Energy-based Control of a Wave Equation with Boundary Anti-damping

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Abstract: In this paper, we consider the asymptotic boundary stabilisation of a one-dimensional wave equation subject to anti-damping at its free end and with control at the opposite one. The control action, implemented through a state feedback or a dynamic controller, is derived by using the port-Hamiltonian framework. More precisely, the standard energy-shaping approach plus damping assignment is adapted to cope with infinite dimensional systems with anti-damping boundary conditions. It is shown how to modify the equivalent dynamic controller to account for the instability propagation along the domain.

Keywords: distributed parameter systems, port-Hamiltonian systems, unstable wave equation, passivity-based control

1. INTRODUCTION

Port-Hamiltonian systems have been originally introduced to represent lumped parameter physical systems, van der Schaft (2017). Later, they have been generalised to the infinite dimensional scenario: distributed port-Hamiltonian systems, see van der Schaft and Maschke (2002); Le Gorrec et al. (2005), have proved to be a powerful framework for modelling, simulation and control of physical systems described by PDEs. For this class of systems, the main control synthesis methodologies proposed so far deal with the linear case and are based on the so-called energy-shaping *plus* damping injection paradigm, Ortega et al. (2001). The control action, usually applied at the boundary of the spatial domain, is designed so that the energy function is modified to shift the equilibrium, and / or some dissipative effect is added. Then, the stability proof relies on the passivity properties of the controlled system, and on the use of the total energy as Lyapunov function.

In the early works, see e.g. Rodriguez et al. (2001); Macchelli and Melchiorri (2005), the development of the state-feedback loop responsible for shaping the Hamiltonian function is based on energy-balancing considerations, i.e. on the design of a finite dimensional passive controller that is in charge of providing the right amount of energy to drive the plant towards the equilibrium. As a consequence, the control system is not able to deal with pervasive dissipation in the plant. This limitation is known as *dissipation obstacle,* Ortega et al. (2001). To enlarge the class of systems that can be stabilised via energy-shaping, for the boundary control systems in port-Hamiltonian form studied in Le Gorrec et al. (2005); Jacob and Zwart (2012), in Macchelli et al. (2017) it has been proposed to design a state-feedback action able to map the original dynamic into a target one, characterised by a "desired" Hamiltonian function and for which the equilibrium is, at least, simply stable. Convergence of the trajectories is again obtained via damping injection. In any case, independently from the approach, the control design always relies on the hypothesis that the distributed port-Hamiltonian system is passive. Roughly speaking, this means that the plant is characterised by a sort of "stable behaviour" since if the control input is set equal to zero, then the total energy is not increasing.

In this paper, we start to study the stabilisation problem of boundary control systems in port-Hamiltonian form that are not passive. In particular, the focus is on the one-dimensional wave equation with anti-damping, a system that has been already studied e.g. in Freitas and Zuazua (1996); Smyshlyaev and Krstic (2009); Hassine (2017). More specifically, the anti-damping is at the uncontrolled side of the spatial domain. To asymptotically stabilise the system, a procedure similar to the one discussed in Macchelli (2016) is adopted. The idea is at first to rely on the control by interconnection paradigm to design a state-feedback law or a dynamical controller that is able to shape the Hamiltonian function of the closed-loop system or, more precisely, that provides a control action and a related Lyapunov function to be employed in the stability analysis. As a second step, asymptotic stability is achieved via damping injection. In particular, a control action whose effect on the closed-loop system is to dissipate energy in order to reach the equilibrium is designed. The rationale behind this second step is to modify the energy-shaping controller to take into account the instability propagation along the domain, and to compensate it by adding sufficient damping. The result is another dynamical controller that is now able to asymptotically stabilise the system. Then, the stabilising damping injection law is obtained thanks to a direct comparison of the control actions that these two dynamical systems generate.

The paper is organised as follows. In Section 2, the control problem is discussed and the port-Hamiltonian formulation of the wave equation with anti-damping boundary condition is introduced. In Section 3, the problem of designing the energy-shaping law is addressed, while in Section 4 it is shown how to construct a damping injection control action that leads to an asymptotically stable closed-loop system. Finally, conclusions and some ideas about future related activities are reported in Section 5.

