Observer based nonlinear control of a rotating flexible beam ⋆

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Abstract: This paper presents an observer based nonlinear control for a flexible beam clamped on a rotating inertia. The considered model is composed by a set of Partial Differential Equations (PDEs) interconnected with an Ordinary Differential Equation (ODE), with control input in the ODE. The control problem consists in orienting the beam at the desired position, maintaining the flexible vibrations as low as possible. To this end, it is presented a nonlinear controller that depends on the beam's state. An Observer is designed to reconstruct the infinite dimensional state, and the estimated state is used in the nonlinear controller instead of the real one. Assuming well-posedness of the closed loop system, it is shown the exponential convergence of the estimated state, and the asymptotic stability of the closed loop system. Numerical simulations are presented to characterize the closed loop behaviour with different choices of observer's parameters.

Keywords: Distributed-parameter system, Nonlinear control, Observers, Asymptotic stability, port-Hamiltonian system.

1. INTRODUCTION

Control of flexible robots has been an highly investigated topic over the last 30 years. The needs of precise controllers and stability requirements made necessary to take into consideration distributed flexible phenomena. These processes are modelled using Partial Differential Equations (PDEs), where the state variables are space and time dependent. In the specific case of a rotating flexible beam, the inertia of the hub to which the beam is connected (i.e. the rotor of a motor) cannot be neglected. This scenario brings to a system modelled by an interconnection between a set of PDEs and an ODE, with control input on the ODE. In the literature, the control of a coupled set of PDEs and ODEs is often referred as control of Hybrid systems (Luo et al., 1999). The design of stabilizing controllers for rotating flexible beams can been addressed with the use of a PD controller (Luo and Feng, 1999), but different control methods have been employed to have a faster vibration suppression. In (Morgul, 1991) is shown the asymptotic stability of a rotating Euler-Bernoulli beam with a PD + strain feedback control, while in (Wang et al., 2017) is used a feed-forward control law obtained by model inversion to minimize the flexible vibrations during motion. Another possible strategy is to include in the controller the information about the deformation of the beam. To do so in a passive preserving way, it is necessary to design a nonlinear dynamic controller (Luo and Feng, 1999). This control law have been rewritten as a passively interconnected port-Hamiltonian system in (Aoues et al., 2019), where Lyapunov stability has been proved.
Functional analysis is a powerful tool for studying the asymptotic behaviour of dynamical system described by PDEs or Hybrid systems (Curtain and Zwart, 1995). In the last decades, the port-Hamiltonian framework has been extended from finite to infinite dimensional systems (van der Schaft and Maschke, 2002), making possible the use of functional analysis theory for studying the stability of port-Hamiltonain systems (Jacob and Zwart, 2012; Villegas, 2007; Le Gorrec et al., 2005). In a more general fashion, the control problem of nonlinear feedback for a class of infinite dimensional port-Hamiltonian systems has been presented in (Ramirez et al., 2017), where conditions for asymptotic and exponential stability are given. In (Mileti et al., 2016), pre-compactness of trajectories combined with the existence of a limit set is used to prove asymptotic stability for a Euler-Bernoulli beam subject to a class of nonlinear feedbacks.

In this manuscript we propose a similar control law as proposed in (Luo and Feng, 1999) where, since in practical applications the state of the beam may not be directly available, the controller makes use of an observed state instead of the original one. The beam is modelled using the Timoshenko's beam assumptions, and the closed loop system results composed by two linear sets of PDEs interconnected with a nonlinear set of ODEs, with the nonlinearity depending on the infinite dimensional state.

The paper is organized as follows. In Section 2 the port-Hamiltonian model of a flexible beam clamped on a rotating inertia and the observer based control design are given. In section 3, assuming the well-posedness of the closed loop system, is proven the exponential convergence of the observer and the asymptotic stability of the closed-loop system. In Section 4 are shown numerical simulations, while some concluding remarks and comments on future works are given in Section 5.

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2. MODELLING AND CONTROL DESIGN

In the following we propose the equations of a rotating flexible beam using Timoshenko's assumptions in the port-Hamiltonian framework.

2.1 Modelling

For sake of clarity, we define the variables and the parameters that will be used afterwards. The rotor angle $\theta(t)$ is a real function of time defined as $\theta : \mathbb{R} \to \mathbb{R} : \theta \to \theta(t)$. With $\xi \in [0, L]$ we identify the spatial coordinate of the beam. We identify the deflection of the beam with respect to his axis z with $w(t,\xi) \in \mathcal{L}_2(0,L)$, while with $\phi(t,\xi) \in \mathcal{L}_2(0,L)$ we identify the relative rotation of the beam cross section. All the physical parameters are positive real. I_h represents the rotary inertia of the hub to which the beam is connected. E, I are respectively the Young's modulus and the moment of inertia of the beam's cross section. The beam's cross section is assumed to be rectangular, hence his inertia is defined to be $I = \frac{L_w^3 L_t}{12}$, where L_w and L_t are respectively the width and the thickness of the beam. ρ, I_ρ are respectively the density and the mass moment of inertia of the beam's cross section. The mass moment of inertia of the cross section is defined as $I_\rho = I\rho$. K is defined as $K = kGA$, where k is a constant depending on the shape of the cross section ($k = 5/6$ for rectangular cross sections), G is the Shear modulus and A is the cross sectional area.

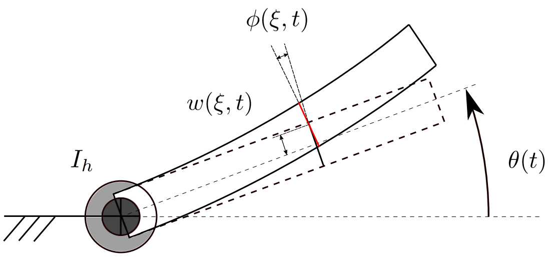

Fig. 1. Rotating flexible beam.

