov approach. Contrary to existing works on the same topic no assumption is made on the external well-posedness of the considered class of PDEs. The results are applied to robust tracking of a piezo actuated tube used in atomic force imaging.**

***Index Terms*—Distributed port-Hamiltonian system, boundary control system, robust output regulation, controller design.**

I. Introduction

We consider robust output regulation for a class of linear partial differential equations (PDEs) with boundary control and observation, namely, *port-Hamiltonian systems* (PHS) [11], [13]

$$\frac{\partial x}{\partial t}(z,t) = P_2\frac{\partial^2}{\partial z^2}\left(\mathcal{H}(z)x(z,t)\right) + P_1\frac{\partial}{\partial z}\left(\mathcal{H}(z)x(z,t)\right) \quad \text{(1a)}$$

$$+ \left(P_0 - G_0\right)\left(\mathcal{H}(z)x(z,t)\right) + B_d(z)w_{dist,1}(t), \quad \text{(1b)}$$

$$W_1\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = u(t) + w_{dist,2}(t), \;\; W_2\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = w_{dist,3}(t) \quad \text{(1c)}$$

$$y(t) = \tilde{W}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} \quad \text{(1d)}$$

on a one-dimensional spatial domain $[a,b]$ (see Section II for detailed assumptions). In robust regulation, the purpose of the control $u(t) \in \mathbb{R}^p$ is to achieve the asymptotic convergence of the output $y(t) \in \mathbb{R}^p$ of (1) to a given reference signal $y_{ref}(t)$, i.e., $\|y(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$, despite external disturbance signals $w_{dist}(t) := (w_{dist,1}(t), w_{dist,2}(t), w_{dist,3}(t))$. The signals $y_{ref}(t)$ and $w_{dist}(t)$ are assumed to have the forms

$$y_{ref}(t) = a_0 + \sum_{k=1}^{q}\left[a_k^1\cos(\omega_k t) + a_k^2\sin(\omega_k t)\right], \quad \text{(2a)}$$

$$w_{dist}(t) = b_0 + \sum_{k=1}^{q}\left[b_k^1\cos(\omega_k t) + b_k^2\sin(\omega_k t)\right], \quad \text{(2b)}$$

for known frequencies $0 = \omega_0 < \omega_1 < \cdots < \omega_q$ and unknown amplitudes $\{a_k^1\}_{k=0}^q, \{a_k^2\}_{k=1}^q \subset \mathbb{R}^p$, and $\{b_k^1\}_{k=0}^q, \{b_k^2\}_{k=1}^q \subset \mathbb{R}^{n_{d,1}+p+n_{d,3}}$.

Several recent articles have considered output regulation for individual linear PDEs, such as 1D heat equations [4], beam equations [12] and wave equations [6]. In this paper we solve the control problem for a *class* of boundary controlled 1D PDEs (1), which covers many particular hyperbolic PDE systems such as boundary controlled wave equations, Schrödinger equations, Timoshenko and Euler–Bernoulli beam models with spatially varying physical parameters, and is used in modeling and control of flexible structures, heat exchangers, and chemical reactors. We focus here on *impedance passive* PHS (1), and solve the output regulation problem using a finite-dimensional dynamic error feedback controller

$$\dot{x}_c(t) = J_c x_c(t) + \delta_c B_c(y_{ref}(t) - y(t)), \quad x_c(0) \in X_c \quad \text{(3a)}$$

$$u(t) = \delta_c B_c^* x_c(t) + D_c(y_{ref}(t) - y(t)) \quad \text{(3b)}$$

where J_c is skew-symmetric, $B_c \in \mathbb{R}^{p\times n_c}$, and $D_c \in \mathbb{R}^{p\times p}$ satisfies $D_c \geq 0$. Finally, $\delta_c > 0$ is a gain parameter. In studying the class (1) of PDEs our aim is to design the controller (3) under assumptions that can be verified directly based on the properties of the original PDE (1) and the matrices $(P_0, P_1, P_2, G_0, W_1, W_2, \tilde{W})$, without the need to reformulate (1) as an abstract system.

Our results for the class (1) are based on the theoretical results on robust output regulation of *abstract boundary control and observation systems* [3], [21] presented in this paper. They extend the theory related to internal model based controllers for passive *well-posed linear systems* and PHS in [7]–[10], [18], and they compose the main technical contributions of the paper. In particular, we introduce a new Lyapunov-type argument for the stability analysis of the closed-loop system consisting of the boundary control system and the controller (extending our earlier results in [16] for PHS with distributed control and observation). In addition, the controller design is done without assuming well-posedness of the original control system (which was assumed in [18]) and the analysis is completed directly in the abstract boundary control system framework (whereas in [9], [10] the boundary control inputs were first reformulated as distributed inputs using a state extension). The class (1) includes models which are not wellposed (in the sense of [20, Sec. 2]). The stability analysis of the closed-loop system is also related to references [14], [17] studying the stability of coupled impedance passive systems in a different context *i.e.* when the infinite dimensional system is undamped and the controller strictly input passive.

The paper is organised as follows. In Section II we define the considered class of boundary controlled PHS and state

This work was supported by the Academy of Finland Grants number 298182 and 310489 held by L. Paunonen, the Agence Nationale de la Recherche/Deutsche Forschungsgemeinschaft (ANR-DFG) project INFIDHEM, ID ANR-16-CE92-0028, and the EUROPEAN COMMISSION through H2020-ITN-2017-765579 — ConFlex, and AC3E basal project FB0008.

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our main result for the PDEs (1) (these are proved later in Section V). In Sections III–V we present our main results for abstract boundary control systems. The results are applied in solving a concrete output regulation problem in Section VI. The paper ends with some conclusions and perspectives.

Notation. If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{L}(X)$ on a Hilbert space X we define $\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$. $H^k(a,b;\mathbb{R}^n)$ is the kth order Sobolev space of functions $f : [a,b] \to \mathbb{R}^n$. For $T \in \mathcal{L}(X)$ we denote $T > 0$ if $T - \varepsilon I \geq 0$ for some $\varepsilon > 0$.

II. The Main Results for PHS

In this section we summarise our main results for the class (1) of boundary controlled PDEs. The parameters $P_2, P_1, P_0, G_0 \in \mathbb{R}^{n\times n}$ are assumed to satisfy $P_2 = -P_2^T$, $P_1 = P_1^T$, $P_0 = -P_0^T$, $G_0 = G_0^T \geq 0$, and $\mathcal{H}(\cdot)$ is a bounded and Lipschitz continuous matrix-valued function such that $\mathcal{H}(z) = \mathcal{H}(z)^T$ and $\mathcal{H}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a,\, b]$. The distributed disturbance input profile is assumed to satisfy $B_d(\cdot) \in L^2(a,b;\mathbb{R}^{n\times n_{d,1}})$ and can be unknown.

We consider first and second order PHS by assuming that either P_2 is invertible (the system (1) is of order $N = 2$) or $P_2 = 0$ and P_1 is invertible (the system is of order $N = 1$). The boundary inputs and ouputs are determined using the following *boundary port variables*.

Definition II.1. *The* boundary port variables $f_\partial(t)$ *and* $e_\partial(t)$ *associated to the system* (1) *are defined as*

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = R_{ext}\Phi(\mathcal{H}x(t)), \qquad \text{with} \quad R_{ext} = \frac{1}{\sqrt{2}}\begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}$$

where $Q \in \mathbb{R}^{2nN\times 2nN}$ *and* $\Phi(\cdot) : H^N(a,b;\mathbb{R}^n) \to \mathbb{R}^{2nN}$ *are defined so that*

- *if* $N = 2$*, then*

$$Q = \begin{bmatrix} P_1 & P_2 \\ -P_2 & 0 \end{bmatrix}, \qquad \Phi(\mathcal{H}x) := \begin{bmatrix} (\mathcal{H}x)(b) \\ \frac{\partial(\mathcal{H}x)}{\partial z}(b) \\ (\mathcal{H}x)(a) \\ \frac{\partial(\mathcal{H}x)}{\partial z}(a) \end{bmatrix},$$

  whenever $\mathcal{H}x \in H^2(a,b;\mathbb{R}^n)$.
- *if* $N = 1$*, then* $Q = P_1$ *and* $\Phi(\mathcal{H}x) = \begin{bmatrix} \mathcal{H}x(b) \\ \mathcal{H}x(a) \end{bmatrix}$ *whenever* $\mathcal{H}x \in H^1(a,b;\mathbb{R}^n)$.

