Modelling and stability analysis of a flexible rotating beam in collision

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Abstract— **This paper considers the problem of a rotating flexible beam in collision with an external object. The model's equations for the flexible beam in collision exhibits instant changes during impact times, therefore the model is cast in the class of switched infinite dimensional operator systems. The aim is to study the stability of the closed loop system with a PD control law, making use of the semigroup formalism together with Lyapunov stability theory. To this end, we present a new stability result making use of multiple Lyapunov functions obtained as an adaptation of a theorem from finite dimensional hybrid systems theory. We show the port-Hamiltonian modelling procedure for the colliding rotating flexible beam, using distributed parameter equations to describe the beam's dynamic. Then, we compute the equilibrium position of the closed loop systems, and using the shifted variables with respect to the equilibrium position, we cast the system in the class of switched infinite operator systems. Finally we select the Lyapunov functions for the contact and noncontact phases and we show, through numerical simulations, that they respect the assumptions of the proposed stability theorem.**

I. INTRODUCTION

A lot of critical tasks in robotics involve the contact between the manipulator and an external object or the environment. In some cases, flexible manipulators are preferable to rigid ones due to their lightweight and because they can assure smooth contact force in impact scenario. This is why they can be encountered in many application fields ranging from spatial [1] to micro-manipulation applications [2]. The major challenge is to come up with a suitable model for control purposes that is enough accurate in taking into account the impact dynamics.

The main difficulty is that the distributed parameter nature of the flexible beam system would require an infinite dimensional analysis. A finite dimensional analysis provides a good approximation of the flexible phenomena in case of unconstrained conditions, but it can bring to misleading results in presence of impact, where a large bandwidth of frequencies will be excited. While there exist many studies on the control of flexible manipulators in impact scenario using finite dimensional models [3], [4], [5], very few have discussed the collision issue using infinite-dimensional models [6]. The dynamical model of a colliding flexible beam is expected to have instant changes in impact times. Therefore the model will combine behaviours that are typical of continuous-time dynamical systems with behaviours that are typical of discrete-time dynamical systems. This definition perfectly fits into the class of Hybrid dynamical systems. The stability as well as the control design theory have been extensively studied for finite dimensional hybrid systems [7]. On the other side, some results have been established for infinite dimensional hybrid systems. In [8] are presented some general results on Lagrange, asympotic and exponential stability (in all their variation) for the class of hybrid infinite dimensional systems, that do not require the determination of Lyapunov functions, as well as results which do involve Lyapunov functions. In [9] some conditions for obtaining exponential stability are given for a subclass of hybrid systems, namely switched operator systems. Other characterizations of exponentially stable switched operator equations can be founded in [10], [11].

In this prelimanary work we are interested in studying Lagrange stability of a flexible beam in impact scenario. To do so, we cast the proposed model in the class of switched linear operator systems and we study its stability using an adaptation for infinite dimensional systems of the Lagrange stability result, that makes use of multiple Lyapunov functions, proposed in [12]. Numerical simulations are provided to validate the theoretical development. The remainder of this paper is organized as follows. In the next section we give some background on infinite dimensional switching linear systems; In section III we propose a model for the flexible beam and we show the equilibrium position computation together with the stability study; in section IV are given numerical simulations. We conclude the paper with some final remarks and comments on future research.

II. PRELIMINARIES

In this section we provide the necessary background concerning dynamical systems determined by switching operator equations. Consider the general operator equation

$$\dot{x}(t) = f(x,m), \tag{1}$$

where $x \in X$ is the *continuous state* and belongs to an appropriate Hilbert space, and $m \in M = \{1,2,\dots,N\}$ is the *discrete state*. The couple defined as the composition of the continuous and discrete state (x,m) is called the *hybrid state*. The discrete state depends in general on the continuous state x and on the previous discrete state m_{i-1}, *i.e.* $m_i = \phi(x,m_{i-1})$ where $\phi : X \times M \to M$ is a discrete transition. If for each $x \in X$, only one $m \in M$ is possible, then the system is called a switching system, otherwise is an hybrid system. Here, we consider switched systems, then we partition the state space

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in N disjoint regions

$$\Omega_1 \ldots \Omega_N \subset X \tag{2}$$

where $\Omega_i \bigcap \Omega_j = \emptyset, \; i \neq j$.
Consider a family $\mathbb{A} = \{\mathscr{A}_i, \, i \in I\}$ of linear operators defined on a common domain $\mathscr{D}(\mathscr{A}_i) = \mathscr{D}(\mathscr{A}_j)$ for $i, j \in I$. The considered switched operator system is given by

$$\dot{x}(t) = \mathscr{A}_\phi x. \tag{3}$$

The assumption on the common domain for all the considered operators avoids wellposedness problems of (3) during switching times. We point out that the operators $\mathscr{A}_i \in \mathbb{A}$ are not required to generate a strongly continuous semigroup in the space in which they are defined, as assumed in [9]. The continuous state evolution of (3) can be described as: starting at (x_0, m_0) at time t_0, the continuous trajectory evolves according to $\dot{x} = \mathscr{A}_{m_0} x$. Let us assume that at time t_1, x reaches a value $x_1$that triggers a discrete change from m_0 to m_1; then the process evolves according to $\dot{x} = \mathscr{A}_{m_1} x$. Here, we consider hybrid systems with continuous state that doesn't change during switching and therefore the hybrid state (x, m_i) becomes (x, m_j). The changes of discrete state happens at the so called switching sets

$$S_{i,j} = \{x \in X | \, m_j = \phi(x, m_i)\}. \tag{4}$$

We define a *switching sequence* anchored to a certain initial state

$$\{S_n(x_0)\} = (m_0, t_0), \, (m_1, t_1), \ldots, (m_n, t_n), \, \ldots. \tag{5}$$