From now on we will not explicit the dependency from time and space of the variables when it is clear from the context. The kinetic energy and the potential energy, using Timoshenko's assumptions, write

$$H_k = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2}\int_0^L \left[\rho_1 \left(\xi\dot{\theta} + \dot{w}\right)^2 + I_\rho \left(\dot{\theta} + \dot{\phi}\right)^2\right] d\xi,$$
$$H_p = \frac{1}{2}\int_0^L \left[K\left(\frac{\partial w}{\partial \xi} - \phi\right)^2 + EI_1\left(\frac{\partial \phi}{\partial \xi}\right)^2\right] d\xi.$$

To obtain the dynamic equations we use the Hamilton's principle. We denote with $L = H_k - H_p$ the Lagrangian, while $W_{nc} = u(t)\theta$ denotes the work of non-conservative forces. The Hamilton principle writes

$$\int_{t_1}^{t_2} (\delta L - \delta W_{nc})\, dt = 0,$$

from which we can derive the following set of mixed partial and ordinal differential equations

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho\left(z\dot{\theta} + \dot{w}\right)\right) = \frac{\partial}{\partial \xi}\left(K\left(\frac{\partial w}{\partial \xi} - \phi\right)\right) \\ \frac{\partial}{\partial t}\left(I_\rho\left(\dot{\theta} + \dot{\phi}\right)\right) = \frac{\partial}{\partial \xi}\left(EI\frac{\partial \phi}{\partial \xi}\right) + K\left(\frac{\partial w}{\partial \xi} - \phi\right), \\ I\ddot{\theta} = +EI\frac{\partial \phi(0,t)}{\partial \xi} + u(t). \end{cases} \tag{1}$$

With boundary conditions

$$w(0,t) = 0, \qquad \phi(0,t) = 0,$$
$$\frac{\partial w}{\partial \xi}(L,t) - \phi(L,t) = 0, \quad \frac{\partial \phi}{\partial \xi}(L,t) = 0.$$

The system will be presented as the interconnection of an infinite dimensional system with a finite dimensional one. Hence, the two subsystems will be presented separately and eventually interconnected, such to be equivalent to (1). The energy states of the infinite dimensional part of the system are defined as:

$$\varepsilon_t = \frac{\partial w}{\partial \xi} - \phi, \; p_t = \rho\left(\frac{\partial w}{\partial t} + z\dot{\theta}\right),$$
$$\varepsilon_r = \frac{\partial \phi}{\partial \xi}, \qquad p_r = I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right),$$

while the state related to the finite dimensional part writes $p = I\dot{\theta}$. The equations describing the infinite dimensional system can be written as a port-Hamiltonian system

$$\dot{x}_b = \mathcal{J} x_b = \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b x_b) + P_0(\mathcal{H}_b x_b), \tag{2}$$

with $x_b = [p_t \; p_r \; \varepsilon_t \; \varepsilon_r]^T \in X_b \subset \mathcal{L}_2([0,L], \mathbb{R}^4)$ representing the system's state. The matrices in equation (2) are defined as

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \mathcal{H}_b = \begin{bmatrix} \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \frac{1}{I_\rho} & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & EI \end{bmatrix}$$

We equip the state space X_b with the $\mathcal{L}_2$ inner product $\langle x_b, x_b \rangle_{X_b} = \langle x_b, \mathcal{H}_b x_b \rangle_{\mathcal{L}_2}$, such to express the energy related to the flexible part of the system as $H_b = \frac{1}{2}\langle x_b, x_b \rangle_{X_b}$. The boundary variables are defined as

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}} U \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}_b x_b)(t,0) \\ (\mathcal{H}_b x_b)(t,L) \end{bmatrix}$$

with U unitary matrix such that $U^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. The boundary input and output operators can be defined as:

$$\begin{aligned} u_{b,1} &= \mathcal{B}_1(\mathcal{H}_b x_b) = W_1 \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = -\frac{1}{I_\rho} p_r(0,t) \\ u_{b,2} &= \mathcal{B}_2(\mathcal{H}_b x_b) = W_2 \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} -\frac{1}{\rho} p_t(0,t) \\ K\varepsilon_t(L,t) \\ EI\varepsilon_r(L,t) \end{bmatrix} \\ y_{b,1} &= \mathcal{C}_1(\mathcal{H}_b x_b) = \tilde{W}_1 \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = +EI\varepsilon_r(0,t) \\ y_{b,2} &= \mathcal{C}_2(\mathcal{H}_b x_b) = \tilde{W}_2 \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} K\varepsilon_t(0,t) \\ \frac{1}{\rho} p_t(L,t) \\ \frac{1}{I_\rho} p_r(L,t) \end{bmatrix} \end{aligned} \tag{3}$$

where $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ and $\tilde{W} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix}$ are matrices such that $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ is a non-singular matrix. The total boundary input-output operators are defined as

$$\begin{aligned} \mathcal{B}(\mathcal{H}_b x_b) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}_b x_b) \\ \mathcal{B}_2(\mathcal{H}_b x_b) \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}(\mathcal{H}_b x_b) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}_b x_b) \\ \mathcal{C}_2(\mathcal{H}_b x_b) \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \end{aligned} \tag{4}$$

The finite dimensional part of the system is composed of only one state, and his dynamics can be represented by a single integral equation

$$\begin{cases} \dot{p} = +u_r(t), \\ y_r(t) = \dfrac{1}{I} p. \end{cases}$$

Finally, the interconnection relations between the infinite and finite dimensional parts write

$$u_{b,1} = -y_r, \; u_r = +y_{b,1} + u, \tag{5}$$

where u is the control input that have to be designed. The remaining boundary conditions of (2) are $u_{b,2} = [0\ 0\ 0]^T$.