The input $u(t) \in \mathbb{R}^p$, output $y(t) \in \mathbb{R}^p$ (the numbers of inputs and outputs are the same) and the disturbance inputs $w_{dist}(t) = (w_{dist,1}(t), w_{dist,2}(t), w_{dist,3}(t))^T \in \mathbb{R}^{n_{d,1}+p+n_{d,3}}$ of the system are defined as in (1). We assume the matrices W_1, W_2, and $\tilde{W}$ determining the inputs and outputs satisfy the following (concrete and checkable) conditions. As shown later in Lemma V.3, part (b) of Assumption II.2 guarantees that (1) is impedance passive.

Assumption II.2. *Denote* $\Sigma := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \in \mathbb{R}^{2nN\times 2nN}$*. We assume* $W_1 \in \mathbb{R}^{p\times 2nN}$ *and* $W_2 \in \mathbb{R}^{n_{d,3}\times 2nN}$ *with* $n_{d,3} = nN - p$ *and* $\tilde{W} \in \mathbb{R}^{p\times 2nN}$ *satisfy the following*

(a) $W := \left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right] \in \mathbb{R}^{nN\times 2nN}$ *has full rank and* $W\Sigma W^T \geq 0$

(b) $\langle (W_1^T\tilde{W} + \tilde{W}^T W_1 - \Sigma)g, g\rangle \geq 0$ *for all* $g \in \mathcal{N}(W_2)$.

Our second assumption concerns stabilizability properties of (1). The system (1) is *exponentially stable* if there exist $M, \alpha > 0$ such that with $u(t) \equiv 0$ and $w_{dist}(t) \equiv 0$ we have

$$\|x(\cdot,t)\|_{L^2(a,b)} \leq Me^{-\alpha t}\|x(\cdot,0)\|_{L^2(a,b)}$$

for all $x(\cdot,0) \in L^2(a,b;\mathbb{R}^n)$ such that $\mathcal{H}x(\cdot,0) \in H^N(a,b;\mathbb{R}^n)$ and for which (1c) hold for $t = 0$.

Assumption II.3. *For any* $K \in \mathbb{R}^{p\times p}$*,* $K > 0$*, system* (1) *becomes exponentially stable with output feedback* $u(t) = -Ky(t)$*.*

The output feedback $u(t) = -Ky(t)$ alters the boundary conditions of the PDE (1) by changing W_1 in (1c) to $W_1 + K\tilde{W}$. By [10, Lem. 7] Assumption II.3 holds in particular if $W_1 \in \mathbb{R}^{nN\times 2nN}$ (i.e., (1) has $p = nN$ inputs) and if Assumption II.2 holds. For further results on stability of (1), see [1].

Definition II.4 contains the construction of the controller (3). The controller has an *internal model* of the frequencies in (2) in the sense that $\{\pm i\omega_k\}_{k=1}^q \cup \{0\}$ are eigenvalues of J_c with geometric multiplicities equal to p (see also Section IV).

Definition II.4. *Given* $0 < \omega_1 < \cdots < \omega_q$ *in* (2)*, choose the parameters of the controller* (3) *on* $X_c = \mathbb{R}^{p(2q+1)}$ *so that* $D_c > 0$*,* $\delta_c > 0$*,*

$$J_c = \mathrm{blockdiag}(J_c^0, J_c^1, \ldots, J_c^q), \tag{4a}$$

$$J_c^0 = 0_p, \quad J_c^k = \begin{bmatrix} 0 & \omega_k I_p \\ -\omega_k I_p & 0 \end{bmatrix}, \tag{4b}$$

$$B_c = \begin{bmatrix} B_c^0 \\ \vdots \\ B_c^q \end{bmatrix}, \qquad B_c^0 = I_p, \quad B_c^k = \begin{bmatrix} I_p \\ 0 \end{bmatrix}. \tag{4c}$$

The following theorem is the main result of this section.

Theorem II.5. *Let Assumptions* II.2 *and* II.3 *be satisfied and let* $0 = \omega_0 < \omega_1 < \cdots < \omega_q$*. Assume* (1) *has no transmission zeros at* $\{\pm i\omega_k\}_{k=0}^q \subset i\mathbb{R}$*. For every* $D_c > 0$ *there exists* $\delta_c^* > 0$ *such that for all* $\delta_c \in (0, \delta_c^*)$ *the controller in Definition* II.4 *achieves output tracking and disturbance rejection for all signals in* (2)*. In particular, there exists* $\alpha > 0$ *(depending on* $\delta_c \in (0, \delta_c^*)$*) such that*

$$e^{\alpha t}\|y(t) - y_{ref}(t)\| \to 0 \qquad \text{as} \quad t \to \infty \tag{5}$$

for all $y_{ref}(t)$ *and* $w_{dist}(t)$ *in* (2) *and for all initial states* $x(\cdot,0) \in L^2(a,b;\mathbb{R}^n)$ *and* $x_c(0) \in \mathbb{R}^{p(2q+1)}$ *such that* $\mathcal{H}x(\cdot,0) \in H^N(a,b;\mathbb{R}^n)$ *and which satisfy the boundary conditions* (1c) *at* $t = 0$*.*

The controller is robust in the sense that the tracking (5) *is achieved (with a modified* $\alpha > 0$*) also if the parameters* $(P_2, P_1, P_0, G_0, W_1, W_2, \tilde{W}, \mathcal{H}, B_d)$ *of* (1) *are perturbed in such a way that Assumption* II.2 *continues to hold and the closed-loop system remains exponentially stable.*

The proof of Theorem II.5 is presented in Section V. If $\omega_0 = 0$ is a transmission zero, then J_c^0 and B_c^0 can be removed from the controller parameters in (4) and Theorem II.5 holds for $y_{ref}(t)$ and $w_{dist}(t)$ with $a_0 = 0$ and $b_0 = 0$.

III. Background on Boundary Control Systems

Our main abstract results are formulated for the general class of *boundary control and observation systems* [3], [19]

$$\dot{x}(t) = \mathfrak{A}_0 x(t) + B_d w_{dist,1}(t), \qquad x(0) = x_0 \in Z \quad \text{(6a)}$$
$$\mathfrak{B}x(t) = u(t) + w_{dist,2}(t) \quad \text{(6b)}$$
$$\mathfrak{B}_d x(t) = w_{dist,3}(t) \quad \text{(6c)}$$
$$y(t) = \mathfrak{C}x(t) \quad \text{(6d)}$$

on a Hilbert space X. We present these abstract results *only in the case* $D_c = 0$. This simplification does not result in loss of generality, because if $D_c \neq 0$, then (6b) becomes

$$(\mathfrak{B} + D_c\mathfrak{C})x(t) = \tilde{u}(t) + (w_{dist,2}(t) + D_c y_{ref}(t)) \quad (7)$$

(which has the same structure as (6b)) where $\tilde{u}(t)$ is the control produced by the controller (3) with $D_c = 0$. We make the following standard assumptions on the parameters of (6).

Assumption III.1. *We assume X and $Z \subset X$ are (complex) Hilbert spaces and $\mathfrak{A}_0 \in \mathcal{L}(Z, X)$, $\mathfrak{B} \in \mathcal{L}(Z, \mathbb{C}^p)$, $B_d \in \mathcal{L}(\mathbb{C}^{n_{d,1}}, X)$, $\mathfrak{B}_d \in \mathcal{L}(Z, \mathbb{C}^{n_{d,3}})$ and $\mathfrak{C} \in \mathcal{L}(Z, \mathbb{C}^p)$ have the properties:*

(a) *The operator $A := \mathfrak{A}_0|_{\mathcal{D}(A)}$ with $\mathcal{D}(A) = \mathcal{N}(\mathfrak{B}) \cap \mathcal{N}(\mathfrak{B}_d)$ generates a contraction semigroup $T(t)$ on X.*

(b) *The operator $\left[\begin{smallmatrix} \mathfrak{B} \\ \mathfrak{B}_d \end{smallmatrix}\right] \in \mathcal{L}(Z, \mathbb{C}^{p+n_{d,3}})$ is surjective.*

(c) $\mathrm{Re}\langle \mathfrak{A}x, x\rangle \leq \mathrm{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^p}$ *for all $x \in Z$.*

By [15, Thm. 3.4] part (c) of Assumption III.1 is equivalent to the system (6) being impedance passive in the sense that

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_X^2 \leq \langle u(t), y(t)\rangle_{\mathbb{C}^p}.$$

We also denote $\mathfrak{A} := \mathfrak{A}_0|_{\mathcal{D}(\mathfrak{A})}$ with $\mathcal{D}(\mathfrak{A}) = \mathcal{N}(\mathfrak{B}_d)$, and in this notation we have $A = \mathfrak{A}|_{\mathcal{D}(A)}$ and $\mathcal{D}(A) = \mathcal{D}(\mathfrak{A}) \cap \mathcal{N}(\mathfrak{B})$.

