About dissipative and pseudo Port-Hamiltonian Formulations of irreversible Newtonian Compressible Flows

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Abstract: In this paper we consider the problem of obtaining a general port-Hamiltonian formulation of Newtonian fluids. We propose the port-Hamiltonian models to describe the energy flux of rotational three-dimensional isentropic and non-isentropic fluids, whose boundary flows and efforts can be used for control purposes or for power-preserving interconnection with other physical systems. In case of two-dimensional flows, we include the considerations about the operators associated with fluid vorticity, preserving the port-Hamiltonian structure of the models proposed.

Keywords: Port-Hamiltonian systems, Compressible Fluids, Entropy, Newtonian fluids, Vorticity

1. INTRODUCTION

In control theory, the control methods requires of models that describes the plant dynamics with sufficient precision and simplicity. In particular, in energy-based control methods, such as energy-shaping (Macchelli et al., 2017), IDA-PBC (Vu et al., 2015), observer-based control (Toledo et al., 2019), among others, are necessary models that describes the energy flux of the physical phenomena to control. In this sense, the models are commonly formulated using the port-Hamiltonian (PH) framework.

Port-Hamiltonian systems provides useful properties for the control theory, such as passivity, stability in the Lyapunov sense and a power-preserving connectivity by ports (van der Schaft and Jeltsema, 2014). For infinite-dimensional systems a PH formulation based in a Stoke-Dirac structure is proposed by Le Gorrec et al. (2005) and a extension to include dissipative effects is presented in Villegas et al. (2006). Similarly, a irreversible-PH formulation for thermodynamic systems is proposed by Ramirez et al. (2013). In this work we focus in the dynamics and thermodynamics of non-reactive Newtonian fluids. These fluid kinds are studied in different engineering areas, from biomedical systems, as the phono-respiratory modeling (Mora et al., 2018), to Fluid-Structure-Interaction problems (Cardoso-Ribeiro et al., 2017).

In literature appears different energy-based approaches to describe Newtonian fluids. However, these approaches are constrained to a kind of fluid due to the assumptions that were considered. For example, for ideal isentropic fluids, a one-dimensional port-hamiltonian models are proposed by Macchelli et al. (2017) for inviscid fluids and Kotyczka (2013) with friction dissipation, for control purposes and pipe network modeling, respectively, where the voricity effects are neglect as a consequence of the one-dimensional assumption. In this sense, a Hamiltonian model based in stream functions to describe the vorticity dynamics of two-dimensional fluid is presented in Swaters (2000), however the model is limited to potential flows. A port-Hamiltonian model of 3D irrotational fluid with dissipation is proposed by Matignon and Hélie (2013) and a general Hamiltonian model for inviscid fluid is presented by van der Schaft and Maschke (2002). For non-isentropic fluids a one-dimensional model for reactive flows is proposed by Altmann and Schulze (2017), neglecting the vorticity effects.

In this work we present an general energy-based formulation for isentropic and non-isentropic three-dimensional compressible fluids using the port-Hamiltonian framework, including the vorticity effects in the velocity field. First we develop a pseudo-PH model for non-isentropic fluids, focused in non-reactive flows. Later, we describe the treatment of terms associated with viscous tensor under an isentropic assumption for the fluid, to obtain a dissipative PH

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model. Finally, we describe the necessary considerations to conserve the PH structure of the models proposed for 2D fluids.

This paper is organized as following. In Section 2 an infinite-dimensional port-Hamiltonian model is developed for non-isentropic fluids. In Section 3, we consider an isentropic assumption, rewriting the term associated with viscous stress tensor to obtain a port-Hamiltonian model with dissipation. Section 4 describes the considerations over the operators associated with the vorticity, to conserve the same structure of port-Hamiltonian models developed in previous sections. Finally, the conclusions are presented in Section 5. The notation and mathematical identities are summarized in the Appendix.

2. NON-ISENTROPIC FLUID

In this section we describe the energy-based formulation for non-insentropic fluids. Denote by ρ, $\mathbf{v}$, s and T the density, velocity field, entropy per unit of mass and temperature of the fluid, respectively. The fluid dynamics are described by following governing equations:

$$\partial_t \rho = -div\left(\rho \mathbf{v}\right) \tag{1a}$$
$$\rho \partial_t \mathbf{v} = -\rho \mathbf{v} \cdot Grad\left(\mathbf{v}\right) - grad\left(p\right) - Div\left(\boldsymbol{\tau}\right) \tag{1b}$$
$$\rho T \partial_t s = -\rho T \mathbf{v} \cdot grad\left(s\right) - \boldsymbol{\tau} : Grad\left(\mathbf{v}\right) - div\left(\mathbf{q}\right) \tag{1c}$$

where (1a) and (1b) are the continuity and motion equations, respectively, and (1c) is the general equation of heat transfer (Landau and Lifshitz, 1987); p is the static pressure, $\boldsymbol{\tau}$ is the viscosity tensor and $\mathbf{q}$ is the heat flux. In this work, we consider non-reactive Newtonian fluids. Then, $\boldsymbol{\tau}$ and $\mathbf{q}$ are defined as:

$$\boldsymbol{\tau} = -\mu\left(Grad\left(\mathbf{v}\right) + Grad\left(\mathbf{v}\right)^T - \frac{2}{3} div\left(\mathbf{v}\right) I\right) - \kappa div\left(\mathbf{v}\right) I \tag{2}$$
$$\mathbf{q} = -K grad\left(T\right) \tag{3}$$

where μ and κ are the shear and dilatational viscosities (Bird et al., 2015), respectively, I is the identity matrix and K is a non-negative matrix that describes the thermal conductivity of the fluid (Öttinger, 2005).