2.2 Observer based Control design

The aim of the proposed control law is firstly to orient the beam in the desired configuration, and secondly to use the state of the infinite dimensional part of the system to change the elastic behaviour of the closed loop system. The nonlinear control law makes use of an observed state instead of the original one. The control law writes

$$\begin{cases} \dot{x}_c = -r_c x_c + g(\hat{x}_b)\dot{\theta}(t) \\ u(t) = -k_1(\theta(t) - \theta^*) - g(\hat{x}_b) k_2 x_c - k_3 \dot{\theta}(t) \end{cases} \tag{6}$$

The first term is responsible of the orientation in the desired angular configuration. Without loss of generality, we consider the stabilization problem to the origin $\theta^* = 0$. If we want to stabilize to a different desired configuration, a change of coordinate can cast the problem to origin stabilization. The second term is the nonlinear term depending on the linear function of the observed state $g(\hat{x}_b)$ and on the controller variable x_c. The x_c dynamics is again nonlinear in the input entrance. This controller construction makes possible the dependence of the controller dynamics on observed infinite dimensional state. The last term is the derivative term corresponding to a damping injection.

For the infinite dimensional state reconstruction we propose the "Simple observer": starting from the boundary observation of the infinite dimensional system, it asymptotically reconstructs the original state. The controller design and the stability proof are carried out with an observer described by a set of partial differential equations. It is assumed that all the physical parameters of the infinite dimensional system are known. The observer equations have the same form of the original system

$$\dot{\hat{x}}_b = \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b \hat{x}_b) + P_0(\mathcal{H}_b \hat{x}_b), \tag{7}$$

with $\hat{x}_b \in \hat{X}_b \subset \mathcal{L}_2([0, L], \mathbb{R}^4)$, and boundary inputs and observations:

$$\mathcal{B}(\mathcal{H}_b \hat{x}_b) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \hat{u}_b(t) = u_b(t) - L(\hat{y}_b(t) - y_b(t)),$$

$$\mathcal{C}(\mathcal{H}_b \hat{x}_b) = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \hat{y}_b(t),$$

where, $L = diag([l_1\ l_2\ l_3\ l_4]) \geq 0$ since $l_1, l_2, l_3, l_4 \geq 0$. For analysis purposes, we perform a change of coordinate defining the error state $\tilde{x}_b = \hat{x}_b - x_b$ and his dynamics

$$\begin{aligned} \dot{\tilde{x}}_b &= \dot{\hat{x}}_b - \dot{x}_b \\ &= \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b \hat{x}_b) + P_0(\mathcal{H}_b \hat{x}_b) - \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b x_b) - P_0(\mathcal{H}_b x_b) \\ &= \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b(\hat{x}_b - x_b)) + P_0(\mathcal{H}_b(\hat{x}_b - x_b)) \\ &= \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b \tilde{x}_b) + P_0(\mathcal{H}_b \tilde{x}_b), \end{aligned} \tag{8}$$

where the operators linearity has been used, and the state space defined as $\tilde{x}_b \in \tilde{X}_b \subset \mathcal{L}_2([0, L], \mathbb{R}^4)$. As input and output boundary operators we use the same as for the original infinite dimensional system and for his observer

$$\begin{aligned} \mathcal{C}(\mathcal{H}_b \tilde{x}_b) &= \mathcal{C}(\mathcal{H}_b(\hat{x}_b - x_b)) \\ &= \mathcal{C}(\mathcal{H}_b \hat{x}_b) - \mathcal{C}(\mathcal{H}_b x_b) \\ &= \hat{y}_b - y_b = \tilde{y}_b, \end{aligned}$$

and,

$$\begin{aligned} \tilde{u}_b = \mathcal{B}(\mathcal{H}_b \tilde{x}_b) &= \mathcal{B}(\mathcal{H}_b \hat{x}_b) - \mathcal{B}(\mathcal{H}_b x_b) \\ &= u_b(t) - L(\hat{y}_b - y(t)) - u_b(t) \\ &= -L\tilde{y}_b. \end{aligned} \tag{9}$$

The control law (6) can be rewritten as a dynamic port-Hamiltonian system of the form:

$$\begin{cases} \dot{q} = u_c, \\ \dot{x}_c = -r_c k_2 x_c + g(\hat{x}_b) u_c, \\ y_c = +k_1 q + k_2 g(\hat{x}_b) x_c + k_3 u_c \end{cases}$$

To connect the controller to the system we make use of a power preserving interconnection:

$$u_c = y_r, \; u = -y_c. \tag{10}$$

To make the analysis more clear, we maintain the x_c dynamics separated from the rest of the system. Hence, we define the closed loop semilinear equation

$$\begin{aligned} \dot{x} &= \mathcal{A}x + f(x) \\ &= \begin{bmatrix} \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b x_b) + P_0(\mathcal{H}_b x_b) \\ \frac{\partial}{\partial \xi} P_1(\mathcal{H}_b \tilde{x}_b) + P_0(\mathcal{H}_b \tilde{x}_b) \\ (J_r - R_r) Q_r x_r + g_r \mathcal{C}_1(\mathcal{H}_b x_b) \\ -r_c k_2 x_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -g_r g(\tilde{x}_b + x_b) k_2 x_c \\ +g(\tilde{x}_b + x_b) g_r^T Q_r x_r \end{bmatrix} \end{aligned} \tag{11}$$