For $\lambda \in \rho(A)$ we denote the transfer function (from the input $u(t)$ to the output $y(t)$) of the system (1) by $P(\lambda)$. By [3, Thm. 2.9], for any $u \in U$ and $\lambda \in \rho(A)$ we have $P(\lambda)u = \mathfrak{C}x$ where $x \in Z$ is such that $(\lambda - \mathfrak{A})x = 0$ and $\mathfrak{B}x = u$. If we denote $\mathrm{Re}\, T = \frac{1}{2}(T + T^*)$, then the passivity of the system implies that $\mathrm{Re}\, P(i\omega) \geq 0$ for all $i\omega \in \rho(A) \cap i\mathbb{R}$, see [20].

We assume the controller (3) on $X_c = \mathbb{C}^{n_c}$ satisfies $J_c^* = -J_c \in \mathbb{C}^{n_c \times n_c}$, $B_c \in \mathbb{C}^{n_c \times p}$, $D_c \in \mathbb{C}^{p \times p}$ with $D_c \geq 0$ and $\delta_c > 0$ (as mentioned above, in Sections III–V we let $D_c = 0$). We now show that the closed-loop system consisting of (6) and the controller (3) on $X_c = \mathbb{C}^{n_c}$ leads to a well-defined closed-loop state $x_e(t) := (x(t), x_c(t))^T$ and regulation error $e(t)$ for all reference and disturbance signals in (2). The closed-loop system (with $D_c = 0$) has the form

$$\dot{x}_e(t) = \begin{bmatrix} \mathfrak{A}_0 & 0 \\ -\delta_c B_c \mathfrak{C} & J_c \end{bmatrix} x_e(t) + \begin{bmatrix} B_d & 0 \\ 0 & \delta_c B_c \end{bmatrix} \begin{bmatrix} w_{dist,1}(t) \\ y_{ref}(t) \end{bmatrix}$$
$$\begin{bmatrix} \mathfrak{B} & -\delta_c B_c^* \\ \mathfrak{B}_d & 0 \end{bmatrix} x_e(t) = \begin{bmatrix} w_{dist,2}(t) \\ w_{dist,3}(t) \end{bmatrix}$$
$$e(t) = \begin{bmatrix} \mathfrak{C}, & 0 \end{bmatrix} x_e(t) - y_{ref}(t)$$

with state $x_e(t) = (x(t), x_c(t))^T \in X_e := X \times X_c$. We denote

$$\mathfrak{A}_e = \begin{bmatrix} \mathfrak{A}_0 & 0 \\ -\delta_c B_c \mathfrak{C} & J_c \end{bmatrix}, \; \mathfrak{B}_e = \begin{bmatrix} \mathfrak{B} & -\delta_c B_c^* \\ \mathfrak{B}_d & 0 \end{bmatrix},$$

$B_e = \left[\begin{smallmatrix} B_d & 0 \\ 0 & \delta_c B_c \end{smallmatrix}\right]$, and $\mathfrak{C}_e = \begin{bmatrix} \mathfrak{C}, & 0 \end{bmatrix}$.

Proposition III.2. *Under Assumption III.1 and for $J_c^* = -J_c$ and $D_c = 0$ the operator $A_e := \mathfrak{A}_e|_{\mathcal{N}(\mathfrak{B}_e)}$ generates a strongly continuous contraction semigroup $T_e(t)$ on X_e. For any $y_{ref}(\cdot) \in C^2([0,\infty); \mathbb{C}^p)$ and $w_{dist}(\cdot) \in C^2([0,\infty); \mathbb{C}^{n_{d,1}+p+n_{d,3}})$ and for all initial states $x(0) \in Z$ and $x_c(0) \in X_c$ satisfying the compatibility conditions $\mathfrak{B}x(0) = \delta_c B_c^* x_c(0) + w_{dist,2}(0)$ and $\mathfrak{B}_d x(0) = w_{dist,3}(0)$ the closed-loop system has a state*

$$x(\cdot) \in C(0, T; Z) \cap C^1(0, T; X), \qquad x_c(\cdot) \in C^1(0, T; X_c)$$

and $e(t) = y(t) - y_{ref}(t) \in C(0, T; \mathbb{C}^p)$ for all $T > 0$.

Proof. The closed-loop system is a boundary control and observation system on the spaces $Z \times X_c$ and $X_e = X \times X_c$. The operator $\mathfrak{B}_e$ is surjective due to Assumption III.1(b). Our aim is to show that A_e generates a contraction semigroup on X_e. Since $\mathfrak{C}_e \in \mathcal{L}(Z \times X_c, \mathbb{C}^p)$ and B_d and B_c are bounded, the properties of the closed-loop system's state then follow (due to linearity) from [21, Prop. 4.2.10 and Prop. 10.1.8]. We now use the Lumer–Phillips Theorem. Let $x_e := (x, x_c)^T \in \mathcal{N}(\mathfrak{B}_e)$. Then $\mathfrak{B}x = \delta_c B_c^* x_c$ and $\mathfrak{B}_d x = 0$. In particular $x \in \mathcal{D}(\mathfrak{A})$ and $\mathfrak{A}_0 x = \mathfrak{A}x$. The impedance passivity of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ implies $\mathrm{Re}\langle \mathfrak{A}x, x\rangle \leq \mathrm{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle$ for all $x \in Z$ [15, Thm. 3.4]. Thus

$$\begin{aligned}\mathrm{Re}\langle A_e x_e, x_e\rangle &= \mathrm{Re}\langle \mathfrak{A}x, x\rangle + \mathrm{Re}\langle J_c x_c - \delta_c B_c \mathfrak{C}x, x_c\rangle \\ &\leq \mathrm{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle - \mathrm{Re}\langle \mathfrak{C}x, \delta_c B_c^* x_c\rangle = 0,\end{aligned}$$

since J_c is skew-adjoint and $\delta_c B_c^* x_c = \mathfrak{B}x$. Therefore A_e is dissipative, and it remains to show that $\lambda - A_e$ is surjective for some $\lambda > 0$. Let $\lambda > 0$, $y_1 \in X$, and $y_2 \in X_c$ be arbitrary. We will construct $x_e := (x, x_c)^T \in \mathcal{N}(\mathfrak{B}_e)$ such that $(y_1, y_2)^T = (\lambda - A_e)x_e$. Recall that $P(\lambda)$ is the transfer function of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and denote $P_c(\lambda) = \delta_c^2 B_c^* R(\lambda, J_c) B_c$. Since $\lambda > 0$ is real, we have $P_c(\lambda) \geq 0$ and $P(\lambda) \geq 0$, and it can be shown that $Q_1(\lambda) := I + P(\lambda)P_c(\lambda)$ and $Q_2(\lambda) := I + P_c(\lambda)P(\lambda)$ are boundedly invertible. Denote $R_\lambda = R(\lambda, A)$ and $R_\lambda^c = R(\lambda, J_c)$ for brevity. Due to the theory in [3], [21, Ch. 10] the "abstract elliptic problem"

$$\begin{aligned}(\lambda - \mathfrak{A})x &= y_1 \\ \mathfrak{B}x &= Q_2(\lambda)^{-1}(\delta_c B_c^* R_\lambda^c y_2 - P_c(\lambda)\mathfrak{C}R_\lambda y_1)\end{aligned}$$

has a solution $x \in Z$. Now [3, Thm. 2.9] and linearity imply

$$\begin{aligned}\mathfrak{C}x &= \mathfrak{C}R_\lambda y_1 + P(\lambda)Q_2(\lambda)^{-1}(\delta_c B_c^* R_\lambda^c y_2 - P_c(\lambda)\mathfrak{C}R_\lambda y_1) \\ &= Q_2(\lambda)^{-1}(\mathfrak{C}R_\lambda y_1 + \delta_c P(\lambda) B_c^* R_\lambda^c y_2).\end{aligned}$$

If we now define

$$x_c = R_\lambda^c y_2 - \delta_c R_\lambda^c B_c Q_1(\lambda)^{-1}(\mathfrak{C}R_\lambda y_1 + \delta_c P(\lambda) B_c^* R_\lambda^c y_2),$$

then

$$\begin{aligned}\delta_c B_c^* x_c &= \delta_c B_c^* R_\lambda^c y_2 \\ &\quad - P_c(\lambda)Q_1(\lambda)^{-1}(\mathfrak{C}R_\lambda y_1 + \delta_c P(\lambda) B_c^* R_\lambda^c y_2) \\ &= Q_2(\lambda)^{-1}(\delta_c B_c^* R_\lambda^c y_2 - P_c(\lambda)\mathfrak{C}R_\lambda y_1) = \mathfrak{B}x\end{aligned}$$

and thus $x_e := (x, x_c)^T$ satisfies $\mathfrak{B}_e x_e = 0$. A direct computation also shows that $-\delta_c B_c \mathfrak{C} x + (\lambda - J_c) x_c = y_2$, and thus indeed $(y_1, y_2)^T = (\lambda - A_e) x_e$. □

IV. Robust tracking and disturbance rejection

In this section we formulate the robust output regulation problem and present a general condition for a controller (3) to solve this problem.