In fluid dynamics the tendency to rotate is characterized by the vorticity $\boldsymbol{\omega} = \mathbf{curl}\left(\mathbf{v}\right)$ and the term $\boldsymbol{\omega} \times \mathbf{v}$, from the point of view of energy, describes the power exchange between the velocity field components given by the fluid rotation.

Definition 1. Let $\boldsymbol{\omega} = \left[\omega_1\ \omega_2\ \omega_3\right]^T$ the vorticity vector of the fluid. We define the fluid Gyroscope $G_{\boldsymbol{\omega}}$ as the skew-symmetric matrix, such that $G_{\boldsymbol{\omega}} \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$. For 3D fluids, the Gyroscope is given by:

$$G_{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4}$$

On the other hand, the term $Div\left(\mathbf{q}\right)$ in(1c) can be rewritten as

$$Div\left(\mathbf{q}\right) = T Div\left(\mathbf{q}_s\right) + \mathbf{q}_s \cdot grad\left(T\right) \tag{5}$$

where $\mathbf{q}_s$ is the entropy flux by heat conduction (Bird et al., 2015). For non-reactive fluids $\mathbf{q}_s$ is defined as

$$\mathbf{q}_s = -\frac{K}{T} grad\left(T\right) \tag{6}$$

Then, considering the Gibbs equation

$$du = -pd\left(\frac{1}{\rho}\right) + T ds \tag{7}$$

that describes the change of the specific internal energy u with respect to changes of ρ and s, the fluid enthalpy $h = u + p/\rho$ and the relationships $grad\left(\frac{1}{\rho}p\right) = \frac{1}{\rho} grad\left(p\right) + grad\left(\frac{1}{\rho}\right) p$ and $T = \partial_s u$. Then, we can rewrite the fluid dynamics in terms of the state variables and the temperature, namely:

$$\partial_t \rho = -div\left(\rho \mathbf{v}\right) \tag{8a}$$
$$\partial_t \mathbf{v} = -grad\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + h\right) - G_{\boldsymbol{\omega}} \mathbf{v} + T grad\left(s\right) - \frac{1}{\rho} Div\left(\boldsymbol{\tau}\right) \tag{8b}$$
$$\partial_t s = -\mathbf{v} \cdot grad\left(s\right) - \frac{\boldsymbol{\tau}}{\rho T} : Grad\left(\mathbf{v}\right) - \frac{\mathbf{q}_s}{\rho T} \cdot grad\left(T\right) - \frac{1}{\rho} div\left(\mathbf{q}_s\right) \tag{8c}$$

Note that entropy generation, second law of thermodynamics, is given by the following non-negative condition (Öttinger, 2005)

$$-\frac{1}{\rho T}\boldsymbol{\tau} : grad\left(\mathbf{v}\right) - \frac{\mathbf{q}_s}{\rho T} \cdot grad\left(T\right) \geq 0 \tag{9}$$

where $-\frac{1}{\rho T}\boldsymbol{\tau} : grad\left(\mathbf{v}\right)$ is the rate of entropy creation by the kinetic energy dissipated into heat by viscosity friction, and $-\frac{\mathbf{q}_s}{\rho T} \cdot grad\left(T\right)$ is the rate of entropy creation by heat flux.

2.1 Port-Hamiltonian description of non-isentropic fluids

Consider the fluid domain Ω with boundary $\partial\Omega$. The total energy of the fluid described in (8) is given by:

$$\mathcal{H} = \int_\Omega \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} + \rho u\left(\rho, s\right) \tag{10}$$

Then, the fluid efforts $\mathbf{e} = \left[e_\rho\ \mathbf{e}_\mathbf{v}^T\ e_s\right]^T$ are given by the variational derivative of the energy, namely

$$\begin{bmatrix} e_\rho \\ \mathbf{e}_\mathbf{v} \\ e_s \end{bmatrix} = \begin{bmatrix} \delta_\rho \mathcal{H} \\ \delta_\mathbf{v} \mathcal{H} \\ \delta_s \mathcal{H} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + h \\ \rho \mathbf{v} \\ \rho T \end{bmatrix} \tag{11}$$

Note that $\delta_\rho \mathcal{H} = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + u + \rho \partial_\rho u$. Given the relationship $p = \rho^2 \partial_\rho u$, we obtain $\delta_\rho \mathcal{H} = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + u + p/\rho = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + h$. Using (11), the fluid dynamics in (8) can be related with energy through the fluid efforts, i.e.,

$$\partial_t \rho = -div\left(\mathbf{e}_\mathbf{v}\right) \tag{12a}$$
$$\partial_t \mathbf{v} = -grad\left(e_\rho\right) - \frac{G_{\boldsymbol{\omega}}}{\rho}\mathbf{e}_\mathbf{v} + grad\left(s\right)\frac{e_s}{\rho} - \frac{1}{\rho} Div\left(\frac{\boldsymbol{\tau}}{\rho T} e_s\right) \tag{12b}$$
$$\partial_t s = -grad\left(s\right) \cdot \frac{\mathbf{e}_\mathbf{v}}{\rho} - \frac{1}{\rho T}\boldsymbol{\tau} : grad\left(\frac{\mathbf{e}_\mathbf{v}}{\rho}\right) + \frac{1}{\rho T}\left\|grad\left(\frac{e_s}{\rho}\right)\right\|^2_{\frac{K}{T}} + \frac{1}{\rho} div\left(\frac{K}{T} grad\left(\frac{e_s}{\rho}\right)\right) \tag{12c}$$