2. PROBLEM FORMULATION

For simplicity, let us consider the lossless transmission line equation with unitary capacitance and inductance distributions that can be written in the port-Hamiltonian form as follows, see e.g. van der Schaft and Maschke (2002); Le Gorrec et al. (2005):

$$\begin{aligned}\frac{\partial x_1}{\partial t}(t,z) &= \frac{\partial x_2}{\partial z}(t,z)\\ \frac{\partial x_2}{\partial t}(t,z) &= \frac{\partial x_1}{\partial z}(t,z)\end{aligned} \tag{1}$$

Here, $z \in [0,1]$ denotes the spatial coordinate, and $x := (x_1, x_2) \in L^2(0,1;\mathbb{R}^2)$ the state variable. Let us assume that in $z = 1$ we have an anti-damping boundary condition, i.e. given $r_1 > 0$ and $r_1 \neq 1$

$$x_2(t,1) = r_1 x_1(t,1) \tag{2}$$

and also that

$$u(t) = x_2(t,0) \tag{3}$$

is the boundary actuation. The case $r_1 = 1$ corresponds to the situation in which the transmission line is terminated in $z = 1$ on a load that is in "anti-adaptation" with respect to the line impedance, and this would lead to a finite escape time. Under these conditions, the origin of (1) with the boundary condition (2) is unstable when $u(t) = 0$. Moreover, if

$$H(x(t,\cdot)) = \frac{1}{2}\int_0^1 \left[x_1^2(t,z) + x_2^2(t,z)\right] \mathrm{d}z \tag{4}$$

denotes the total energy and under the hypothesis that (1) with the boundary condition (2) and actuation (3) is well-posed, i.e. it is a boundary control system in the sense of Fattorini (1968), we have that

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t,\cdot)) = r_1 x_1^2(t,1) - x_1(t,0)u(t) \tag{5}$$

It is easy to show that thanks to the coordinate transformation $(x_1, x_2) \mapsto (\xi_1, \xi_2)$ defined as

$$x_1 := \frac{1}{\sqrt{2}}(\xi_1 + \xi_2) \quad x_2 := \frac{1}{\sqrt{2}}(\xi_1 - \xi_2) \tag{6}$$

the PDE (1) can be rewritten in the equivalent form

$$\begin{aligned}\frac{\partial \xi_1}{\partial t}(t,z) &= \frac{\partial \xi_1}{\partial z}(t,z)\\ \frac{\partial \xi_2}{\partial t}(t,z) &= -\frac{\partial \xi_2}{\partial z}(t,z)\end{aligned} \tag{7}$$

In (7), $\xi := (\xi_1, \xi_2)$ are the so-called wave (or scattering) variables, see e.g. van der Schaft (2017), with ξ_1 associated to the propagation from $z = 1$ to $z = 0$, and ξ_2 in the opposite direction. The boundary condition (2) transforms into

$$\xi_1(t,1) = \gamma_1 \xi_2(t,1), \quad \gamma_1 = \frac{1 + r_1}{1 - r_1} \tag{8}$$

while the balance relation (5) into

$$\frac{\mathrm{d}H}{\mathrm{d}t}(\xi(t,\cdot)) = \frac{1}{2}\left(\gamma_1^2 - 1\right)\xi_2^2(t,1) - \frac{1}{2}\xi_1^2(t,0) + \frac{1}{2}\xi_2^2(t,0) \tag{9}$$

Note that, due to the fact that $r_1 > 0$ and $r_1 \neq 1$, we have that $|\gamma_1| > 1$. Moreover, if $r_1 \to 1$, then $|\gamma_1| \to \infty$. This makes clear why for $r_1 = 1$ the system presents a finite escape time.

The problem tackled in this paper is to design

$$u(t) = \beta(x(t,\cdot)) + u'(t) \tag{10}$$

so that the origin of (1) or, equivalently, of (7), is asymptotically stable. In (10), the function $\beta(\cdot)$ is a state-feedback control action that is responsible to shape the Hamiltonian function (4). Intuitively speaking, its effect is similar to the one of the proportional term in a PD controller. Differently, the auxiliary control input $u'(t)$ is designed to assure the desired convergence property to the equilibrium, so it is in charge of dissipating energy. It is shown that it is given by the sum of a derivative action plus an integral term.