The Robust Output Regulation Problem. *Let $0 < \omega_1 < \cdots < \omega_q$. Choose a controller* (3) *in such a way that the following hold.*

(a) *The semigroup $T_e(t)$ generated by $A_e = \mathfrak{A}_e|_{\mathcal{N}(\mathfrak{B}_e)}$ is exponentially stable.*

(b) *There exists $\alpha > 0$ such that for all $y_{ref}(t)$ and $w_{dist}(t)$ of the form* (2) *and for all initial states $x(0) \in Z$ and $x_c(0) \in X_c$ satisfying the boundary conditions of* (6) *the regulation error satisfies*

$$e^{\alpha t} \| y(t) - y_{ref}(t) \| \to 0 \qquad \text{as} \quad t \to \infty.$$

(c) *If $(\mathfrak{A}_0, \mathfrak{B}, \mathfrak{B}_d, B_d, \mathfrak{C})$ in* (6) *are perturbed in such a way that Assumption* III.1 *is satisfied and the perturbed closed-loop operator generates an exponentially stable semigroup, then* (b) *continues to hold for some $\tilde{\alpha} > 0$.*

The robust output regulation problem only has a solution if the control system does not have transmission zeros at $\{\pm i\omega_k\}_{k=0}^q$ (a transmission zero at $\lambda \in \rho(A)$ is equivalent to $P(\lambda) \in \mathbb{C}^{p \times p}$ being singular). For impedance passive systems it is natural to make the following stronger assumption.

Assumption IV.1. *Let $0 = \omega_0 < \omega_1 < \cdots < \omega_q$. We assume $\pm i\omega_k \in \rho(A)$ and $\operatorname{Re} P(\pm i\omega_k) > 0$ for all $k \in \{0, \ldots, q\}$.*

The following theorem shows that a controller incorporating an *internal model* (in the sense of conditions (8) below) will solve the robust output regulation problem provided that the closed-loop system is exponentially stable. The result generalises [10, Thm. 4] by removing the assumption of regularity (and well-posedness) of the closed-loop system, and the proof is completed without reformulating (6) as a system with extended state and distributed inputs.

Theorem IV.2. *Let $0 = \omega_0 < \omega_1 < \cdots < \omega_q$. A controller* (3) *with $J_c^* = -J_c$, $D_c = 0$ and $\delta_c > 0$ solves the robust output regulation problem if $A_e = \mathfrak{A}_e|_{\mathcal{N}(\mathfrak{B}_e)}$ generates an exponentially stable semigroup and*

$$\mathcal{R}(\pm i\omega_k - J_c) \cap \mathcal{R}(B_c) = \{0\}, \quad \forall k \in \{0, \ldots, q\} \tag{8a}$$

$$\mathcal{N}(B_c) = \{0\}. \tag{8b}$$

Then there exists $\alpha > 0$ such that

$$e^{\alpha t} \| y(t) - y_{ref}(t) \| \to 0, \qquad \text{as} \quad t \to \infty$$

for any $y_{ref}(t)$ and $w_{dist}(t)$ of the form (2) *and for all $x(0) \in Z$ and $x_c(0) \in X_c$ satisfying the compatibility conditions $\mathfrak{B} x(0) = \delta_c B_c^* x_c(0) + w_{dist,2}(0)$ and $\mathfrak{B}_d x(0) = w_{dist,3}(0)$.*

Proof. Assume the closed-loop system is exponentially stable and (8) are satisfied. Then there exist $M_e, \omega_e > 0$ such that $\|T_e(t)\| \le M_e e^{-\omega_e t}$. Let $\{\mu_k\}_{k=-q}^q$ be such that $\mu_k = \omega_k$ for $k > 0$, $\mu_0 = 0$, and $\mu_k = -\omega_{|k|}$ for $k < 0$. We can then write

$$y_{ref}(t) = \sum_{k=-q}^{q} y_r^k e^{i\mu_k t}, \qquad w_{dist}(t) = \sum_{k=-q}^{q} \begin{bmatrix} w_{1k} \\ w_{2k} \\ w_{3k} \end{bmatrix} e^{i\mu_k t}$$

for some constant elements $\{y_r^k\}_k$, $\{w_{1k}\}_k$, $\{w_{2k}\}_k$, and $\{w_{3k}\}_k$. Since $i\mu_k \in \rho(A_e)$ for all k, we have from [21, Sec. 10.1] that we can choose $\Sigma_k \in Z$ such that

$$(i\mu_k - \mathfrak{A}_e) \Sigma_k = B_e \begin{bmatrix} w_{1k} \\ y_r^k \end{bmatrix} \tag{9a}$$

$$\mathfrak{B}_e \Sigma_k = \begin{bmatrix} w_{2k} \\ w_{3k} \end{bmatrix}. \tag{9b}$$

Consider initial conditions $x(0) \in Z$ and $x_c(0) \in X_c$ satisfying the compatibility conditions $\mathfrak{B} x(0) = \delta_c B_c^* x_c(0) + w_{dist,2}(0)$ and $\mathfrak{B}_d x(0) = w_{dist,3}(0)$. If we define $\Sigma(t) = \sum_{k=-q}^q e^{i\mu_k t} \Sigma_k \in Z$, then

$$\begin{aligned} &\frac{d}{dt}(x_e(t) - \Sigma(t)) \\ &\quad = \mathfrak{A}_e x_e(t) + B_e \begin{bmatrix} w_{dist,1}(t) \\ y_{ref}(t) \end{bmatrix} - \sum_{k=-q}^{q} i\mu_k e^{i\mu_k t} \Sigma_k \\ &\quad = \mathfrak{A}_e (x_e(t) - \Sigma(t)) \end{aligned}$$

due to (9a). For all $t \ge 0$ we also have from (9b) that

$$\mathfrak{B}_e (x_e(t) - \Sigma(t)) = \begin{bmatrix} w_{dist,2}(t) \\ w_{dist,3}(t) \end{bmatrix} - \sum_{k=-q}^{q} e^{i\mu_k t} \mathfrak{B}_e \Sigma_k = 0.$$

Thus $x_e(t) - \Sigma(t) \in \mathcal{D}(A_e)$ is a classical solution of the abstract Cauchy problem $\frac{d}{dt}(x_e(t) - \Sigma(t)) = A_e (x_e(t) - \Sigma(t))$, and therefore $\|x_e(t) - \Sigma(t)\| = \|T_e(t)(x_e(0) - \Sigma(0))\| \le M_e e^{-\omega_e t} \|x_e(0) - \Sigma(0)\|$.