where $x = [x_b\ \tilde{x}_b\ x_r\ x_c]^T \in X \subset \mathcal{L}_2([0, L], \mathbb{R}^4) \times \mathcal{L}_2([0, L], \mathbb{R}^4) \times \mathbb{R}^2 \times \mathbb{R}$, and $x_r = [p\ q]^T$. We define $g : X_b \times \tilde{X}_b \to \mathbb{R}$ and we assume that is linear. The new matrices are defined as

$$J_r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; R_r = \begin{bmatrix} k_3 & 0 \\ 0 & 0 \end{bmatrix} \; Q_r = \begin{bmatrix} \frac{1}{I} & 0 \\ 0 & k_1 \end{bmatrix}$$

with $J_r = -J_r^T, \; R_r = R_r^T \geq 0, \; Q_r = Q_r^T > 0$. The linear operator domain is defined as

$$D(\mathcal{A}) = \{x \in X | x_b, \tilde{x}_b \in H^1([0, L], \mathbb{R}^4), W_x x = 0\} \tag{12}$$

where,

$$W_x x = \begin{bmatrix} \mathcal{B}_{b,1}(\mathcal{H}_b x_b) + g_r^T Q_r x_r \\ \mathcal{B}_{b,2}(\mathcal{H}_b x_b) \\ \mathcal{B}_b(\mathcal{H}_b \tilde{x}_b) + \mathcal{C}_b(\mathcal{H}_b \tilde{x}_b) \end{bmatrix}.$$

We equip the state space with the inner product $\langle x, x \rangle_X = \langle x_b, \mathcal{H}_b x_b \rangle_{\mathcal{L}_2} + \langle \tilde{x}_b, \mathcal{H}_b \tilde{x}_b \rangle_{\mathcal{L}_2} + x_r^T Q_r x_r + k_2 x_c^2$. The energy of the closed loop system can be thought as half of the previously defined inner product. In this new closed-loop energy appears also the square of the controller state x_c that in turns has his dynamics depending on the function $g(\hat{x})$. Hence, we added to the energy a term depending on the the estimation of the flexible state. Choosing the function $g(\hat{x})$, we are able to change the elastic behaviour of the beam in the direction on which we are more interested. In this control scenario there are two problems to address: convergence of the observed state $\hat{x}_b$ to the original infinite dimensional state and asymptotic stability of the infinite dimensional system. The use of the error state in the closed loop equation allow to study both of them as a stabilization to the origin problem.

3. ASYMPTOTIC STABILITY OF THE CLOSED-LOOP SYSTEM

Theorem 1. The linear operator $\mathcal{A}$ with domain (12) generates a contraction semigroup on X. Moreover, $\mathcal{A}$ has a compact resolvent.

Proof. The contraction C0-semigroup generation is proved applying the Lummer-Phillips Theorem. To use this theorem we have to verify two properties: that the operator $\mathcal{A}$ is dissipative and that $ran(\lambda - \mathcal{A}) = X$. To show dissipativity we have that

$$\begin{aligned}\Re\langle \mathcal{A}x, x\rangle_X &= \langle \mathcal{A}x, x\rangle_X + \langle x, \mathcal{A}x\rangle_X \\ &= \langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{\mathcal{L}_2} + \langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{\mathcal{L}_2} \\ &\quad +2(Q_r x_r)^T (J_r - R_r) Q_r x_r \\ &\quad +2\mathcal{C}_{b,1}(\mathcal{H}_b x_b) g_r^T Q_r x_r - 2 r_c k_2^2 x_c^2 \\ &\quad +\langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b \tilde{x}_b\rangle_{\mathcal{L}_2} + \langle \mathcal{H}_b \tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{\mathcal{L}_2}\end{aligned}$$

Thanks to the port variable selection we have

$$\begin{aligned}\langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{\mathcal{L}_2} + \langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{\mathcal{L}_2} &= 2u_{1,b}y_{1,b} \\ \langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b \tilde{x}_b\rangle_{\mathcal{L}_2} + \langle \mathcal{H}_b \tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{\mathcal{L}_2} &= 2\langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned} \tag{13}$$

hence,

$$\Re\langle \mathcal{A}x, x\rangle_X = -(Q_r x_r)^T R_r Q_r x_r - r_c k_2^2 x_c^2 - \langle L\tilde{y}, \tilde{y}\rangle \le 0$$

Since $R_c \ge 0$, $r_c > 0$, $k_2 > 0$, $L \ge 0$. The range condition consists in finding $(x_b, \tilde{x}_b, x_r, x_c) \in D(\mathcal{A}) \setminus \{0\}$ such that

$$\lambda \begin{bmatrix} x_b \\ \tilde{x}_b \\ x_r \\ x_c \end{bmatrix} - \mathcal{A} \begin{bmatrix} x_b \\ \tilde{x}_b \\ x_r \\ x_c \end{bmatrix} = \begin{bmatrix} f_b \\ \tilde{f}_b \\ f_r \\ f_c \end{bmatrix}$$

for all $[f_b,\ \tilde{f}_b,\ f_r,\ f_c]^T \in X$. We note that this problem is actually divided in two different parts because the error system is not affected from the rest of the system. The range condition for the part of the system related to the error system follows from Theorem 2.26 of (Villegas, 2007). For the remaining equations, the range condition relies on the existence of the right inverse of the operator $\mathcal{B}_b$ subjected to a perturbation of the form $(\mathcal{B}_b + K\mathcal{C}_b)$, with $\mathcal{B}, \mathcal{C}$ operators defined in equation (4), and K singular matrix. The existence of this right inverse follows from the non-singularity of $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$.