To obtain the port-Hamiltonian formulation, is necessary set the interconnections between the components of the fluid dynamics. In the case of the velocity field and the entropy, they are interconnected through the operator $\mathcal{J}_{\boldsymbol{\tau}}$ and the corresponding adjoint $\mathcal{J}_{\boldsymbol{\tau}}^*$ in the effort space. These operators are defined as follows:

Lemma 1. Let $\boldsymbol{\tau}$ be a symmetric second order tensor and $\mathcal{J}_{\boldsymbol{\tau}} = grad\,(s)\left(\frac{\cdot}{\rho}\right) - \frac{1}{\rho}Div\left(\frac{\boldsymbol{\tau}}{\rho T}\cdot\right)$ an operator on the entropy effort e_s. Then, the adjoint operator $\mathcal{J}_{\boldsymbol{\tau}}^*$ in the effort space of the fluid is given by $\mathcal{J}_{\boldsymbol{\tau}}^* = grad\,(s)\cdot\left(\frac{\cdot}{\rho}\right) + \frac{\boldsymbol{\tau}}{\rho T} : Grad\left(\frac{\cdot}{\rho}\right)$, such that

$$\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s\rangle_\Omega - \langle e_s, \mathcal{J}_{\boldsymbol{\tau}}^* \mathbf{e_v}\rangle_\Omega = -\int_{\partial\Omega} \boldsymbol{\tau} : \left(\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T\right) \quad (13)$$

Proof. Consider the inner product

$$\begin{aligned}\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s\rangle_\Omega &= \int_\Omega \mathbf{e_v}\cdot\mathcal{J}_{\boldsymbol{\tau}} e_s \\ &= \int_\Omega \mathbf{e_v}\cdot\left[grad\,(s)\,\frac{e_s}{\rho} - \frac{1}{\rho}Div\left(\frac{\boldsymbol{\tau}}{\rho T}e_s\right)\right]\end{aligned}$$

For a symmetric tensor, the formal adjoint of Div is given by $-Grad$ (Brugnoli et al., 2019). Then, using the identity (A.2), where $\boldsymbol{\sigma} = \boldsymbol{\tau} e_s/\rho T$ and $\mathbf{u} = \mathbf{e_v}/\rho$, the inner product in previous equation can be rewritten as:

$$\begin{aligned}\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s\rangle_\Omega &= \int_\Omega e_s\left[grad\,(s)\cdot\frac{\mathbf{e_v}}{\rho} + \frac{\boldsymbol{\tau}}{\rho T} : Grad\left(\frac{\mathbf{e_v}}{\rho}\right)\right] \\ &\quad - \int_\Omega div\left(\frac{\boldsymbol{\tau} e_s}{\rho T}\cdot\frac{\mathbf{e_v}}{\rho}\right) \\ &= \langle e_s, \mathcal{J}_{\boldsymbol{\tau}}^*\mathbf{e_v}\rangle_\Omega - \int_{\partial\Omega}\left[\frac{\boldsymbol{\tau} e_s}{\rho T}\cdot\frac{\mathbf{e_v}}{\rho}\right]\cdot\mathbf{n} \quad (14)\end{aligned}$$

where $\mathcal{J}_{\boldsymbol{\tau}}^* = grad\,(s)\cdot\left(\frac{\cdot}{\rho}\right) + \frac{\boldsymbol{\tau}}{\rho T} : Grad\left(\frac{\cdot}{\rho}\right)$ and $\mathbf{n}$ is the normal outward unitary vector to the boundary $\partial\Omega$. Finally, using the mathematical identity $(\boldsymbol{\tau}\cdot\mathbf{v})\cdot\mathbf{n} = \boldsymbol{\tau} : \mathbf{v}\mathbf{n}^T$ we obtain

$$\langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s\rangle_\Omega - \langle e_s, \mathcal{J}_{\boldsymbol{\tau}}^* \mathbf{e_v}\rangle_\Omega = -\int_{\partial\Omega} \boldsymbol{\tau} : \left(\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T\right) \quad (15)$$

where $\left(\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T\right)$ is the tangential projection of the velocity field. □

Lemma 2. Let $\mathcal{J}_{\mathbf{q}}$ be a operator on space of the entropy effort e_s, defined as

$$\mathcal{J}_{\mathbf{q}} = \mathcal{Q}_T - \mathcal{G}_T^* S_T \mathcal{G}_T, \quad (16)$$

where $S_T = \frac{K}{T} \geq 0$, $\mathcal{Q}_T = \frac{1}{\rho T}\left\|\frac{\cdot}{\rho}\right\|_{S_T}^2$ describes the entropy creation by the heat flux, such that $\mathcal{Q}_T e_s \geq 0, \forall e_s$, and the operator $\mathcal{G}_T^* = \frac{1}{\rho}div$ is the formal adjoint of $\mathcal{G}_T = -grad\left(\frac{\cdot}{\rho}\right)$. Then, the rate of entropy addition by heat flux can be expressed as

$$-\frac{1}{\rho T}div\,(\mathbf{q}) = \mathcal{J}_{\mathbf{q}} e_s \quad (17)$$

satisfying

$$\langle e_s, \mathcal{J}_{\mathbf{q}} e_s\rangle_\Omega = -\int_{\partial\Omega} T\,(\mathbf{q}_s\cdot\mathbf{n}) \quad (18)$$

Proof. Note that $\frac{1}{\rho T}div\,(\mathbf{q}) = \frac{\mathbf{q}_s}{\rho T}\cdot grad\,(T) + \frac{1}{\rho}div\,(\mathbf{q}_s)$. Defining $S_T = K/T$, from (12c) we obtain

$$-\frac{\mathbf{q}_s}{\rho T}\cdot grad\,(T) = \frac{1}{\rho T}\left\|grad\left(\frac{e_s}{\rho}\right)\right\|_{S_T}^2 \quad (19)$$

$$-\frac{1}{\rho}div\,(\mathbf{q}_s) = \frac{1}{\rho}div\left(S_T grad\left(\frac{e_s}{\rho}\right)\right) \quad (20)$$