The switching sequence along (3) describes completely the trajectory of the system according to the following rule: (m_i, t_i) means that the system evolves according to $\dot{x}(t) = \mathscr{A}_{m_i} x$ for $t_i \leq t \leq t_{i+1}$. We denote this trajectory by $x_{S(x_0)}(t)$. Throughout we assume that this sequence is minimal. We can take projections of the defined sequence

$$\begin{array}{l} \Pi_1(\{S_n(x_0)\}) = m_0, m_1, \ldots, m_i, \ldots \\ \Pi_2(\{S_n(x_0)\}) = t_0, t_1, \ldots, t_i, \ldots \end{array} \tag{6}$$

We denote by $S(x_0)|_m$ the endpoints of times for which the system m is active. The internal completion $\mathscr{I}(T)$ of a strictly increasing sequence of time $T = t_0, \, t_1, \ldots, t_i, \ldots$ is the set

$$\mathscr{I}(T) = \bigcup_{j \in \{0,1,2,\ldots\}} [t_{2j}, t_{2j+1}] \tag{7}$$

Here, $\mathscr{I}(S(x_0)|_m)$ is the set of time that the $m-th$ system is active. Finally, let $\mathscr{E}$ denote the even sequence of T : t_0, t_2, $t_4, \ldots$, and $\mathscr{O}$ denote the odd sequence of T : t_1, t_3, $t_5, \ldots$.

Definition 2.1: An Hybrid state (x_{eq}, m_{eq}) is said to be an *hybrid equilibrium* of (1) if it has the property that whenever the hybrid system starts at (x_{eq}, m_{eq}), it will remain there for all future time.
The hybrid equilibrium points may be obtained by finding the states satisfying

$$\mathscr{A}_m x = 0 \quad \forall m \in M. \tag{8}$$

All the continuous states satisfying (8) are not hybrid equilibrium because there may be not possible hybrid states. *Example*: one solution of (8) (x_{eq}, m_i) may not be possible in the sense that x_{eq} is not contained in the region of the state space that is associated with the discrete state m_i.
Without loss of generality the origin is assumed to be a continuous equilibrium for which stability is investigated. Now, we can define a single candidate Lyapunov's function V_m for a certain system's dynamic $\mathscr{A}_m x$.

Definition 2.2: A continuous functional $V_m : X \to [0, \infty)$ such that $\forall x \in \Omega_m \; \alpha(||x||) \leq V_m(x) \leq \beta(||x||)$, where $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are class-K functions, is a Lyapunov functional for $\mathscr{A}_m x$ and the trajectory $x_{S(x_0)}(t)$ if:

- $V_m(x_{S(x_0)}(t))$ is Dini differentiable;
- $\dot{V}_{m,+}(x_0) := \limsup_{t \to 0} \frac{V_m(x(t,x_0)) - V_m(x_0)}{t} \leq 0 \quad \forall x_0 \in \Omega_m$.

Since the Dini derivative is usually difficult to compute, we introduce in the next Lemma an easy way to compute it. Note that for a functional V_m to be considered a Lyapunov functional for $\mathscr{A}_m x$, it is necessary that $\dot{V}_{m,+}(x_0)$ is non positive only in the region Ω_m, but in principle $\dot{V}_{m,+}(x_0)$ can be computed in the whole state space X.

Lemma 2.1: If the functional V_m is Frechet differentiable, then for $x_0 \in \Omega_i \bigcap D(\mathscr{A}_i) \; i \in M$, $V_m(x_{S(x_0)(t)})$ is differentiable for $t = 0$ and

$$\dot{V}_{m,t}(x_0) = \frac{dV_m(x_{S(x_0)}(t, x_0))}{dt}|_{t=0} = dV_m(x_0)\mathscr{A}_i x_0 \quad \forall x_0 \in \Omega_i \tag{9}$$

where dV_m denotes the Frechet derivative of V_m.

Proof: Divide the state space in the different subspaces Ω_i. Then, the time derivative equality in each Ω_i is shown to hold as in Lemma 11.2.5 of [13]. □

In the previous lemma, we gave the formula for computing the time derivative of the Lyapunov function V_m in any subspace Ω_i. At this point we are in position to state the bounded trajectory theorem for switched linear operator systems, that is an adaptation for infinite dimensional systems of theorem 2.3 in [12].