To prove the compactness of the resolvent we define the sequence

$$\{z_n\} = (\lambda I - \mathcal{A})^{-1}\{x_n\} \tag{14}$$

where, without loss of generality, we assume $\{x_n\}$ bounded $\forall n \in \mathbb{N}$. For the compact operator definition, we have to show that $\{z_n\}$ has a converging subsequence. We define $\{z_n\} = [\{z_{n,1}\}\ \{z_{n,2}\}\ \{z_{n,3}\}]^T \in H^1([0,L], \mathbb{R}^4) \times H^1([0,L], \mathbb{R}^4) \times \mathbb{R}^3$ and $\{x_n\} = [\{x_{n,1}\}\ \{x_{n,2}\}\ \{x_{n,3}\}]^T \in \mathcal{L}_2([0,L], \mathbb{R}^4) \times \mathcal{L}_2([0,L], \mathbb{R}^4) \times \mathbb{R}^3$. We already shown that the operator $\mathcal{A}$ generates a contraction $C0-$Semigroup, hence by the Hille-Yoshida theorem (Curtain and Zwart, 1995) (Theorem 2.1.12, pag 26) we know that $||(\lambda I - \mathcal{A})^{-1}|| < \frac{1}{\lambda}$. This implies that also $\{z_n\}$ is bounded. Since $\{z_{n,3}\}$ belongs to a finite dimensional space, it follows that has a convergent subsequence. For both $\{z_{n,1}\}$ and $\{z_{n,2}\}$ we have

$$||z_{n,i}||_{H^1}^2 = ||\frac{\partial}{\partial z} z_{n,i}||_{\mathcal{L}_2} + ||z_{n,i}||_{\mathcal{L}_2}^2 \quad i = \{1,2\} \tag{15}$$

Using the $\mathcal{J}$ definition and equation (14), it holds

$$\begin{aligned}||\frac{\partial}{\partial z} z_{n,i}||_{\mathcal{L}_2}^2 &= ||P_1^{-1}\mathcal{J}z_{n,i} + P_1^{-1}P_0 z_{n,i}||_{\mathcal{L}_2}^2 \\ &\le ||P_1^{-1}(x_{n,i} + \lambda z_{n,i})||_{\mathcal{L}_2}^2 + ||P_1^{-1}P_0 z_{n,i}||_{\mathcal{L}_2}^2 \\ &< \infty\end{aligned} \tag{16}$$

Then, $\{z_{n,i}\}$ is bounded in H^1 and from the Sobolev embedding theorem it implies that $\{z_{n,i}\}$ has a converging subsequence in $\mathcal{L}_2$ for $i = \{1,2\}$. Therefore, $\mathcal{A}$ has compact resolvent.

Theorem 2. The solution of system (11) is bounded in every interval $[0,t]$, $t > 0$. Moreover, $p, x_c \in \mathcal{L}_2([0,t])\ \forall t > 0$.

Proof. Boundedness of solutions follows from the existence of a Lyapunov function. This means that we have to show that exists a function $V : X \to \mathbb{R}^+$ such that $V(0) = 0$ and with time derivative $\dot{V}(x_0) \le 0$, $\forall x_0 \in D(\mathcal{A})$. To this end we take as candidate Lyapunov function half of the defined inner product on the State space $V(x_0) = \frac{1}{2}\langle x_0, x_0\rangle_X$. The time derivative is defined as

$$\dot{V}(x_0) = \lim_{t \to 0} \frac{V(x(t, x_0)) - V(x_0)}{t}$$

and it can be proved that $\dot{V}(x_0) = dV(x_0)\mathcal{A}_{nl}x$, where $dV(x_0)$ is the Fréchet derivative of the candidate Lyapunov function in x_0. Then,

$$\begin{aligned}dV(x_0)\mathcal{A}_{nl}x &= +\frac{1}{2}\langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{\mathcal{L}_2} + \frac{1}{2}\langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{\mathcal{L}_2} \\ &\quad +\frac{1}{2}\dot{x}_r^T Q_r x_r + \frac{1}{2}x_r Q_r \dot{x}_r^T + k_c x_c \dot{x}_c \\ &\quad +\frac{1}{2}\langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b \tilde{x}_b\rangle_{\mathcal{L}_2} + \frac{1}{2}\langle \mathcal{H}_b \tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{\mathcal{L}_2}\end{aligned}$$

Similarly to proof of Theorem 1, and substituting x_r and x_c dynamics we obtain

$$\begin{aligned}\dot{V}(x_0) &= u_{b,1}y_{b,1} + \frac{1}{2}\left((J_r - R_r)Q_r x_r + g_r \mathcal{C}_{b,1}(\mathcal{H}_b x_b)\right. \\ &\quad \left.-g_r g(\hat{x}) k_2 x_c\right) Q_r x_r + \frac{1}{2}(Q_r x_r)^T((J_r - R_r)Q_r x_r \\ &\quad +g_r \mathcal{C}_{b,1}(\mathcal{H}_b x_b) - g_r g(\hat{x}) k_2 x_c) + k_2 x_c(-r_c k_2 x_c \\ &\quad +g(\hat{x}) g_r^T Q_r x_r) + \langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} \\ &= u_{b,1}y_{b,1} + y_r y_{b,1} - (Q_r x_r)^T R_r Q_r x_r - r_c k_2^2 x_c^2 \\ &\quad +\langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned}$$

thanks to the skew symmetry of $J_r = -J_r^T$ and using the boundary input-output definitions (3).Using the interconnection law definition (10), and the error system input definition (9), we obtain

$$\dot{V}(x_0) = -(Q_r x_r)^T R_r (Q_r x_r) - r_c k_2^2 x_c^2 - \langle L\tilde{y}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} \le 0 \tag{17}$$