3. ENERGY-SHAPING CONTROL DESIGN

To design the energy-shaping control action $\beta(\cdot)$ in (10), the idea is to rely on the control by interconnection paradigm, discussed in the finite dimensional case e.g. in Ortega et al. (2001) and extended to the distributed parameter case e.g. in Macchelli et al. (2017). Differently from what has been already presented in literature, the plant is characterised by the presence of boundary anti-damping, i.e. by an "(anti-)dissipative" effect that leads to instability. Bearing in mind the balance relation (5), let us define the dual output of $u(t)$ as

$$y(t) = -x_1(t,0) \tag{11}$$

Moreover, let us consider the linear control system

$$\begin{cases}\dot{x}_c(t) = -r_c q_c x_c(t) + g_c u_c(t)\\ y_c(t) = g_c q_c x_c(t)\end{cases} \tag{12}$$

in which $x_c, u_c, y_c \in \mathbb{R}$ are the state variable, and the input and the output signals, respectively, and $q_c, r_c \in \mathbb{R}$, with $q_c > 0$, two parameters. System (12) is in port-Hamiltonian form, and it is passive with storage function $H_c(x_c) := \frac{1}{2}q_c x_c^2$ if and only if $r_c \geq 0$. In fact, we have that

$$\frac{\mathrm{d}H_c}{\mathrm{d}t}(x_c(t)) = -r_c\left[q_c x_c(t)\right]^2 + y_c(t)u_c(t) \tag{13}$$

The closed-loop system resulting from the power conserving interconnection of (1) and (12), i.e.:

$$\begin{pmatrix}u(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix}u_c(t)\\ y_c(t)\end{pmatrix} \tag{14}$$

has a total energy $H_d(x, x_c) := H(x) + H_c(x_c)$.

The idea is to shape $H_d(x, x_c)$ by acting on the free gain q_c that defines the energy of (12). However, due to the fact that the state variables of (1) and of (12), namely x

and x_c, respectively, are independent, it is not clear how to influence the x coordinates by acting on the energy function $H_c(x_c)$, that depends on x_c. When plant and controller are both passive, the problem has been solved by selecting the "structure" of (12) so that the closed-loop system is characterised by (a set of) invariant functions that take the following form:

$$C(x_c, x) := x_c + \int_0^1 [\psi_1 x_1(z) + \psi_2 x_2(z)]\, \mathrm{d}z \tag{15}$$

in which $\psi_1, \psi_2 \in \mathbb{R}$. Such invariant functions are called Casimir functions, see e.g. Ortega et al. (2001); van der Schaft (2017) for their definition in the final dimensional case, and e.g. Macchelli (2014) for the extension to distributed parameter systems. Note that, if $C(x_c, x)$ defined as in (15) is invariant along the trajectories of the closed-loop system for all the $q_c > 0$, then x_c is in fact a function of x. Then, $H_d(x, x_c)$ depends on x only, and can be shaped by acting on $H_c(x_c)$, i.e. on the gain q_c. Moreover, the output equation in (12) would provide the expression of the energy shaping control action $\beta(\cdot)$ in (10).

Proposition 1. Let us consider the PDE (1) with input and output u and y defined in (3) and (11), respectively, that is interconnected as in (14) to the control system (12). The functional $C(x_c, x)$ introduced in (15) is invariant along the trajectories of the closed-loop system independently from the Hamiltonian $H_c(x_c) = \frac{1}{2} q_c x_c^2$ of (12) if

$$r_c = \psi_1 = \frac{1}{r_1} \qquad g_c = -\psi_2 = 1 \tag{16}$$

Proof. Along the trajectories of the closed-loop system, the time derivative of (15) is given by

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} &= \dot{x}_c + \int_0^1 \left[\psi_1 \frac{\partial x_1}{\partial t} + \psi_2 \frac{\partial x_2}{\partial t}\right] \mathrm{d}z \\ &= -r_c q_c x_c - g_c x_1(0) + \psi_1 [x_2(1) - x_2(0)] + \\ &\quad + \psi_2 [x_1(1) - x_1(0)] \\ &= (-r_c + \psi_1 g_c) q_c x_c - (g_c + \psi_2) x_1(0) + \\ &\quad + (r_1 \psi_1 + \psi_2) x_1(1)\end{aligned}$$

where, beside (1) and (12), the boundary condition (2), the definition of input and output (3) and (11), respectively, and the interconnection constraint (14) have been taken into account. Since it is required that $\dot{C}(x_c(t), x(t)) = 0$ for all $q_c > 0$, conditions (16) immediately follow.