If we write $\Sigma_k = \left[\begin{smallmatrix} \Pi_k \\ \Gamma_k \end{smallmatrix}\right] \in Z \times X_c$, then (9a) and the conditions (8) imply

$$\begin{bmatrix} i\mu_k - \mathfrak{A}_0 & 0 \\ \delta_c B_c \mathfrak{C} & i\mu_k - J_c \end{bmatrix} \begin{bmatrix} \Pi_k \\ \Gamma_k \end{bmatrix} = \begin{bmatrix} B_d w_{1k} \\ \delta_c B_c y_r^k \end{bmatrix}$$

$$\Rightarrow \quad (i\mu_k - J_c)\Gamma_k = \delta_c B_c (y_r^k - \mathfrak{C}\Pi_k)$$

$$\overset{(8a)}{\Rightarrow} \quad B_c (y_r^k - \mathfrak{C}\Pi_k) = 0 \quad \overset{(8b)}{\Rightarrow} \quad y_r^k = \mathfrak{C}\Pi_k = \mathfrak{C}_e \Sigma_k.$$

Using $\mathfrak{C}_e \Sigma_k = y_r^k$, we can write $e(t) = y(t) - y_{ref}(t)$ as

$$\begin{aligned} e(t) &= \mathfrak{C}_e x_e(t) - \sum_{k=-q}^{q} y_r^k e^{i\mu_k t} = \mathfrak{C}_e x_e(t) - \sum_{k=-q}^{q} \mathfrak{C}_e \Sigma_k e^{i\mu_k t} \\ &= \mathfrak{C}_e (x_e(t) - \Sigma(t)). \end{aligned}$$

Finally, since $\mathfrak{C}_e A_e^{-1} \in \mathcal{L}(X, \mathbb{C}^p)$ for boundary control systems, we have

$$\begin{aligned} \|e(t)\| &= \|\mathfrak{C}_e (x_e(t) - \Sigma(t))\| \\ &= \|\mathfrak{C}_e A_e^{-1} T_e(t) A_e (x_e(0) - \Sigma(0))\| \\ &\le M_e e^{-\omega_e t} \|\mathfrak{C}_e A_e^{-1}\| \cdot \|A_e (x_e(0) - \Sigma(0))\| \end{aligned}$$

and thus $e^{\alpha t}\|e(t)\| \to 0$ as $t \to \infty$ for any $0 < \alpha < \omega_e$.

Since the proof can be repeated analogously for any perturbations of $(\mathfrak{A}_0, \mathfrak{B}, \mathfrak{B}_d, B_d, \mathfrak{C})$ for which Assumption III.1 is satisfied and the closed-loop semigroup is exponentially stable, the controller satisfies part (c) of the robust output regulation problem. □

V. A Passive Robust Controller

In this section we prove that if the system (6) is exponentially stable and the parameters of the controller (3) on $X_c = \mathbb{C}^{p(2q+1)}$ are chosen as (real) matrices $D_c = 0$,

$$J_c = \text{blockdiag}(J_c^0, J_c^1, \ldots, J_c^q), \tag{10a}$$

$$J_c^0 = 0_p, \quad J_c^k = \begin{bmatrix} 0 & \omega_k I_p \\ -\omega_k I_p & 0 \end{bmatrix}, \tag{10b}$$

$$B_c = \begin{bmatrix} B_c^0 \\ \vdots \\ B_c^q \end{bmatrix}, \qquad B_c^0 = I_p, \quad B_c^k = \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \tag{10c}$$

then the controller solves the robust output regulation problem for a range of gain parameters $\delta_c > 0$. The following theorem is the main abstract result of the paper, and it is also used in proving Theorem II.5 at the end of this section.

Theorem V.1. *Let $0 = \omega_0 < \omega_1 < \cdots < \omega_q$. Assume A generates an exponentially stable semigroup $T(t)$, $C := \mathfrak{C}|_{\mathcal{D}(A)}$ is admissible with respect to $T(t)$, and Assumption* IV.1 *holds. Then there exists $\delta_c^* > 0$ such that for all $\delta_c \in (0, \delta_c^*)$ the controller* (3) *on $X_c = \mathbb{C}^{p(2q+1)}$ with parameters* (10) *and $D_c = 0$ solves the robust output regulation problem for all $y_{ref}(t)$ and $w_{dist}(t)$ in* (2).

The main part of the proof of Theorem V.1 consists of showing the exponential stability of the closed-loop system for $\delta_c \in (0, \delta_c^*)$, and for this we use a new Lyapunov argument. Similar methods have been used in study of stability of coupled PHS especially in [14], [17]. Our situation is different from the previous references due to the fact that the infinite-dimensional system (6) is exponentially stable and the unstable controller (3) is finite-dimensional. The proof of Theorem V.1 begins with the definition of a component $H \in \mathcal{L}(X_c, X)$ of the Lyapunov candidate function in Lemma V.2. For the proofs we define a block-diagonal similarity transform $T = \text{blockdiag}(T_0, T_1, \ldots, T_q) \in \mathbb{C}^{n_c \times n_c}$ where $n_c = p(2q+1)$ such that for $k \in \{1, \ldots, q\}$

$$T_0 = I_p, \qquad T_k = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}, \quad T_k^{-1} = \frac{1}{2}\begin{bmatrix} I & -iI \\ I & iI \end{bmatrix}.$$

Moreover, we define $G_1 = T^{-1} J_c T \in \mathbb{C}^{p(2q+1) \times p(2q+1)}$ and $G_2 = T^{-1} B_c \in \mathbb{C}^{p \times p(2q+1)}$. A direct computation shows that

$$G_1 = \text{blockdiag}(i\omega_0 I_p, i\omega_1 I_p, -i\omega_1 I_p, \ldots, i\omega_q I_p, -i\omega_q I_p)$$
$$G_2 = \frac{1}{2}\left[I_p, I_p, \ldots, I_p\right]^T.$$

Lemma V.2. *Let Assumption* IV.1 *hold and assume A generates an exponentially stable semigroup on X. Let $X_c = \mathbb{C}^{p(2q+1)}$ and let J_c and B_c be as in* (10). *Then there exists $H \in \mathcal{L}(X_c, X)$ satisfying $\mathcal{R}(H) \subset Z$ such that*

$$HJ_c = \mathfrak{A}H \qquad \text{and} \qquad \mathfrak{B}H = -B_c^*, \tag{11}$$

and we have $\mathfrak{C}H \in \mathcal{L}(X_c, \mathbb{C}^p)$. Moreover, there exist constants $\delta_0^*, M_c > 0$ such that for any $\delta_c \in (0, \delta_0^*)$ we can choose $P_{c0} > 0$ such that $\|P_{c0}\| \le M_c$ and

$$P_{c0}(J_c + \delta_c^2 B_c \mathfrak{C}H) + (J_c + \delta_c^2 B_c \mathfrak{C}H)^* P_{c0} = -\delta_c^2 I.$$

Proof. Since $J_c = TG_1T^{-1}$, an operator $H \in \mathcal{L}(X_c, X)$ with $\mathcal{R}(H) \subset Z$ satisfies (11) if and only if $HTG_1 = \mathfrak{A}HT$ and $\mathfrak{B}HT = -B_c^*T$. Due to the block-diagonal structure of G_1, the operator HT has the form $HT = (H_0, H_1, H_{-1}, \ldots, H_q, H_{-q})$. Since $B_c^*T = \left[I, \ldots, I\right]$, for each $k \in \{0, \ldots, q\}$ the operators $H_{\pm k} : \mathbb{C}^p \to X$ are determined by $z_{\pm k} = H_{\pm k} y$ for all $y \in \mathbb{C}^p$ where $z_{\pm k}$ are the solutions of the abstact elliptic equations

$$\begin{cases} (\pm i\omega_k - \mathfrak{A}) z_{\pm k} = 0 \\ \mathfrak{B} z_{\pm k} = -y. \end{cases}$$

By [21, Prop. 10.1.2, Rem. 10.1.3 & 10.1.5] the above equations have unique solutions and $H_k \in \mathcal{L}(\mathbb{C}^p, X)$ and $\mathcal{R}(H_k) \subset Z$ for all $k \in \{-q, \ldots, q\}$. Thus $H \in \mathcal{L}(X_c, X)$ and $\mathcal{R}(H) \subset Z$. We further have from [3, Thm. 2.9] that $\mathfrak{C}H_{\pm k} y = \mathfrak{C} z_{\pm k} = -P(\pm i\omega_k) y$ for all $y \in \mathbb{C}^p$ and $k \in \{0, \ldots, q\}$. Because of this, we have

$$\mathfrak{C}HT = -\Big[P(i\omega_0), P(i\omega_1), P(-i\omega_1), \ldots, P(i\omega_q), P(-i\omega_q)\Big],$$

which in particular implies $\mathfrak{C}H \in \mathcal{L}(X_c, \mathbb{C}^p)$.

To prove the second claim, we first note that Assumption IV.1 implies that $\operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(P(\pm i\omega_k))$ and k. Indeed, if $k \in \{0, \ldots, q\}$ and $\operatorname{Re} \lambda \le 0$, then $\operatorname{Re} P(\pm i\omega_k) > 0$ implies $\operatorname{Re}(P(\pm i\omega_k) - \lambda) = |\operatorname{Re} \lambda| + \operatorname{Re} P(\pm i\omega_k) > 0$, and thus $P(\pm i\omega_k) - \lambda$ is nonsingular.