From this it follows that $V(x)$ is a Lyapunov function and then that trajectories are bounded. To obtain the second part of the statement we integrate both members of (17) and we note that $(Q_r x_r)^T R_r (Q_r x_r) = \frac{k_3}{I^2} p^2$ to obtain

$$V(x(t,x_0)) = V(x_0) - \int_0^t \frac{k_3}{I^2} p^2 ds - \int_0^t r_c k_2^2 x_c^2 ds - \int_0^t \langle L\tilde{y}_b, \tilde{y}_b \rangle_{\mathbb{R}^4} ds$$

Since the Lyapunov function $V(x)$ is bounded from below, the statement follows.

Theorem 3. The error system defined by equation (8) and boundary conditions (9) is exponentially stable if $l_1, l_2 > 0,\ l_3, l_4 \geq 0$ or $l_3, l_4 > 0,\ l_1, l_2 \geq 0$.

Proof. Assume that $l_1, l_2 > 0,\ l_3, l_4 \geq 0$ and define the function $\tilde{E} = \frac{1}{2}\langle \tilde{x}, \mathcal{H}_b \tilde{x} \rangle_{\mathcal{L}_2}$. Take its time derivative to obtain

$$\begin{aligned} \frac{1}{2}\dot{E}(x(t,\tilde{x}_0)) &= \frac{1}{2}\langle \mathcal{J}\tilde{x}, \mathcal{H}_b\tilde{x} \rangle_{\mathcal{L}_2} + \frac{1}{2}\langle \mathcal{H}_b\tilde{x}, \mathcal{J}\tilde{x} \rangle_{\mathcal{L}_2} \\ &= \langle \tilde{u}_b, \tilde{y}_b \rangle_{\mathbb{R}^4} = -\langle L\tilde{y}_b, \tilde{y}_b \rangle_{\mathbb{R}^4}, \end{aligned}$$

Where equations (9) and (13) have been used. Then, defining $k = max\{\frac{1+l_1^2}{l_1^2}, \frac{1+l_2^2}{l_2^2}\}$ and using equation (9),

$$\begin{aligned} ||(\mathcal{H}_b\tilde{x})(0)||^2_{\mathbb{R}^4} &= \left(\frac{1}{\rho}\tilde{p}_t(0)\right)^2 + \left(\frac{1}{I_\rho}\tilde{p}_r(0)\right)^2 + (K\tilde{\varepsilon}_t(0))^2 \\ &\quad +(EI\tilde{\varepsilon}_r(0))^2 \\ &= (l_1 K\tilde{\varepsilon}_t(0))^2 + (l_2 EI\tilde{\varepsilon}_r(0))^2 + (K\tilde{\varepsilon}_t(0))^2 \\ &\quad +(EI\tilde{\varepsilon}_r(0))^2 \\ &= \frac{1+l_1^2}{l_1^2}(l_1 K\tilde{\varepsilon}_t(0))^2 + \frac{1+l_2^2}{l_2^2}(l_2 EI\tilde{\varepsilon}_r(0))^2 \\ &\leq k((l_1 K\tilde{\varepsilon}_t(0))^2 + (l_2 EI\tilde{\varepsilon}_r(0))^2) \\ &\leq k((l_1 K\tilde{\varepsilon}_t(0))^2 + (l_2 EI\tilde{\varepsilon}_r(0))^2 \\ &\quad + \left(l_3\frac{1}{\rho}\tilde{p}_t(L)\right)^2 + \left(l_4\frac{1}{I_\rho}\tilde{p}_r(L)\right)^2) \\ &= k\langle L\tilde{y}_b, \tilde{y}_b \rangle_{\mathbb{R}^4} \end{aligned}$$

The statement follows from Theorem 5.19 of (Villegas, 2007). Exponential stability assuming $l_3, l_4 > 0,\ l_1, l_2 \geq 0$ follows from very similar arguments computing $||(\mathcal{H}_b\tilde{x})(L)||^2_{\mathbb{R}^4}$ instead of $||(\mathcal{H}_b\tilde{x})(0)||^2_{\mathbb{R}^4}$.

Remark 4. The previous theorem states that we have an exponentially converging observer state also in case we have boundary observation only at one side of the beam.

Theorem 5. Consider the closed loop system (11), and assume that the system

$$\begin{aligned} \dot{z} &= \tilde{\mathcal{A}}z + Bu(t) \\ &= \begin{bmatrix} \frac{\partial}{\partial \xi}P_1(\mathcal{H}_b x_b) + P_0(\mathcal{H}_b x_b) \\ (J_r - R_r)Q_r x_r + g_r \mathcal{C}(\mathcal{H}_b x_b) \\ -r_c k_c x_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g_r & 0 \\ 0 & 1 \end{bmatrix} u(t) \end{aligned} \tag{18}$$

with $z = [x_b\ x_r\ x_c]^T \in Z \subset \mathcal{L}^2([0,L],\mathbb{R}^4) \times \mathbb{R}^2 \times \mathbb{R}$ and

$$\begin{aligned} D(\tilde{\mathcal{A}}) = \{&z \in Z | x_b \in H^1([0,L],\mathbb{R}^4), \\ &\mathcal{B}_1(\mathcal{H}_b x_b) + g_r Q_r x_r = 0, \mathcal{B}_2(\mathcal{H}_b x_b) = 0\} \end{aligned}$$

is weakly controllable on infinite time, and that the function $g : X_b \times \tilde{X}_b \to \mathbb{R}$ is linear. Then, the closed loop system is asymptotically stable.