Given that the formal adjoint of divergence is minus the gradient, it is easy to proof that $\mathcal{G}_T^* = \frac{1}{\rho}div\,()$ is the formal adjoint of operator $\mathcal{G}_T = -grad\left(\frac{\cdot}{\rho}\right)$. Then, from (19) and (20) the entropy addition by heat flux can be expressed as

$$-\frac{1}{\rho T}div\,(\mathbf{q}) = \left(\mathcal{Q}_T - \mathcal{G}_T^* S_T \mathcal{G}_T\right) e_s = \mathcal{J}_{\mathbf{q}} e_s \quad (21)$$

With respect to the inner product in the left-hand side of (18), we obtain

$$\begin{aligned}\langle e_s, \mathcal{J}_{\mathbf{q}} e_s\rangle_\Omega &= \int_\Omega \frac{e_s}{\rho T}\left\|grad\left(\frac{e_s}{\rho}\right)\right\|_{S_T}^2 \\ &= +\int_\Omega \frac{e_s}{\rho}div\left(S_T grad\left(\frac{e_s}{\rho}\right)\right) \\ &= -\int_\Omega \mathbf{q}_s\cdot grad\left(\frac{e_s}{\rho}\right) + \frac{e_s}{\rho}div\,(\mathbf{q}_s) \\ &= -\int_\Omega div\left(\frac{e_s}{\rho}\mathbf{q}_s\right) = -\int_\Omega div\,(T\mathbf{q}_s) \quad (22)\end{aligned}$$

Finally, applying the Gauss divergence theorem the relationship in (18) is obtained. □

Proposition 1. Let be a non-isentropic Newtonian compressible fluid, the governing equations in (8) can be expressed as pseudo infinite-dimensional port-Hamiltonian system

$$\partial_t \boldsymbol{x} = \mathcal{J}\mathbf{e} \quad (23)$$

where $\boldsymbol{x} = \begin{bmatrix}\rho & \mathbf{v}^T & s\end{bmatrix}^T$ is the state vector, $\mathbf{e} = \begin{bmatrix}e_\rho & \mathbf{e_v}^T & e_s\end{bmatrix}^T$ is the fluid effort vector described in (11) and $\mathcal{J}$ is a operator given by

$$\mathcal{J} = \begin{bmatrix} 0 & -div & 0 \\ -grad & -\frac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{J}_{\boldsymbol{\tau}} \\ 0 & -\mathcal{J}_{\boldsymbol{\tau}}^* & \mathcal{J}_{\mathbf{q}}\end{bmatrix} \quad (24)$$

satisfying

$$\dot{\mathcal{H}} = \langle \mathbf{e}_\partial, \mathbf{f}_\partial\rangle_{\partial\Omega} \quad (25)$$

where $\langle \mathbf{e}_\partial, \mathbf{f}_\partial\rangle_{\partial\Omega}$ is the power supplied through the boundary $\partial\Omega$ and the boundary flows $\mathbf{f}_\partial$ and efforts $\mathbf{e}_\partial$ are given by

$$\mathbf{f}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \boldsymbol{\tau}|_{\partial\Omega} \\ T|_{\partial\Omega}\end{bmatrix} \text{ and } \mathbf{e}_\partial = \begin{bmatrix} -(\mathbf{e_v}\cdot\mathbf{n})|_{\partial\Omega} \\ -\mathbf{v}\mathbf{n}^T|_{\partial\Omega} \\ -(\mathbf{q}_s\cdot\mathbf{n})|_{\partial\Omega}\end{bmatrix}.$$

Proof. The fluid governing equations in (8) can be rewritten as function of the fluid efforts described in (11), as shown in (12). Then, using the operators defined in Lemmas 1 and 2 we obtain

$$\underbrace{\begin{bmatrix}\partial_t\rho \\ \partial_t\mathbf{v} \\ \partial_t s\end{bmatrix}}_{\partial_t\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 & -div & 0 \\ -grad & -\frac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{J}_{\boldsymbol{\tau}} \\ 0 & -\mathcal{J}_{\boldsymbol{\tau}}^* & \mathcal{Q}_T - \mathcal{G}_T^* S_T \mathcal{G}_T\end{bmatrix}}_{\mathcal{J}} \underbrace{\begin{bmatrix} e_\rho \\ \mathbf{e_v} \\ e_s\end{bmatrix}}_{\mathbf{e}} \quad (26)$$

The energy balance of this system is given by:

$$\dot{\mathcal{H}} = \langle \mathbf{e}, \mathcal{J}\mathbf{e}\rangle_\Omega = \int_\Omega \mathbf{e}\cdot\mathcal{J}\mathbf{e}$$
$$= -\int_\Omega e_\rho div(\mathbf{e_v}) + \mathbf{e_v}\cdot grad(e_\rho) - \frac{\mathbf{e_v}}{\rho}\cdot G_{\boldsymbol{\omega}}\mathbf{e_v}$$
$$+ \langle \mathbf{e_v}, \mathcal{J}_{\boldsymbol{\tau}} e_s\rangle_\Omega - \langle e_s, \mathcal{J}^*_{\boldsymbol{\tau}}\mathbf{e_v}\rangle_\Omega + \langle e_s, \mathcal{J}_{\mathbf{q}} e_s\rangle_\Omega \quad (27)$$