Remark 2. The invariant function obtained in Proposition 1 is not a Casimir function in classical sense. In fact, Casimir functions are invariants independently from the Hamiltonian *and* the dissipative properties of the system, (van der Schaft, 2017, Proposition 6.4.2). This is not what happens in this situation, since $C(x_c, x)$ is computed for the particular (anti-)dissipative boundary condition (2). Note, in fact, that r_1 appears in the definition of $C(x_c, x)$. Moreover, it is interesting to note that, to keep (15) invariant under the effect of the regenerative effect at the boundary in $z = 1$, some dissipation has to be present in the dynamical extension (12).

If the initial condition for (12) is selected so that

$$x_c(0) + \int_0^1 \left[\frac{1}{r_1} x_1(0, z) - x_2(0, z)\right] \mathrm{d}z = 0 \tag{17}$$

being $(x_1(0, z), x_2(0, z))$ the initial condition for (1), then for the closed-loop system we have that $C(x_c(t), x(t, \cdot)) = 0$ for all $t \geq 0$. This implies that for the closed-loop system we have that

$$x_c(t) = \int_0^1 \left[x_2(t, z) - \frac{1}{r_1} x_1(t, z)\right] \mathrm{d}z \tag{18}$$

and then the Hamiltonian $H_d(x, x_c)$ can be expressed in terms of the state of (1) only, i.e.:

$$\begin{aligned}H_d(x) = \frac{1}{2} \int_0^1 \left[x_1^2(z) + x_2^2(z)\right] \mathrm{d}z + \\ + \frac{1}{2} q_c \left\{\int_0^1 \left[x_2(z) - \frac{1}{r_1} x_1(z)\right] \mathrm{d}z\right\}^2\end{aligned} \tag{19}$$

Moreover, the corresponding energy-shaping control action $\beta(\cdot)$ in (10) follows from the output equation of (12) and the interconnection constraint (14). Having in mind (18), the result is that

$$\begin{aligned}\beta(x(t, \cdot)) &= -q_c x_c(t) \\ &= -q_c \int_0^1 \left[x_2(t, z) - \frac{1}{r_1} x_1(t, z)\right] \mathrm{d}z\end{aligned} \tag{20}$$

This control action is in state-feedback and it can be implemented directly without relying on the dynamical extension (12). Its effect on (1) with boundary condition (2) is to shape the Hamiltonian according to (19). With simple calculations, it is possible to show that, for the closed-loop system, the following energy-balance relation holds true:

$$\begin{aligned}\frac{\mathrm{d}H_d}{\mathrm{d}t} &= r_1 x_1^2(1) - \\ &- \frac{1}{r_1} q_c^2 \left[\int_0^1 \left(x_2(z) - \frac{x_1(z)}{r_1}\right) \mathrm{d}z\right]^2 - x_1(0) u'\end{aligned} \tag{21}$$

Note that, when $u' = 0$, the total energy can increase along system trajectories. The idea is now to properly select the auxiliary input u' to add dissipation in the system. This further control loop is usually called damping injection, see e.g. Ortega et al. (2001). This topic is investigated in the next section.

Remark 3. A different approach for the design of an energy-shaping control action is to directly look at the function $\beta(x(t, \cdot))$ in (10) that maps (1) with boundary condition (2) into the port-Hamiltonian system

$$\begin{aligned}\frac{\partial x_1}{\partial t}(t, z) &= \frac{\partial}{\partial z} \frac{\delta H_d}{\delta x_2}(x(t, z)) \\ \frac{\partial x_2}{\partial t}(t, z) &= \frac{\partial}{\partial z} \frac{\delta H_d}{\delta x_1}(x(t, z))\end{aligned} \tag{22}$$

with boundary conditions

$$u'(t) = \frac{\delta H_d}{\delta x_2}(x(t, 0)) \qquad \frac{\delta H_d}{\delta x_2}(x(t, 1)) = r_1 \frac{\delta H_d}{\delta x_1}(x(t, 1))$$

being $u'(t)$ the auxiliary input that appears in (10), and in which $H_d(x) = H(x) + H_a(x)$, being $H_a(x)$ a function that has to be determined. As in van der Schaft and Maschke (2002), with $\frac{\delta H_d}{\delta x}$ we denote the variational derivative (see Olver (1993)) of the functional $H_d(x)$. By following Macchelli et al. (2017), a possible solution is to have $H_a(x_a) = \frac{1}{2} q_a x_a^2$, with $q_a > 0$ and

$$x_a(t) = \int_0^1 \left[x_2(t, z) + \frac{x_1(t, z)}{r_1}\right] \mathrm{d}z \tag{23}$$

The corresponding energy-shaping action is $\beta(x(t, \cdot)) = -q_a x_a(t)$. Such control action looks very similar to (20), but the expressions of $x_c(t)$ and $x_a(t)$ differ. The advantage of the procedure described here is that, if $\beta(x(t,\cdot)) = -q_a x_a(x(t))$ with $x_a(t)$ given as in (23), the closed-loop system is described by a port-Hamiltonian system with energy function $H_d(x) = H(x) + H_a(x)$. However, the stabilising law $u'(t)$ is more complicated to be computed. More details on this point in Remark 7.