Proof. We first notice that we don't have any control on the error system, but with Theorem 3 we already proven that the error state $\tilde{x}$ is exponentially stable. Hence, it remains to show that the other part of the system described by (18) with $u(t) = [g(\tilde{x}_b + x_b)k_2 x_c\ \ g(\tilde{x}_b + x_b)Q_r x_r]^T$, is asymptotically stable. With very similar arguments as in Theorem 1, it is possible to show that $\tilde{\mathcal{A}}$ generates a contraction C0-semigroup and it has compact resolvent. To show asymptotic stability of (18) we apply Lemma 2.1.3 and Lemma 2.2.6 of (Oostveen, 2000). Firstly, we define

$$B = \begin{bmatrix} 0 & 0 \\ g_r & 0 \\ 0 & 1 \end{bmatrix},\ B^* = B^T \begin{bmatrix} \mathcal{H}_b & 0 & 0 \\ 0 & Q_r & 0 \\ 0 & 0 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & g_r^T Q_r & 0 \\ 0 & 0 & k_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_3 & 0 \\ 0 & r_c \end{bmatrix}. \tag{19}$$

Hence, we can define the weighted input-output matrices as $\tilde{B} = B\sqrt{K}$ and $\tilde{B}^* = \sqrt{K}B^*$. Then, the system dynamic can be rewritten as $\dot{z} = (\tilde{\mathcal{A}}' - \tilde{B}\tilde{B}^*)z + \tilde{B}\tilde{u}(t)$, with $\tilde{u}(t) = \sqrt{K}^{-1}u(t)$. With very similar arguments as in the proof of Theorem 1, it is possible to show that the operator $\tilde{\mathcal{A}}'$ generates a contraction $C0-$semigroup. From this, we conclude that $\tilde{A}$ is strongly stable. Then, it remains to show that the considered nonlinearity $\tilde{u}(t)$ is square integrable in infinite time. By definition of $\tilde{u}(t)$, proving its square integrability is the same as proving the square integrability of $u(t)$, hence

$$\begin{aligned} \int_0^\infty (g(\hat{x})k_2 x_c)^2 dt &= \int_0^\infty (g(x_b + \tilde{x}_b)k_2 x_c)^2 dt \\ &\leq M_g^2 k_2^2 \int_0^\infty x_c^2 dt < \infty \end{aligned}$$

where for the first inequality it has been used the boundedness of $x_b, \tilde{x}$ and the linearity of $g(\cdot)$, while for the second it has been used the square integrability of x_c shown in Theorem 2. In a similar manner, the square integrability of $g(\hat{x}_b)g_r^T Q_r x_r$ follows from the square integrability of p.

Remark 6. The weak controllability of the rotating Timoshenko beam connected to a hub has been proven in (Krabs and Sklyar, 1999).

4. NUMERICAL SIMULATIONS

To perform the numerical simulations, it has been considered a finite dimensional approximation of the system. In particular, it has been used the finite element discretization for infinite dimensional port-Hamiltonian system presented in (Golo et al., 2004). This allows to spatially approximate the resulting linear PDEs with linear systems of dimensions depending on the number of discretizing elements. Simulations were made in the Simulink® environment using the "ode23t" time integration algorithm. The set of parameters used for simulation are listed in Table 1, where a Polyethilene HDPE material has been considered for the beam. For isotropic materials, the Shear modulus is related to the Young's modulus $G = \frac{E}{2(1+\nu)}$, where $\nu = \frac{1}{2} - \frac{E}{6K}$ is the Poisson's ratio. To show the observer action, we initialize the flexible beam to the zero initial state, while we set the observer initial condition different from the origin $\hat{x}_0 = [0.01 \cdot \mathbb{1}(z)\ 0\ 0.01 \cdot \mathbb{1}(z)\ 0]^T$, where $\mathbb{1}(z)$ is the characteristic function on the interval $[0, L]$. As weighting function for the nonlinear controller we select the Beam's tip deformation, that can be reconstructed from the system's state

$$g(\hat{x}_b) = \hat{w}(L,t) = \int_0^L \hat{\varepsilon}_t(z,t) + \left(\int_0^z \hat{\varepsilon}_r(\xi,t)d\xi\right) dz \tag{20}$$

Table 1. Simulation Parameters

Name	Variable	Value
Beam's Length	L	$1\ m$
Beam's Width	L_w	$0.1\ m$
Beam's Thickness	L_t	$0.02\ m$
Density	ρ	$950\ \frac{kg}{m^3}$
Young's modulus	E	$8 \times 10^8\ \frac{N}{m^2}$
Bulk's modulus	K	$1.7 \times 10^9\ \frac{N}{m^2}$
Hub's inertia	I	$1\ kg \cdot m^2$
Beam's discretizing elements	n_b	20
Observer's discretizing elements	$\tilde{n}_b$	20
Proportional control constant	k_1	5×10^2
Nonlinear control constant	k_2	1×10^2
Damping injection constant	k_3	5×10^4
Controller dissipation	r_c	0.001