Note that given the skew-symmetry property of the gyroscope $\frac{\mathbf{e_v}}{\rho}G_{\boldsymbol{\omega}}\mathbf{e_v} = 0$. Then, using (13) and (18), the equation (27) can be rewritten as

$$\dot{\mathcal{H}} = -\int_{\partial\Omega} e_\rho(\mathbf{e_v}\cdot\mathbf{n}) + \boldsymbol{\tau} : \left[\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T\right] + T[\mathbf{q}_s\cdot\mathbf{n}] \quad (28)$$

Defining the boundary flows and efforts as:

$$\mathbf{e}_\partial = \begin{bmatrix} e_\rho|_{\partial\Omega} \\ \boldsymbol{\tau}|_{\partial\Omega} \\ \left.\frac{e_s}{\rho}\right|_{\partial\Omega} \end{bmatrix} \qquad \mathbf{f}_\partial = \begin{bmatrix} -(\mathbf{e_v}\cdot\mathbf{n})|_{\partial\Omega} \\ \left.-\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T\right|_{\partial\Omega} \\ -(\mathbf{q}_s\cdot\mathbf{n})|_{\partial\Omega} \end{bmatrix} \quad (29)$$

where $\mathbf{e_v}\cdot\mathbf{n}$ is the normal projection of the momentum density, $\frac{\mathbf{e_v}}{\rho}\mathbf{n}^T$ is the tangential projection of the velocity field. Then, the rate of change of the total energy is given by:

$$\dot{\mathcal{H}} = \langle e_\partial, f_\partial\rangle_{\partial\Omega} \quad (30)$$

Remark 1. Note that the effort $e_s = \rho T$ appear in operators $\mathcal{J}_{\boldsymbol{\tau}}$ and $\mathcal{J}_{\mathbf{q}}$ (see Lemmas 1 and 2), then, $\mathcal{J} = \mathcal{J}(\mathbf{x}, \mathbf{e})$. This implies that the system in (23) does not generate a Stoke-Dirac structure.

3. ISENTROPIC FLUID

In this section we describe the port-Hamiltonian formulation for ideal isentropic fluids. The governing equations are reduced to the continuity and motion equations described in (1a) and (1b), respectively. Similarly, the Gibbs equation is reduced to

$$du = -pd\left(\frac{1}{\rho}\right) \quad (31)$$

In isentropic fluids the internal energy is a function that depends only on the density, as shown in (31). Then, the total energy is described as

$$\mathcal{H} = \int_\Omega \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho u(\rho) \quad (32)$$

and the fluid efforts $\mathbf{e} = \left[e_\rho \ \mathbf{e}_\mathbf{v}^T\right]^T$ are given by

$$\begin{bmatrix} e_\rho \\ \mathbf{e_v}\end{bmatrix} = \begin{bmatrix} \delta_\rho\mathcal{H} \\ \delta_\mathbf{v}\mathcal{H}\end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + h \\ \rho\mathbf{v}\end{bmatrix} \quad (33)$$

Then, the fluid dynamics can be expressed as

$$\partial_t\rho = -div(\mathbf{e_v}) \quad (34a)$$
$$\partial_t\mathbf{v} = -grad(e_\rho) - \frac{1}{\rho}G_{\boldsymbol{\omega}}\mathbf{e_v} - \frac{1}{\rho}Div(\boldsymbol{\tau}) \quad (34b)$$

The term $\frac{1}{\rho}div(\boldsymbol{\tau})$ in (34b) represents the friction effects over the velocity field, given the fluid viscosity. In previous section, the velocity field and the entropy of the fluid are interconnected through the heat generated by this friction, by means of the operators $\mathcal{J}_{\boldsymbol{\tau}}$ and $\mathcal{J}^*_{\boldsymbol{\tau}}$. In this case, given the isentropic assumption, we can interpret $\frac{1}{\rho}div(\boldsymbol{\tau})$ as the dissipation associated with heat generation as a consequence of the viscosity friction of the fluid. According to Villegas et al. (2006), in infinite-dimensional port-Hamiltonian systems the dissipative terms are expressed as $\mathcal{G}^*S\mathcal{G}\mathbf{e}$ where $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$, and $S = S^T \geq 0$. Then, for isentropic fluids $\frac{1}{\rho}div(\boldsymbol{\tau})$ can be expressed as a port-Hamiltonian dissipation term, as shown in the following Lemma. .

Lemma 3. Let be a viscous Newtonian fluid. Defining the operators $\mathcal{G}_r = \mathbf{curl}\left(\frac{\cdot}{\rho}\right)$ and $\mathcal{G}_d = div\left(\frac{\cdot}{\rho}\right)$ and the corresponding adjoints $\mathcal{G}_r^* = \frac{1}{\rho}\mathbf{curl}(\cdot)$ and $\mathcal{G}_d^* = -\frac{1}{\rho}grad(\cdot)$. Then, the rate of velocity addition associated with the viscous tensor, $\frac{1}{\rho}div(\boldsymbol{\tau})$, can be expressed as a dissipative port-Hamiltonian terms associated with the velocity effort, namely,

$$\frac{1}{\rho}Div(\boldsymbol{\tau}) = \mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}}\mathbf{e_v} \quad (35)$$

where $\mathcal{G}^*_{\boldsymbol{\tau}} = [\mathcal{G}_r^* \ \mathcal{G}_d^*]$, $S_{\boldsymbol{\tau}} = \begin{bmatrix}\mu & 0 \\ 0 & \frac{4}{3}\mu+\kappa\end{bmatrix}$ and $\mathcal{G}_{\boldsymbol{\tau}} = \begin{bmatrix}\mathcal{G}_r \\ \mathcal{G}_d\end{bmatrix}$.