4. DAMPING INJECTION AND STABILITY

In the previous section, the state-feedback control action (20) that is capable to transform the open-loop energy-function of (1) into (19) has been obtained. But, at present stage and due to the anti-damping boundary condition (2), the closed-loop system is still unstable. For this reason, in this section it is shown how to design $u'(t)$ in (10) to achieve asymptotic stability. Before presenting the main result, a preliminary step is necessary, and it is discussed in the next proposition.

Proposition 4. Let us consider the wave equation with anti-damping written in scattering form, i.e. the PDE (7) with boundary condition (8) in $z = 1$, and the linear control system

$$\begin{cases} \dot{w}(t) = a_w w(t) + b_w \xi_1(t,0) \\ \xi_2(t,0) = c_w w(t) + d_w \xi_1(t,0) \end{cases} \tag{24}$$

with $w \in \mathbb{R}$, that is interconnected to (7) in $z = 0$. The closed-loop system is asymptotically stable if, given $q_w > 0$, we have that

$$\begin{pmatrix} 2a_w q_w + \gamma_1^2 c_w & q_w b_b + c_w d_w \gamma_1^2 \\ q_w b_b + c_w d_w \gamma_1^2 & d_w^2 \gamma_1^2 - 1 \end{pmatrix} < 0. \tag{25}$$

Proof. Let us consider the Lyapunov function

$$W_1(\xi_1, \xi_2, w) = \underbrace{\frac{1}{2}\int_0^1 \left(\xi_1^2 + \xi_2^2\right) \mathrm{d}z}_{\equiv H(\xi)} + \frac{1}{2} q_w w^2 \tag{26}$$

in which $q_w > 0$. From (9), along the system trajectories, we have that

$$\begin{aligned} \dot{W}_1(\xi, w) &= \frac{1}{2}(\gamma_1^2 - 1)\xi_2^2(1) + \frac{1}{2}\xi_2^2(0) - \frac{1}{2}\xi_1^2(0) + \\ &\quad + q_w w \left[a_w w + b_w \xi_1(0)\right] \\ &= \frac{1}{2}(\gamma_1^2 - 1)\xi_2^2(1) + \frac{1}{2}\left[c_w w + d_w \xi_1(0)\right]^2 - \\ &\quad - \frac{1}{2}\xi_1^2(0) + q_w a_w w^2 + q_w b_w w \xi_1(0) \\ &= \frac{1}{2}(\gamma_1^2 - 1)\xi_2^2(1) + \left(\frac{c_w^2}{2} + a_w q_w\right) w^2 + \\ &\quad + \left(\frac{d_w^2}{2} - \frac{1}{2}\right)\xi_1^2(0) + \left(c_w d_w + q_w b_w\right)\xi_1(0) w. \end{aligned}$$

Now, let us consider the function

$$W_2(\xi_2(0,\cdot)) = \int_{t-1}^{t} \xi_2^2(0,\tau)\,\mathrm{d}\tau \tag{27}$$

for $t \geq 1$. We can check that

$$\begin{aligned} \dot{W}_2(\xi_2(0,t)) &= \xi_2^2(0,t) - \xi_2^2(0,t-1) \\ &= \xi_2^2(0,t) - \xi_2^2(1,t) \end{aligned}$$

since $\xi_2(1)$ is the delayed copy of $\xi_2(0)$ after 1 time unit. As a consequence, for the Lyapunov function

$$W(\xi, \xi_2(0,\cdot), w) := W_1(\xi, w) + \frac{1}{2}(\gamma_1^2 - 1) W_2(\xi_2(0,\cdot)) \tag{28}$$