The results are compared with a PD controller. Since

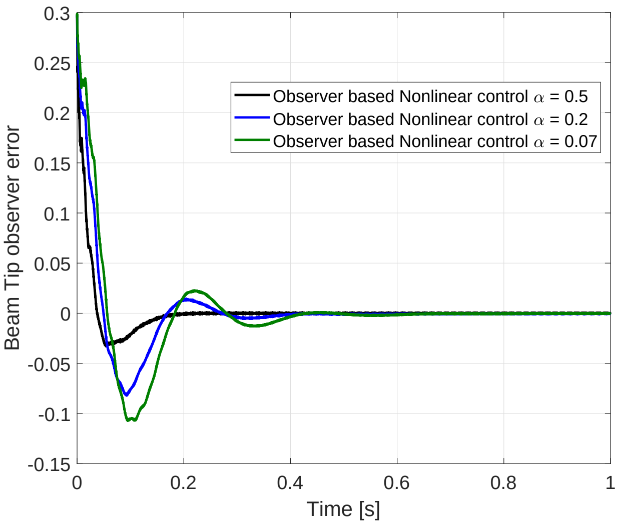

Fig. 2. Beam's tip Observation error $\tilde{w}(L, t)$.

the converging to the "real" values of all the observer's state variables is too dispersive in a single picture, we

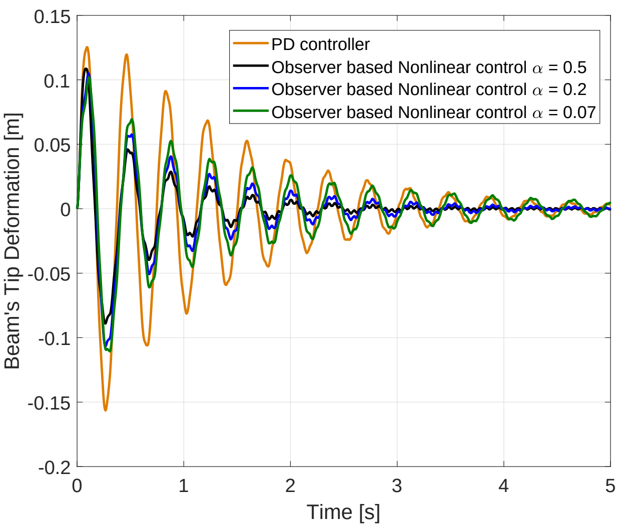

Fig. 3. Tip's deformation $w(L, t)$.

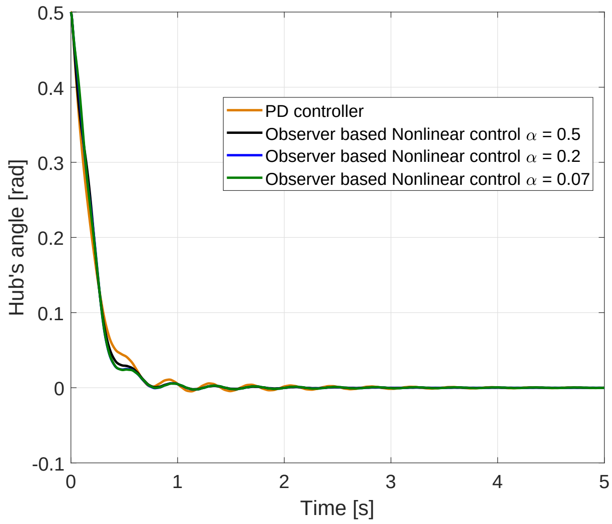

Fig. 4. Hub's angle $q(t)$.

choose to show the error between the real and observed tip's deformation

$$\begin{aligned}\tilde{w_b}(L,t) &= \hat{w}(L,t) - w(L,t)\\ &= \int_0^L \hat{\varepsilon}_t(z,t) - \varepsilon_t(z,t) + \left(\int_0^z \hat{\varepsilon}_r(\xi,t) - \varepsilon_r(\xi,t) d\xi\right)\\ &= \int_0^L \tilde{\varepsilon}_t(z,t) + \left(\int_0^z \tilde{\varepsilon}_r(\xi,t) d\xi\right) dz.\end{aligned} \tag{21}$$

Figure 4 shows that the Beam's tip deformation error converges to zero, and dependently on the value of the diagonal terms of the Observer matrix L we can obtain a faster convergence. This confirm that the Observer's error response has been bounded by an exponential that depends on the Observer matrix's parameters. The set of values used in the simulation are $l_i = \alpha,\ i = \{1, 2, 3, 4\}\ \alpha \in \{0.07, 0.2, 0.5\}$. From Figure 4 we note that as far as the observer converges faster to the original state, the control action is more effective in damping the beam's tip vibrations. In case the observer is not converging fast enough, it is shown that the oscillation are kept smaller and the system asymptotically converge to the origin, but with a rate similar to the PD controller. Finally, Figure 4 shows that the hub's angular displacement has a similar rate of convergence in all the different control law applications.

5. CONCLUSIONS

It has been considered a model of the rotating flexible beam composed by a set of PDEs interconnected with an ODE, with actuation in the ODE. Since the control input is not on the PDEs' boundaries, a passive preserving way of using the deformation information in the controller is through the use of a nonlinear dynamic control law. In this paper, the nonlinear controller makes use of an extimated state instead of the original one. Firstly, it has been proven the exponential stability of the observer's state assured that we have at least the complete observation in one side of the beam. Secondly, the nonlinear closed loop system has been analysed using the operator formalism and asymptotic stability has been formally proved. The observer parameters can be selected to set the convergence speed of the estimated state, while the beam's state function can be chosen to obtain different closed loop deformation behaviours. Numerical simulations have been used to show the closed loop behaviour with the use of different observer's parameters: in case the observer is not converging fast enough, even though the closed loop system is still asymptotically stable, the rate of convergence can became slower than the one of the PD controller. An experimental set-up where it will be possible to test the proposed control law is currently under construction. The future work will deal with the generalization of this type of controller for a class of PDEs-ODE system, that can be frequently encountered in mechanical applications.

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