Proof. The viscosity tensor of Newtonian fluids is described in (2). Then, applying the identities (A.4)-(A.6) we obtain

$$\frac{1}{\rho}Div(\boldsymbol{\tau}) = \frac{1}{\rho}Div\left(-\mu\left(Grad(\mathbf{v}) + Grad(\mathbf{v})^T\right)\right)$$
$$+ \frac{1}{\rho}Div\left(\left(\frac{2}{3}\mu - \kappa\right)div(\mathbf{v})I\right)$$
$$= \frac{1}{\rho}\mathbf{curl}(\mu\mathbf{curl}(\mathbf{v})) - \frac{1}{\rho}grad\left(\left(\frac{4}{3}\mu+\kappa\right)div(\mathbf{v})\right)$$
$$= \frac{1}{\rho}\mathbf{curl}\left(\mu\mathbf{curl}\left(\frac{\mathbf{e_v}}{\rho}\right)\right) - \frac{1}{\rho}grad\left(\hat{\mu}div\left(\frac{\mathbf{e_v}}{\rho}\right)\right) \quad (36)$$

Given that the curl operator is self-adjoint and the adjoint of divergence is minus the gradient. Then, it is easy to check that $\mathcal{G}_r^* = \frac{1}{\rho}\mathbf{curl}$ is the formal adjoint of $\mathcal{G}_r = \mathbf{curl}\left(\frac{\cdot}{\rho}\right)$ and $\mathcal{G}_d^* = -\frac{1}{\rho}grad()$ is the adjoint of $\mathcal{G}_d = div\left(\frac{\cdot}{\rho}\right)$. Thus, the equation (36) can be expressed as the sum of 2 dissipative terms, namely

$$\frac{1}{\rho}div(\boldsymbol{\tau}) = \mathcal{G}_r^*\mu\mathcal{G}_r\mathbf{e_v} + \mathcal{G}_d^*\left(\frac{4}{3}\mu+\kappa\right)\mathcal{G}_d\mathbf{e_v} \quad (37)$$
$$= \underbrace{[\mathcal{G}_r^* \ \mathcal{G}_d^*]}_{\mathcal{G}^*_{\boldsymbol{\tau}}}\underbrace{\begin{bmatrix}\mu & 0 \\ 0 & \frac{4}{3}\mu+\kappa\end{bmatrix}}_{S_{\boldsymbol{\tau}}}\underbrace{\begin{bmatrix}\mathcal{G}_r \\ \mathcal{G}_d\end{bmatrix}}_{\mathcal{G}_{\boldsymbol{\tau}}}\mathbf{e_v} \quad (38)$$

where $S_{\boldsymbol{\tau}}$ satisfies the positive condition $S_{\boldsymbol{\tau}} = S^T_{\boldsymbol{\tau}} \geq 0$. □

Note that $\mathcal{G}^*_{\boldsymbol{\tau}}S_{\boldsymbol{\tau}}\mathcal{G}_{\boldsymbol{\tau}}\mathbf{e_v}$ can be expressed as the sum of two dissipations, as shown in (37). The first dissipation, $\mathcal{G}_r^*\mu\mathcal{G}_r\mathbf{e_v}$, describes the losses associated with the frictions generated by the fluid rotation or vorticity, and it is equal to 0 for under a irrotational assumption. The second dissipation, $\mathcal{G}_d^*(\frac{4}{3}\mu+\kappa)\mathcal{G}_d\mathbf{e_v}$, describes the losses associated with the frictions generated by the dilation or compression of the fluid, and it is equal to 0 under incompressible assumption.

Proposition 2. Let be an isentropic Newtonian fluid in a domain Ω with boundary $\partial\Omega$. Considering the vorticity as a phenomena strictly intern, the governing equations can be expressed as the following port-Hamiltonian system with dissipation:

$$\partial_t \boldsymbol{x} = (\mathcal{J} - \mathcal{G}^* S \mathcal{G})\, \mathbf{e} \tag{39}$$

where $\boldsymbol{x} = \begin{bmatrix}\rho & \mathbf{v}^T\end{bmatrix}^T$ is the state vector, $\mathbf{e} = \begin{bmatrix}e_\rho & \mathbf{e}_\mathbf{v}^T\end{bmatrix}^T$ are the fluid efforts, and

$$\mathcal{J} = \begin{bmatrix} 0 & -div \\ -\mathbf{grad} & -\frac{G_{\boldsymbol{\omega}}}{\rho} \end{bmatrix}, \mathcal{G}^* = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{G}^*_{\boldsymbol{\tau}} \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 0 & S_{\boldsymbol{\tau}} \end{bmatrix}, \mathcal{G} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix}$$

Satisfying the following relationship for the rate of change of the energy:

$$\frac{d\mathcal{H}}{dt} \le \int_{\partial\Omega} \mathbf{f}_\partial \cdot \mathbf{e}_\partial \tag{40}$$

where $\mathbf{f}_\partial = (e_\rho - e_d)\,|_{\partial\Omega}$ and $\mathbf{e}_\partial = -\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n})\,|_{\partial\Omega}$ are the boundary flow and effort, respectively, with e_d as the effort associated with the dissipation by dilatation and $\mathbf{n}$ the normal unitary outward vector to the boundary.