we have that

$$\begin{aligned} \dot{W}(\xi, \xi_2(0,\cdot), w) &= \left(\frac{c_w^2}{2} + a_w q_w\right) w^2 + \left(\frac{d_w^2}{2} - \frac{1}{2}\right)\xi_1^2(0) + \\ &\quad + \left(c_w d_w + q_w b_w\right)\xi_1(0) w + \\ &\quad + \frac{1}{2}(\gamma_1^2 - 1)\left[c_w w + d_w \xi_1(0)\right]^2 \\ &= \left(\frac{1}{2}\gamma_1^2 c_w^2 + a_w q_w\right) w^2 + \frac{1}{2}\left(d_w^2 \gamma_1^2 - 1\right)\xi_1^2(0) + \\ &\quad + \left(q_w b_w + c_w d_w \gamma_1^2\right) w \xi_1(0) \\ &= \frac{1}{2}\begin{pmatrix} w \\ \xi_1(0) \end{pmatrix}^{\mathrm{T}} \cdot \\ &\quad \cdot \begin{pmatrix} 2a_w q_w + \gamma_1^2 c_w & q_w b_b + c_w d_w \gamma_1^2 \\ q_w b_b + c_w d_w \gamma_1^2 & d_w^2 \gamma_1^2 - 1 \end{pmatrix} \begin{pmatrix} w \\ \xi_1(0) \end{pmatrix}. \end{aligned}$$

Because of (25), we have that $\dot{W} \to 0$ as long as $w, \xi_1(0) \to 0$. Under the hypothesis of existence and pre-compactness of the orbits, the La Salle's invariance principle for infinite dimensional systems (Luo et al., 1999, Theorem 3.64) assures that the trajectories converge to the largest invariant set compatible with $\dot{W} = 0$ and $w = \xi_1(0) = 0$. Such invariant set contains only the origin of the closed-loop system, which means that $(\xi, w) \to (0,0)$ asymptotically.

As discussed in Proposition 1, the energy-shaping control law (20) can be thought as been generated by the control system (12) with initial condition (17) and r_c and g_c given as in (16). The same control system can be written in the form (24) in which input and output are the wave variables in $z = 0$. From (6) and since $u(t) = x_2(t,0)$ and $y(t) = -x_1(t,0)$, we obtain that

$$\begin{cases} \begin{aligned} \dot{x}_c(t) &= -\left(1 + \frac{1}{r_1}\right) q_c x_c(t) - \sqrt{2}\xi_1(t,0) \\ &= -\frac{2\gamma_1}{\gamma_1 - 1} q_c x_c(t) - \sqrt{2}\xi_1(t,0) \end{aligned} \\ \xi_2(t,0) = \sqrt{2} q_c x_c(t) + \xi_1(t,0) \end{cases} \tag{29}$$

where the definition of γ_1 in (8) has been taken into account. Note that the requirements stated in Proposition 4 are not met, and this is coherent with the fact that the control action (20) is not able to guarantee stability, but only aims at modifying the shape of the closed-loop energy function. From (29) it is immediate to compute how $x_c(t)$ varies as a function of $\xi_1(t,0)$ and $\xi_2(t,0)$. We get that

$$\dot{x}_c(t) = \frac{\sqrt{2}}{\gamma_1 - 1}\xi_1(t,0) - \frac{\sqrt{2}\gamma_1}{\gamma_1 - 1}\xi_2(t,0). \tag{30}$$

With an eye on (10), the goal is now to compute $u'(t)$ to make the origin of the controlled system asymptotically stable. The idea is to start from (29) that generates the energy-shaping term $\beta(\cdot)$, and to modify it in order to meet the requirements of Proposition 4. Then, by comparing the control actions that both generate, the stabilising term $u'(t)$ can be determined. Among all the possible choices, let us consider the following control system in the form (24):

$$\begin{cases} \begin{aligned} \dot{w}(t) &= -\left(\frac{2\gamma_1}{\gamma_1 - 1} + \frac{\Delta}{q_c}\right) q_c w(t) + \\ &\quad + \sqrt{2}\frac{1 - \delta\gamma_1}{\gamma_1 - 1}\xi_1(t,0) \end{aligned} \\ \xi_2(t,0) = \sqrt{2} q_c w(t) + \delta \xi_1(t,0) \end{cases} \tag{31}$$

in which $q_c > 0$, δ is such that $\delta^2\gamma_1^2 < 1$ and $\Delta > 0$ is sufficiently large so that (25) holds true. As a consequence, the closed-loop system resulting from the power conserving interconnection of (7), with boundary condition (8), and (31) is asymptotically stable. On the other hand, we have also that