In the next step we use the results in [8, App. B] to show that there exist constants $M_0, \omega_0, \delta_0^* > 0$ such that

$$\|\exp((J_c + \delta_c^2 B_c \mathfrak{C}H)t)\| \le M_0 e^{-\omega_0 \delta_c^2 t} \tag{12}$$

for all $\delta_c \in (0, \delta_0^*)$ and $t \ge 0$. If we denote $K = -\mathfrak{C}HT$, then

$$J_c + \delta_c^2 B_c \mathfrak{C}H = T(G_1 - \delta_c^2 G_2 K)T^{-1}.$$

Now $K = [K_0, K_1, K_2, \ldots, K_{2q}]$ where $\operatorname{Re} \lambda < 0$ for all $\lambda \in \sigma(K_k)$ and $k \in \{0, \ldots, 2q\}$, and $G_2 = \frac{1}{2}[I, \ldots, I]^T$. Thus $(G_1 - \delta_c^2 G_2 K)^* = G_1^* - \delta_c^2 K^* G_2^*$ is of the form of $A_c(\varepsilon)$ in [8, App. B] with $\varepsilon = \delta_c^2/2$. The proof of Theorem 1 in [8, App. B] shows that there exist $M_1, \omega_0, \delta_0^* > 0$ such that $\|\exp((G_1^* - \delta_c^2 K^* G_2^*)t)\| \le M_1 e^{-\omega_0 \delta_c^2 t}$ for all $\delta_c \in (0, \delta_0^*)$ and $t \ge 0$. This further implies that if we define $M_0 = M_1 \|T\| \|T^{-1}\|$, then (12) holds for all $\delta_c \in (0, \delta_0^*)$ and $t \ge 0$.

Let $\delta_c \in (0, \delta_0^*)$ and denote $T_{\delta_c}(t) = \exp((J_c + \delta_c^2 B_c \mathfrak{C}H)t)$ for brevity. Since $J_c + \delta_c^2 B_c \mathfrak{C}H$ is Hurwitz, we can choose $\tilde{P}_{c0} > 0$ such that

$$(J_c + \delta_c^2 B_c \mathfrak{C}H)\tilde{P}_{c0} + (J_c + \delta_c^2 B_c \mathfrak{C}H)^* \tilde{P}_{c0} = -I.$$

Here $\tilde{P}_{c0} = \int_0^\infty T_{\delta_c}(t)^* T_{\delta_c}(t) dt$, and thus (12) implies

$$\|\tilde{P}_{c0}\| \le \int_0^\infty \|T_{\delta_c}(t)\|^2 dt \le M_0^2 \int_0^\infty e^{-2\omega_0 \delta_c^2 t} dt = \frac{M_0^2}{2\omega_0 \delta_c^2}.$$

Now the matrix $P_{c0} := \delta_c^2 \tilde{P}_{c0}$ has the required properties. □

Proof of Theorem V.1. The proof of [7, Lem. 12] shows that $\mathcal{R}(\pm i\omega_k - G_1) \cap \mathcal{R}(G_2) = \{0\}$ for all $k \in \{0, \dots, q\}$ and $\mathcal{N}(G_2) = \{0\}$, and by similarity the pair $(J_c, B_c) = (TG_1T^{-1}, TG_2)$ satisfies the conditions (8). By Theorem IV.2 it is thus sufficient to show that the closed-loop system is exponentially stable (in the case $y_{ref}(t) \equiv 0$ and $w_{dist}(t) \equiv 0$).

Let $H \in \mathcal{L}(X_c, X)$ and $\delta_0^*, M_c > 0$ be as in Lemma V.2, and let $\delta_c \in (0, \delta_0^*)$. We choose the Lyapunov function candidate V_e for the closed-loop system by

$$V_e = \langle x + \delta_c H x_c, P(x + \delta_c H x_c)\rangle_X + \langle x_c, P_c x_c\rangle_{X_c}$$

where $x = x(t)$ and $x_c = x_c(t)$ are the states of the plant and the controller, respectively, and P and P_c will be chosen later. Since the coordinate transform $(x, x_c) \to (x + \delta_c H x_c, x_c)$ is boundedly invertible, V_e is a valid Lyapunov function candidate whenever $P > 0$ and $P_c > 0$.

Let $(x(t), x_c(t))^T$ be a classical solution of the closed-loop system with $y_{ref}(t) \equiv 0$ and $w_{dist}(t) \equiv 0$. Since $\mathfrak{B}x(t) = \delta_c B_c^* x_c(t)$ and $\mathfrak{B}H = -B_c^*$, we have $\mathfrak{B}(x(t) + \delta_c H x_c(t)) = 0$. Thus $x(t) + \delta_c H x_c(t) \in \mathcal{N}(\mathfrak{B}) = \mathcal{D}(A)$ and $\mathfrak{A}(x(t) + \delta_c H x_c(t)) = A(x(t) + \delta_c H x_c(t))$. If we denote $\tilde{A} = A - \delta_c^2 H B_c C : \mathcal{D}(A) \subset X \to X$, then a direct computation using (11) shows that

$$\begin{aligned}\frac{1}{2}\dot{V}_e &= \mathrm{Re}\langle \dot{x} + \delta_c H \dot{x}_c, P(x + \delta_c H x_c)\rangle + \mathrm{Re}\langle \dot{x}_c, P_c x_c\rangle \\ &= \mathrm{Re}\langle \mathfrak{A}x + \delta_c H J_c x_c - \delta_c^2 H B_c \mathfrak{C}x, P(x + \delta_c H x_c)\rangle \\ &\quad + \mathrm{Re}\langle J_c x_c - \delta_c B_c \mathfrak{C}x, P_c x_c\rangle \\ &= \mathrm{Re}\langle \tilde{A}(x + \delta_c H x_c), P(x + \delta_c H x_c)\rangle \\ &\quad + \mathrm{Re}\langle (J_c + \delta_c^2 B_c \mathfrak{C} H) x_c, P_c x_c\rangle \\ &\quad + \mathrm{Re}\langle \delta_c^2 B_c \mathfrak{C} H x_c, \delta_c H^* P(x + \delta_c H x_c)\rangle \\ &\quad - \mathrm{Re}\langle C(x + \delta_c H x_c), \delta_c B_c^* P_c x_c\rangle.\end{aligned}$$

Since A generates an exponentially stable semigroup $T(t)$ on X, there exists a unique $P_1 \in \mathcal{L}(X)$ with $P_1 > 0$ such that $A^* P_1 + P_1 A = -2I$. Moreover, the exponential stability also implies that C is infinite-time admissible with respect to $T(t)$, and by [21, Thm. 5.1.1] there exists $P_2 \in \mathcal{L}(X)$ with $P_2 \geq 0$ such that $2\,\mathrm{Re}\langle Ax_1, P_2 x_1\rangle = -2\|Cx_1\|^2$ for all $x_1 \in \mathcal{D}(A)$. Thus if we define $P = P_1 + P_2 \in \mathcal{L}(X)$, then $P > 0$ and

$$2\,\mathrm{Re}\langle Ax_1, Px_1\rangle = -2\|x_1\|^2 - 2\|Cx_1\|^2 \quad \forall x_1 \in \mathcal{D}(A).$$

The scalar inequality $2ab \leq a^2 + b^2$ implies that if $x_1 \in \mathcal{D}(A)$, then

$$\begin{aligned}&2\,\mathrm{Re}\langle \tilde{A}x_1, Px_1\rangle = 2\,\mathrm{Re}\langle Ax_1, Px_1\rangle - 2\delta_c^2\,\mathrm{Re}\langle Cx_1, B_c^* H^* P x_1\rangle \\ &\leq -2\|x_1\|^2 - 2\|Cx_1\|^2 + \delta_c^2\|Cx_1\|^2 + \delta_c^2\|PHB_c\|^2\|x_1\|^2 \\ &= -(2 - \delta_c^2\|PHB_c\|^2)\|x_1\|^2 - (2 - \delta_c^2)\|Cx_1\|^2 \\ &\leq -\|x_1\|^2 - \|Cx_1\|^2\end{aligned}$$

whenever $0 < \delta_c \leq \delta_1^*$ with $\delta_1^* := \min\{1, 1/\|PHB_c\|\} > 0$.