Proof. Considering the Lemma 3, the dynamics in (34) can be rewritten as

$$\partial_t \rho = -div\,(\mathbf{e}_\mathbf{v}) \tag{41a}$$

$$\partial_t \mathbf{v} = -grad\,(e_\rho) - \frac{1}{\rho} G_{\boldsymbol{\omega}} \mathbf{e}_\mathbf{v} - \mathcal{G}^*_{\boldsymbol{\tau}} S_{\boldsymbol{\tau}} \mathcal{G}_{\boldsymbol{\tau}} \mathbf{e}_\mathbf{v} \tag{41b}$$

Thus, regrouping terms the governing equations can be expressed as

$$\partial_t \begin{bmatrix} \rho \\ \mathbf{v} \end{bmatrix} = \left(\begin{bmatrix} 0 & -div \\ -\mathbf{grad} & -\frac{G_{\boldsymbol{\omega}}}{\rho} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathcal{G}^*_{\boldsymbol{\tau}} S_{\boldsymbol{\tau}} \mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix} \right) \begin{bmatrix} e_\rho \\ \mathbf{e}_\mathbf{v} \end{bmatrix}$$

Rewriting the term $\begin{bmatrix} 0 \\ \mathcal{G}_{\boldsymbol{\tau}} S_{\boldsymbol{\tau}} \mathcal{G}_{\boldsymbol{\tau}} \end{bmatrix}$ we obtain the port-Hamiltonian formulation described in (39).

On the other hand, for the rate of change of the total energy we obtain

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \int_\Omega \mathbf{e} \cdot \partial_t \boldsymbol{x} = \int_\Omega \mathbf{e} \cdot \mathcal{J}\mathbf{e} - \mathbf{e} \cdot \mathcal{G}^* S \mathcal{G} \mathbf{e} \\ &= -\int_{\partial\Omega} e_\rho\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) - \int_\Omega \mathbf{e} \cdot \mathcal{G}^* S \mathcal{G} \mathbf{e} \end{aligned} \tag{42}$$

Defining $\mathbf{f}_R = \begin{bmatrix}\mathbf{f}_r^T & f_d\end{bmatrix}^T$ and $\mathbf{e}_R = \begin{bmatrix}\mathbf{e}_r^T & e_d\end{bmatrix}^T$ as the flows and efforts associated with the dissipations, where $\mathbf{f}_r = \mathcal{G}_r \mathbf{e}_\mathbf{v}$, $f_d = \mathcal{G}_d \mathbf{e}_\mathbf{v}$, $\mathbf{e}_r = \mu \mathbf{f}_r$ and $e_d = (\frac{4}{3}\mu + \kappa) f_d$, we obtain $\mathbf{e} \cdot \mathcal{G}^* S \mathcal{G} \mathbf{e} = \mathbf{e}_\mathbf{v} \cdot \mathcal{G}^*_r \mathbf{e}_r + \mathbf{e}_\mathbf{v} \cdot \mathcal{G}^*_d e_d$. Then, considering the vorticity equal to 0 in the boundaries, the equation (42) can be rewritten as

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= -\int_{\partial\Omega} e_\rho\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) - \int_\Omega \mathbf{e}_\mathbf{v} \cdot \mathcal{G}^*_r \mathbf{e}_r + \mathbf{e}_\mathbf{v} \cdot \mathcal{G}^*_d e_d \\ &= -\int_{\partial\Omega} e_\rho\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) - \int_\Omega \mathcal{G}_r \mathbf{e}_\mathbf{v} \cdot \mathbf{e}_r + \mathcal{G}_d \mathbf{e}_\mathbf{v} \cdot e_d \\ &\quad + \int_{\partial\Omega} \frac{e_d}{\rho}\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) \\ &= -\langle \mathcal{G}_{\boldsymbol{\tau}} \mathbf{e}_R, \mathbf{e}_R \rangle_\Omega - \int_{\partial\Omega} \left(e_\rho - \frac{e_d}{\rho} \right) (\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) \\ &= -\,S_{\boldsymbol{\tau}} \langle \mathbf{f}_R, \mathbf{f}_R \rangle_\Omega - \int_{\partial\Omega} \left(e_\rho - \frac{e_d}{\rho} \right) (\mathbf{e}_\mathbf{v} \cdot \mathbf{n}) \end{aligned} \tag{43}$$

Given that $S_{\boldsymbol{\tau}} \ge 0$, then, from (43) we obtain the inequality $\frac{d}{dt}\mathcal{H} \le \int_{\partial\Omega} \mathbf{f}_\partial \cdot \mathbf{e}_\partial$, where $\mathbf{f}_\partial = \left(e_\rho - \frac{e_d}{\rho} \right) |_{\partial\Omega}$ and $\mathbf{e}_\partial = -\,(\mathbf{e}_\mathbf{v} \cdot \mathbf{n})\,|_{\partial\Omega}$. □

Note that, considering different asumption the model of fluid proposed in (39) converge to port-Hamiltonian models of isentropic fluids described in previous works. For example, under an irrotational assumption operators $G_{\boldsymbol{\omega}}$, $\mathcal{G}_r$ and $\mathcal{G}^*_r$ disappear, obtaining the fluid model described by Matignon and Hélie (2013). For inviscid fluids, the operator $\mathcal{G}^* S \mathcal{G}$ is equal to 0. Then, the port-Hamiltonian system in (39) is equivalent to the model proposed in van der Schaft and Maschke (2002).

4. TWO-DIMENSIONAL FLUIDS

The cross product and the curl operator are three-dimensional mathematical operations. Thus, for two-dimensional fluids we need to define appropriately the terms associated with these operations.

Denote by $\{x_1, x_2\}$ the variables associated with the axes of a two-dimensional velocity field $\mathbf{v} = \begin{bmatrix}v_1 & v_2\end{bmatrix}^T$. Then, the vorticity w is a scalar perpendicular to the plane $x_1 \times x_2$, defined as $\omega = -\partial_{x_2} v_1 + \partial_{x_1} v_2$ [ref]. For convenience we rewrite w as

$$w = -div\,(W\mathbf{v}) \tag{44}$$

where $W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation matrix.