$$\begin{aligned}\dot{w}(t) &= -\Delta w(t) + \frac{\sqrt{2}}{\gamma_1 - 1}\xi_1(t,0) - \frac{\sqrt{2}\gamma_1}{\gamma_1 - 1}\xi_2(t,0) \\ &= -\Delta w(t) + \dot{x}_c(t)\end{aligned} \tag{32}$$

where (30) has been taken into account. If we integrate (32), we obtain that

$$w(t) = e^{-\Delta t}w(0) + \int_0^t e^{-\Delta(t-\tau)}\dot{x}_c(\tau)\,\mathrm{d}\tau$$

where the initial condition $w(0)$ can be freely chosen. The integral term is equal to

$$\begin{aligned}&\left[e^{-\Delta(t-\tau)}x_c(\tau)\right]_{\tau=0}^{\tau=t} - \Delta\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau = \\ &\qquad = x_c(t) - e^{-\Delta t}x_c(0) - \Delta\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau\end{aligned}$$

which implies that

$$w(t) = x_c(t) - \Delta\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau \tag{33}$$

if $w(0) = x_c(0)$. Consequently, the stabilising control action that (31) applies to (7) is given by the corresponding output equation, that can be now written as a function of $x_c(t)$. This quantity is the crucial term in the energy-shaping law (20). The result is that

$$\begin{aligned}\xi_2(t,0) &= \sqrt{2}q_c x_c(t) - \\ &\quad - \sqrt{2}q_c\Delta\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau + \delta\xi_1(t,0)\end{aligned} \tag{34}$$

which is the energy-shaping plus damping injection control law applied to (7) or, equivalently, to (1), when the boundary input and output are $\xi_2(t,0)$ and $\xi_1(t,0)$, respectively. The final step, then, consists in rewriting (34) in terms of $u(t)$ and $y(t)$ defined in (3) and (11), respectively. From (6) we have that

$$\xi_1(0) = \frac{1}{\sqrt{2}}(u - y) \qquad \xi_2(0) = -\frac{1}{\sqrt{2}}(u + y)$$

and then (34) gives the following expression for $u(t)$:

$$\begin{aligned}u(t) &= -q_c x_c(t) - \frac{1-\delta}{1+\delta}\left[y(t) - q_c x_c(t)\right] + \\ &\qquad + \frac{2q_c\Delta}{1+\delta}\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{35}$$

If we compare (35) with (10) having in mind that $\beta(\cdot)$ is given by (20), we obtain that

$$u'(t) = -K_\delta y'(t) + K_\Delta\int_0^t e^{-\Delta(t-\tau)}x_c(\tau)\,\mathrm{d}\tau \tag{36}$$

where

$$\begin{aligned}y'(t) &:= y(t) - q_c x_c(t) \\ &= y(t) - q_c\int_0^1\left[\frac{1}{r_1}x_1(t,z) - x_2(t,z)\right]\mathrm{d}z\end{aligned} \tag{37}$$

is sort of "shaped output" for (1), and

$$K_\delta = \frac{1-\delta}{1+\delta} \qquad K_\Delta = \frac{2q_c\Delta}{1+\delta} \tag{38}$$

two positive gains, since $|\delta| < 1$. The result is summarised in the next proposition.

Proposition 5. Let us consider the PDE (1) with boundary condition (2) and input $u(t)$ defined in (3). The energy-shaping plus damping injection control law (10) in which $\beta(x(t,\cdot))$ is given by (20) and $u'(t)$ by (36) makes the origin of (1) asymptotically stable. Here, q_c, K_δ and K_Δ are positive gains, where the latter two are defined in (38), and $y'(t)$ is defined in (37). The constants δ and $\Delta > 0$ have to be selected so that, given

$$\begin{aligned}a_w &= -\frac{2\gamma_1 q_c}{\gamma_1 - 1} - \Delta & b_w &= \sqrt{2}\frac{1-\delta\gamma_1}{\gamma_1 - 1} \\ c_w &= \sqrt{2}q_c & d_w &= \delta\end{aligned}$$

the LMI (25) holds true, with $q_w = q_c$.