Since $\delta_c \in (0, \delta_0^*)$ by assumption, we can choose $P_{c0} > 0$ (corresponding to this δ_c) as in Lemma V.2 and define $P_c = \varepsilon_c P_{c0} > 0$ for some $\varepsilon_c > 0$. Then $\|P_c\| \leq M_c \varepsilon_c$ and

$$P_c(J_c + \delta_c^2 B_c \mathfrak{C} H) + (J_c + \delta_c^2 B_c \mathfrak{C} H)^* P_c = -\varepsilon_c \delta_c^2 I.$$

If $0 < \delta_c < \min\{\delta_0^*, \delta_1^*\}$, we can estimate (using the inequality $2\,\mathrm{Re}\langle z_1, z_2\rangle \leq 2\|z_1\|\|z_2\| \leq \frac{1}{2}\|z_1\|^2 + 2\|z_2\|^2$ in the last term)

$$\begin{aligned}\dot{V}_e &= 2\,\mathrm{Re}\langle \tilde{A}(x + \delta_c H x_c), P(x + \delta_c H x_c)\rangle \\ &\quad + 2\,\mathrm{Re}\langle (J_c + \delta_c^2 B_c \mathfrak{C} H) x_c, P_c x_c\rangle \\ &\quad + 2\,\mathrm{Re}\langle \delta_c^2 B_c \mathfrak{C} H x_c, \delta_c H^* P(x + \delta_c H x_c)\rangle \\ &\quad - 2\,\mathrm{Re}\langle C(x + \delta_c H x_c), \delta_c B_c^* P_c x_c\rangle \\ &\leq -\|x + \delta_c H x_c\|^2 - \|C(x + \delta_c H x_c)\|^2 \\ &\quad - \varepsilon_c \delta_c^2 \|x_c\|^2 + \delta_c^4\|B_c \mathfrak{C} H x_c\|^2 + \delta_c^2\|H^* P(x + \delta_c H x_c)\|^2 \\ &\quad + \frac{1}{2}\|C(x + \delta_c H x_c)\|^2 + 2\delta_c^2\|B_c^* P_c x_c\|^2 \\ &= \Big[-1 + \delta_c^2\|PH\|^2\Big]\|x + \delta_c H x_c\|^2 - \frac{1}{2}\|C(x + \delta_c H x_c)\|^2 \\ &\quad + \delta_c^2\Big[-\varepsilon_c + \delta_c^2\|B_c \mathfrak{C} H\|^2 + 2M_c^2\varepsilon_c^2\|B_c\|^2\Big]\|x_c\|^2.\end{aligned}$$

We can now choose a sufficiently small fixed $\varepsilon_c > 0$ and $\delta_2^* > 0$ such that if $0 < \delta_c < \delta_c^* := \min\{\delta_0^*, \delta_1^*, \delta_2^*\}$, then

$$\begin{aligned}\dot{V}_e &\leq -\tilde{\varepsilon}_e\left(\|x + \delta_c H x_c\|^2 + \|x_c\|^2\right) \\ &\leq -\tilde{\varepsilon}_e \max\{\|P^{-1}\|, \|P_c^{-1}\|\} V_e =: -\varepsilon_e V_e,\end{aligned}$$

where $\varepsilon_e > 0$ depends on the choice of $\delta_c > 0$. Since $T_e(t)$ is contractive, this proves exponential closed-loop stability. □

We now present the proof of Theorem II.5 for PHS. To use Theorem V.1 we formulate (1) as a boundary control system on $X = L^2(a, b; \mathbb{C}^n)$ with norm defined by $\|x\|_{\mathcal{H}} = \sqrt{\langle \mathcal{H}x, x\rangle_{L^2}}$ for $x \in X$ (since $(P_2, P_1, P_0, G_0, \mathcal{H}, W, \tilde{W})$ are real, real-valued initial data for (1) and (3) leads to real-valued solutions). We begin by showing that the condition (b) in Assumption II.2 implies impedance passivity of (1).

Lemma V.3. *If Assumption* II.2 *holds and* $w_{dist}(t) \equiv 0$*, then the classical solutions of* (1) *satisfy* $\frac{1}{2}\frac{d}{dt}\|x(t)\|_{\mathcal{H}}^2 \leq u(t)^T y(t)$.

Proof. Let $w_{dist}(t) \equiv 0$. The proof of [13, Thm. 4.2] and (b) imply that the solution of (1) satisfies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b x(z,t)^T \mathcal{H}(z) x(z,t)dz &\leq \frac{1}{2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}^T \Sigma \begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} \\ &\leq \frac{1}{2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}^T (W_1^T \tilde{W} + \tilde{W}^T W_1)\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = y(t)^T u(t)\end{aligned}$$

where we have used that $\left[\begin{smallmatrix} f_\partial(t) \\ e_\partial(t)\end{smallmatrix}\right] \in \mathcal{N}(W_2)$ by (1c). □

As shown in [13, Sec. 4–5], (1) becomes a boundary control system (6) on X with choices

$$\begin{aligned}\mathfrak{A}_0 x &:= P_2\frac{\partial^2}{\partial z^2}(\mathcal{H}x) + P_1\frac{\partial}{\partial z}(\mathcal{H}x) + (P_0 - G_0)(\mathcal{H}x) \\ \mathcal{D}(\mathfrak{A}_0) &= Z := \{\, x \in L^2(a, b; \mathbb{C}^n) \mid \mathcal{H}x \in H^N(a, b; \mathbb{C}^n)\,\} \\ \mathfrak{B}x &= W_1 R_{ext}\Phi(\mathcal{H}x), \qquad \mathfrak{B}_d x = W_2 R_{ext}\Phi(\mathcal{H}x) \\ \mathfrak{C}x &= \tilde{W} R_{ext}\Phi(\mathcal{H}x), \qquad B_d v = B_d(\cdot)v\end{aligned}$$

where R_{ext} and $\Phi(\cdot)$ are as in Definition II.1. For these definitions the properties in Assumption III.1 follow from [13, Thm. 4.2] and Lemma V.3.

Proof of Theorem II.5. To apply Theorem V.1 we rewrite the feedthrough $D_c > 0$ as in (7), in which case the boundary control system has the input operator $\mathfrak{B} + D_c\mathfrak{C}$ and the controller (3) has no feedthrough. This corresponds to preliminary output feedback $u(t) = -D_c y(t) + \tilde{u}(t)$. Denote by $A_{D_c} = \mathfrak{A}|_{\mathcal{N}(\mathfrak{B}+D_c\mathfrak{C})}$ with $\mathcal{D}(A_{D_c}) = \mathcal{N}(\mathfrak{B} + D_c\mathfrak{C})$.

By Lemma V.3, the original system is impedance passive, and since $D_c > 0$, the output feedback preserves impedance passivity. The operator A_{D_c} is dissipative, and straightforward perturbation arguments (similar to those in the proof of Proposition III.2) show that $\mathcal{R}(1 - A_{D_c}) = X$. Thus A_{D_c} generates a contraction semigroup by the Lumer–Phillips Theorem and this semigroup is exponentially stable by Assumption II.3 (with $K = D_c$). As shown in [9, Prop. II.4]), $C = \mathfrak{C}|_{\mathcal{D}(A_{D_c})}$ is admissible with respect to the semigroup generated by A_{D_c}.

Finally, we need to verify Assumption IV.1, i.e., that the transfer function $P_{D_c}(\lambda)$ of (1) with feedback $u(t) = -D_c y(t) + \tilde{u}(t)$ satisfies $\operatorname{Re} P_{D_c}(\pm i\omega_k) > 0$ for all k. Define $K_0 = \frac{1}{2}D_c > 0$ and denote the transfer function of (1) with output feedback $u(t) = -K_0 y(t) + \tilde{u}(t)$ by $P_{K_0}(\lambda)$. By Assumption II.3 $P_{K_0}(\lambda) \in \mathbb{R}^{p\times p}$ is well-defined for $\lambda \in \{\pm i\omega_k\}_{k=0}^{q}$, and since (1) has no transmission zeros at $\pm i\omega_k$, $P_{K_0}(\pm i\omega_k)$ are nonsingular for all k. Since $D_c = K_0 + K_0$, we have $P_{D_c}(\pm i\omega_k) = P_{K_0}(\pm i\omega_k)(I + K_0 P_{K_0}(\pm i\omega_k))^{-1} = (P_{K_0}(\pm i\omega_k)^{-1} + K_0)^{-1}$ for all k. Since $\operatorname{Re}(P_{K_0}(\pm i\omega_k)^{-1}) + K_0 > 0$, it is easy to show that Assumption IV.1 holds. The claims now follow from Theorem V.1. □

VI. Application to Atomic Force Microscopy

As application example we consider the output tracking trajectory problem for a piezo actuated tube used in positioning systems for Atomic Force Microscopy (see Figure 1 (left)).