Then, the Gyroscope in a two-dimensional velocity field is defined as (Carodo-Ribeiro, 2016):

$$G_{\boldsymbol{\omega}} = \omega W = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \tag{45}$$

On the other hand, with respect to the dissipative terms of the viscosity tensor, operators $\mathcal{G}_r$ and $\mathcal{G}^*_r$ for two-dimensional fluids are defined as:

$$\mathcal{G}_r = \begin{bmatrix} -\partial_{x_2} & \partial_{x_1} \end{bmatrix} \frac{\cdot}{\rho} = -div \left(W \frac{\cdot}{\rho} \right) \tag{46}$$

$$\mathcal{G}^*_r = \frac{1}{\rho} \begin{bmatrix} \partial_{x_2} \\ -\partial_{x_1} \end{bmatrix} = \frac{1}{\rho} W^T grad\,() \tag{47}$$

Thus, given the operator definitions in (45)-(47), the port-Hamiltonian formulations in Propositions 1 and 2 can be used to describe for non-isentropic and isentropic two-dimensional fluids, respectively.

In the case of one-dimensional fluids, all terms associated with the vorticity disappear, and $div = grad = \partial_{x_1}$. Under these conditions the fluid model proposed in (23) is equivalent to the model described in Altmann and Schulze (2017), neglecting the reactive part. Similarly, the model described in Proposition 2 converge to the models used in Kotyczka (2013); Macchelli et al. (2017).

5. CONCLUSION

A pseudo-PH formulation for non-isentropic Newtonian fluid in a three-dimensional space was presented for non-reactive flows. Similarly, under an isentropic assumption the transformation of kinetic energy into heat by viscosity friction is described as dissipative terms associated with fluid rotation and compression, obtaining a dissipative-PH model for three-dimensional isentropic fluids. These models presents a general formulation for non-reactive compressible flows, i.e., a description for inviscid or irrotational fluids can be derived from the proposed models under the appropriated assumptions in the PH structure. Moreover, we describes the necessary considerations on the operators of purposed models to use the same PH structure for two-dimensional and one-dimensional fluid, obtaining equivalent formulations for fluids models in literature.

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Appendix A. NOMENCLATURE AND USEFUL IDENTITIES

The nomenclature used in this paper is summarized in the next Table.

Table A.1. Nomenclature

Symbol	Description
T	Transpose
$\mathbf{v}\cdot\mathbf{u}$	Scalar product between 2 vectors, $\mathbf{v}^T\mathbf{u}$.
$\mathbf{v}\times\mathbf{u}$	Cross product
$\boldsymbol{\tau}:\boldsymbol{\sigma}$	Scalar product between 2 tensors, $Tr\left(\boldsymbol{\tau}^T\boldsymbol{\sigma}\right)$.
$div\left(\mathbf{u}\right)$	Divergence of vector $\mathbf{u}$.
$grad\left(f\right)$	Gradient of scalar f.
$\mathbf{curl}\left(\mathbf{u}\right)$	Curl or rotational of $\mathbf{u}$.
$Grad\left(\mathbf{u}\right)$	Gradient of vector $\mathbf{u}$.
$Div\left(\boldsymbol{\sigma}\right)$	Divergence of tensor $\boldsymbol{\sigma}$.
$\|\mathbf{v}\|_X^2$	Square of the weighted Euclidean norm, $\mathbf{v}^TX^T\mathbf{v}$.
$\int_\Omega f$	Integral in domain Ω, $\int_\Omega fd\Omega$
$\int_{\partial\Omega} f$	Integral in boundary $\partial\Omega$, $\int_{\partial\Omega} f\partial\Omega$

Additionally, the set of mathematical identities used in this work are described below:

$$\mathbf{u}\cdot Grad\left(\mathbf{u}\right)=grad\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+\mathbf{curl}\left(\mathbf{u}\right)\times\mathbf{u} \tag{A.1}$$

$$\boldsymbol{\sigma}:Grad\left(\mathbf{u}\right)=div\left(\boldsymbol{\sigma}\cdot\mathbf{u}\right)-\mathbf{u}\cdot Div\left(\boldsymbol{\sigma}\right) \tag{A.2}$$

$$div\left(f\mathbf{u}\right)=grad\left(f\right)\cdot\mathbf{u}+f div\left(\mathbf{u}\right) \tag{A.3}$$

$$Div\left(Grad\left(\mathbf{u}\right)\right)=grad\left(div\left(\mathbf{u}\right)\right)-\mathbf{curl}\left(\mathbf{curl}\left(\mathbf{u}\right)\right) \tag{A.4}$$

$$Div\left(Grad\left(\mathbf{u}\right)^T\right)=grad\left(div\left(\mathbf{u}\right)\right) \tag{A.5}$$

$$Div\left(div\left(\mathbf{u}\right)I\right)=grad\left(div\left(\mathbf{u}\right)\right) \tag{A.6}$$

where f is a scalar, $\mathbf{u}$ is a vector and $\boldsymbol{\sigma}$ is a symmetric second order tensor.

$$\int_\Omega div\left(\mathbf{u}\right)=\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n} \tag{A.7}$$

$$\int_{\Omega} div\,(\mathbf{u})\, f = -\int_{\Omega} grad\,(f) \cdot \mathbf{u} + \int_{\partial\Omega} f\,(\mathbf{u} \cdot \mathbf{n}) \tag{A.8}$$

$$\int_{\Omega} Div\,(\boldsymbol{\sigma}) \cdot \mathbf{u} = -\int_{\Omega} \boldsymbol{\sigma} : Grad\,(\mathbf{u}) + \int_{\partial\Omega} [\boldsymbol{\sigma} \cdot \mathbf{u}] \cdot \mathbf{n} \tag{A.9}$$