Remark 6. The stabilising action $u'(t)$ defined in (36) results from the sum of two contributions. With an eye on (31), we can see that the term $-K_\delta y'(t)$ is associated to the feedthrough term $\delta\xi_1(t,0)$ in the output equation of (31). From a physical point of view, its effect is to damp the fast dynamics in the plant. On the other hand, the integral action in (36) is strictly linked to the dissipative term $-\Delta w(t)$ that appears in the state equation of (31). This quantity is associated to the internal dissipation of (31), and it is responsible for attenuating the slow dynamics of the controlled system.

Remark 7. The previous result that provides the expression for the stabilising law $u'(t)$ in (10) in which $\beta(x(t,\cdot))$ is given as in (20) can be adapted to cope with the energy-shaping action that results from the approach proposed in Remark 3. The main issue is to find a system similar to (29) or, better, a relation in the form (30) that gives the evolution of $x_a(t)$ defined in (23) in terms of $\xi_1(t,0)$ and $\xi_2(t,0)$. With simple calculations, it is possible to obtain that

$$\dot{x}_a(t) = \frac{\sqrt{2}}{\gamma_1 - 1}\xi_1(t,0) - \frac{\sqrt{2}\gamma_1}{\gamma_1 - 1}\xi_2(t,0) + 2x_1(t,1).$$

As anticipated in Remark 3, it is the last term, i.e. $2x_1(t,1)$, that makes the derivation of $u'(t)$ more involved. In any case, by following the same steps that led to Proposition 5, it is possible to check that $x_1(t,1)$ would appear in the integral term in (36). Similarly to Macchelli et al. (2015), such quantity provides an estimate of the generated power in $z = 1$ because of the anti-damping boundary condition (2). Such an estimate must be known in order to properly dissipate the "right" amount of energy via $u'(t)$.

Remark 8. The energy-shaping control action and (part of) the damping injection term depend on the quantity $x_c(t)$ defined in (18) that requires the knowledge of the whole state $x(t,z)$ of (1). This is a typical limitation of the approach, as discussed e.g. in Macchelli et al. (2017). However, such limitation can be removed for the particular system studied here. At first, from (6), it is immediate that $x_c(t)$ can be equivalently written as

$$x_c(t) = \frac{\sqrt{2}}{1-\gamma_1}\int_0^1\left[\xi_1(t,z) + \gamma_1\xi_2(t,z)\right]\mathrm{d}z$$

where the definition of γ_1 in (8) has been taken into account. On the other hand, from (7) and the boundary condition (8), it is easy to check that

$$\int_0^1 \xi_2(t,z)\,\mathrm{d}z = \int_{t-1}^{t} \xi_2(\tau,0)\,\mathrm{d}\tau$$

$$\int_0^1 \xi_1(t,z)\,\mathrm{d}z = \int_{t-1}^{t} \xi_1(\tau,1)\,\mathrm{d}\tau = \gamma_1 \int_{t-1}^{t} \xi_2(\tau,1)\,\mathrm{d}\tau = \gamma_1 \int_{t-2}^{t-1} \xi_1(\tau,0)\,\mathrm{d}\tau$$

which implies that

$$x_c(t) = \frac{\sqrt{2}\gamma_1}{1-\gamma_1} \int_{t-2}^{t} \xi_2(\tau,0)\,\mathrm{d}\tau \tag{39}$$

As a consequence, for $t \geq 2$, (39) provides the value of $x_c(t)$ to be employed in (10) and in (36) by integrating quantities, namely a linear combination of the state variables of (1), see (6), that are defined in $z = 0$ only. As a matter of fact, (39) is a sort of observer that converges in finite time and provides the basic term in the energy-shaping and damping-injection control action.

5. CONCLUSIONS AND FUTURE WORK

The main contribution of this paper is the design of an energy-shaping plus damping injection control law for a one-dimensional wave equation with boundary anti-damping at the uncontrolled end. This result is quite powerful because it shows for the first time that such techniques can be successfully employed to stabilise distributed parameter systems in port-Hamiltonian form that are not passive. The control action consists of three main terms. The first one is a proportional contribution that is related to energy-shaping, the second and the third one are a derivative and an integral action. The latter twos are designed to add the proper amount of dissipation in the closed-loop system and have asymptotic stability.

Future work is mainly focused on the generalisation of the proposed approach to a wider class of boundary control systems in port-Hamiltonian form for which the anti-damping effect or, in general, the instability source is not only at the boundary of the spatial domain, but also inside the domain itself. For these systems, the goal is to achieve not only asymptotic, but also exponential convergence to the equilibrium.

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