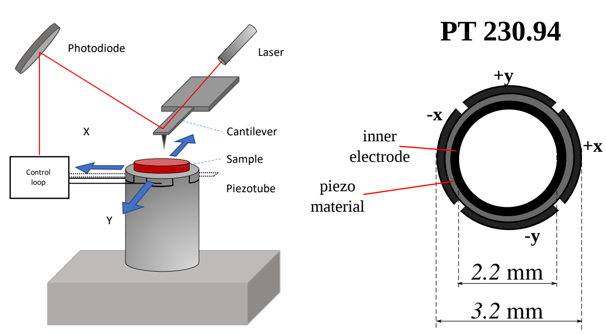

Fig. 1. Atomic Force Microscopy (left). The piezoelectric tube (right).

This actuator provides the high positioning resolution and the large bandwidth necessary for the trajectory control during scanning processes. The active part situated at the tip of the flexible tube is composed of three concentric layers: piezo material in between two cylindric electrodes (Figure 1 (right)). The deformation of the active material subject to an external voltage results in an torque applied at the extremity of the tube.

We consider the motion of the tube in one direction. In this case the structure of the system behaves as a clamped-free beam, represented by the Timoshenko beam model and actuated through boundary control stemming from the piezo-electric action at the tip of the beam. By choosing as state variables the energy variables, namely the shear displacement $x_1(t) = \frac{\partial w}{\partial z}(\cdot,t) - \phi(\cdot,t)$, the transverse momentum distribution $x_2(t) = \rho\frac{\partial w}{\partial t}(\cdot,t)$, the angular displacement $x_3(t) = \frac{\partial \phi}{\partial z}(\cdot,t)$ and the angular momentum distribution $x_4(t) = I_\rho \frac{\partial \phi}{\partial t}(\cdot,t)$ for $t \geq 0$, where $w(z,t)$ is the transverse displacement and $\phi(z,t)$ the rotation angle of the beam, the port-Hamiltonian model of the uncontrolled Timoshenko beam has the form (1a)–(1b) with $\mathcal{H}(\cdot) \equiv \operatorname{diag}\left(K, \frac{1}{\rho}, EI, \frac{1}{I_\rho}\right) \in \mathbb{R}^4$,

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $G_0 = \operatorname{diag}(0, b_w, 0, b_\phi)$ [13]. Here ρ, I_ρ, E, I and K are the mass per unit length, the angular moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively, b_w, b_ϕ the frictious coefficients. From Definition II.1 considering that $N = 1$ and $Q = P_1$ we get

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \\ K\left(\frac{\partial w}{\partial z}(b) - \phi(b)\right) - K\left(\frac{\partial w}{\partial z}(a) - \phi(a)\right) \\ \frac{\partial \phi}{\partial t}(b) - \frac{\partial \phi}{\partial t}(a) \\ EI\frac{\partial \phi}{\partial z}(b) - EI\frac{\partial \phi}{\partial z}(a) \\ K\left(\frac{\partial w}{\partial z}(b) - \phi(b)\right) + K\left(\frac{\partial w}{\partial z}(a) - \phi(a)\right) \\ \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ EI\frac{\partial \phi}{\partial z}(b) + EI\frac{\partial \phi}{\partial z}(a) \\ \frac{\partial \phi}{\partial t}(b) + \frac{\partial \phi}{\partial t}(a) \end{bmatrix}$$

The beam is clamped at point a, i.e., $\frac{1}{\rho}x_2(a,t) = \frac{1}{I_\rho}x_4(a,t) = 0$ for $t \geq 0$ and free/actuated at point b, i.e., $Kx_1(b,t) = 0$ and $EIx_3(b,t) = u(t)$ for $t \geq 0$. The angular velocity $\frac{\partial \phi}{\partial t}(b,t)$ at the tip of the beam is measured. The input and output of the system are then of the form (1) with

$$W_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{W} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix $W := \left[\begin{smallmatrix} W_1 \\ W_2 \end{smallmatrix}\right]$ has full rank and $W\Sigma W^T = 0$. Furthermore $\langle (W_1^T\tilde{W} + \tilde{W}^T W_1 - \Sigma)g, g\rangle = 0$ for all $g \in \mathcal{N}(W_2)$, the system is then impedance passive satisfying Assumption II.2. The system is also exponentially stable and Assumption II.3 holds. From Proposition III.2 the closed loop system has a solution and the regulation error is well defined.

We now build a controller to achieve the robust output tracking for the Piezoelectric tube model. We use the numerical values given in Table I to achieve a realistic approximation of the dynamics of the piezo actuated tube.

For the tracking we consider the reference signal

$$y_{ref}(t) = a\sin(\omega_1 t) + b\cos(\omega_2 t), \qquad a, b \in \mathbb{R}\setminus\{0\}.$$

with two pairs of frequencies $\pm\omega_k$ where $\omega_i > 0$, $k \in \{1, 2\}$. As an input disturbance signal we consider the AC 50 Hz

Beam's parameters	Value	Simulation parameters	Value
Beam length	$5\ cm$	N_f	50
Beam width	$0.3\ cm$	a	200 cm.s^{-1}
Beam thickness	$0.2\ cm$	b	100 cm.s^{-1}
Material Density	$936\ kg.m^{-3}$	c	$0.1\ N.m^{-1}$
Young's modulus	$4.14\ G.Pa$	θ	0.6
Transverse diss. coef.	$10^{-4} N.s.m^{-1}$	ω_1	10 rad.s^{-1}
		ω_2	15 rad.s^{-1}
Rotational diss. coef.	$10^{-4} N.m.s.rd^{-1}$	ω_3	50 rad.s^{-1}
		D_c	0.002
		δ_c	0.2

TABLE I
SIMULATION PARAMETERS.

noise coming from the electrical network, hence $w_{dist,2}(t) = c\sin(2\pi 50t + \theta)$ with unknown $c \in \mathbb{R}$ and $\theta \in [0, 2\pi]$. Since the piezo-actuated tube is a single-input single-output system, we can use a controller of the form (with $e(t) = y_{ref}(t) - y(t)$)

$$\dot{x}_c(t) = \begin{bmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 & -\omega_3 & 0 \end{bmatrix} x_c(t) + \begin{bmatrix} \delta_c \\ 0 \\ \delta_c \\ 0 \\ \delta_c \\ 0 \end{bmatrix} e(t)$$

$$u(t) = \delta_c \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} x_c(t) + D_c e(t)$$

on $X_c = \mathbb{R}^6$. By Theorem II.5 the controller achieves asymptotic output tracking of the reference signal $y_{ref}(t)$ if $i\omega_1$, $i\omega_2$, and $i\omega_3$ are not transmission zeros of the system, if $D_c > 0$, and if $\delta_c > 0$ is sufficiently small.

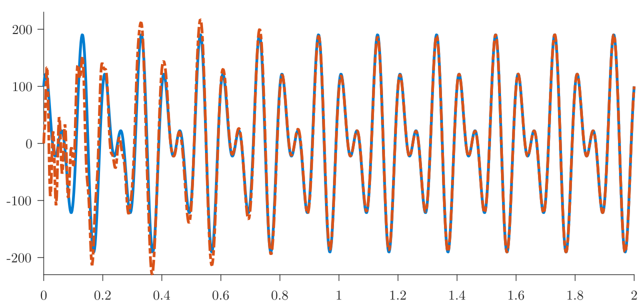

Fig. 2. Simulation results. The controlled output $y(t)$ (dashed red line) and the reference $y_{ref}(t)$ (solid blue line).

For simulation the Timoshenko beam model was discretized using a structure preserving method based on the Mixed Finite Element Method [2], [5]. We denote by N_f the number of basis elements, and consequently the full finite dimensional system has order $4N_f$. All the numerical values of the parameters related to the simulation can be found in table I. Figure 2 depicts the output tracking performance for the zero initial states of the system and the controller, and exhibits steady convergence of the tracking error to zero. Due to robustness the output tracking is achieved even if the physical parameters of the piezo actuated tube model contain uncertainties or experience changes, as long as the closed-loop system stability is preserved.

VII. CONCLUSIONS

In this paper we have proposed a constructive method for the design of impedance passive controllers for robust output regulation of port-Hamiltonian systems with boundary control and observation. Our results use Lyapunov techniques and extend previous results on this topic by removing the assumption of wellposedness, which is often highly challenging to verify for concrete PDE models. Future research topics include the design of robust controllers for nonlinear PHS.

Acknowledgement. The authors are grateful to Jukka-Pekka Humaloja for helpful discussions on PHS.

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