Theorem 2.2: Let assume that there exists a unique local mild solution of (3). If there exist Lyapunov functions V_m for every $\mathscr{A}_m x$ that are non increasing in $\mathscr{E}(S(x_0)|_m) \; \forall m \in M$, then (3) has a global bounded mild solution for every initial condition $x_0 \in X$.

Proof: The proof is very similar to the one of Theorem 2.3 in [12]. The adaptation to infinite dimensional switched operator systems will be given in the journal version of this paper. □

The non-increasing condition of V_m in $\mathscr{E}(S(x_0)|_m)$ concerns the value of each function V_m each time is "switched in". It means that the value of V_m at switching points should be smaller than that of the previous time it has become active or "switched in".

III. FLEXIBLE ROTATING BEAM IN COLLISION

A. Modelling and Control design

For a sake of clarity, we define the variables and the parameters that are used for the modelling of the system

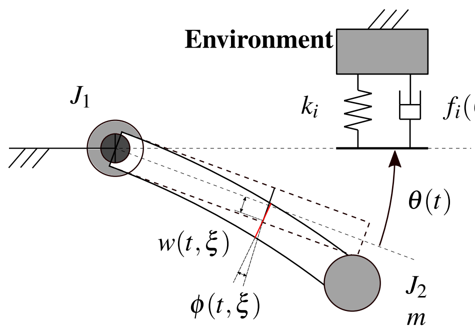

Fig. 1. Rotating flexible Timoshenko's beam in impact with the external environment.

in Figure 1. The rotor angle $\theta(t)$ is a real function of time, while $\xi \in [0, L]$ identifies the spatial coordinate of the beam. The deflection of the beam in the rotating frame is defined with $w(t,\xi)$, while $\phi(t,\xi)$ represents the relative rotation of the beam cross section. All the physical parameters are positive real. J_1 and J_2 represent the rotary inertia of the hub to which the beam is connected and the end effector's rotary inertia, respectively. m is the end effector's mass. E, I are the Young's modulus and the moment of inertia of the beam's cross section, respectively. The beam's cross section is assumed to be rectangular, hence its inertia is defined to be $I = \frac{L_w^3 L_t}{12}$, where L_w and L_t are the width and the thickness of the beam, respectively. ρ, I_ρ are the density and the mass moment of inertia of the beam's cross section, respectively. The mass moment of inertia of the cross section is defined as $I_\rho = I\rho$. K is defined as $K = kGA$, where k is a constant depending on the shape of the cross section ($k = 5/6$ for rectangular cross sections), G is the Shear modulus and A is the cross sectional area.

According to [14] the compliant surface can be considered as a mass-less system composed by a spring and a damper. In this notes we consider a linear spring k_i and a nonlinear damper $f_i(\theta, w) = c_i(L\theta + w(t,L))$, with c_i constant. Note that the quantity $L\theta + w(t,L)$ corresponds to the distance of the end-effector from the external environment when is negative, and the external object deformation when it is positive. From now on we will not explicit the dependency from time and space of the variables when it is clear from the context. The kinetic energy H_k and the potential energy H_p, using Timoshenko's assumptions, write

$$\begin{aligned} H_k &= \tfrac{1}{2}J_1\dot{\theta}^2 + \tfrac{1}{2}J_2(\dot{\theta} + \dot{\phi}(t,L))^2 + \tfrac{1}{2}m(L\dot{\theta} + \dot{w}) \\ &+ \tfrac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right)^2\right] d\xi \\ H_p &= \tfrac{1}{2}\int_0^L \left[K\left(\frac{\partial w}{\partial \xi} - \phi\right)^2 + EI\left(\frac{\partial \phi}{\partial \xi}\right)^2\right] d\xi \\ &+ \tfrac{1}{2}k_i \mathbb{1}(L\theta + w(t,L))(L\theta + w(t,L))^2 \end{aligned}$$

where $\mathbb{1}(\alpha)$ denotes the characteristic function $\mathbb{1} : \mathbb{R} \to \{0,1\}$ defined as

$$\mathbb{1}(\alpha) = \begin{cases} 1 & if\ \alpha \geq 0 \\ 0 & if\ \alpha < 0. \end{cases} \tag{10}$$

The Hamilton's principle is used to obtain the system's dynamical equations, considering $\delta W_{nc} = u(t)\delta\theta - f_i(\theta,w)\mathbb{1}(L\theta + w(t,L))(L\dot{\theta} + \dot{w}(t,L))\delta(L\theta + w(t,L))$ the virtual work of non-conservative forces, where $u(t)$ identifies the external torque, and the other term corresponds to the nonlinear dissipation of the soft-impact model. The derived set of mixed partial and ordinal differential equations writes

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta}\right)\right) = \frac{\partial}{\partial \xi}\left(K\left(\frac{\partial w}{\partial \xi} - \phi\right)\right) \\ \frac{\partial}{\partial t}\left(I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right)\right) = \frac{\partial}{\partial \xi}\left(EI\frac{\partial \phi}{\partial \xi}\right) + K\left(\frac{\partial w}{\partial \xi} - \phi\right) \\ J_1\frac{d}{dt}\dot{\theta} = +EI\frac{\partial \phi(t,0)}{\partial \xi} + u(t) \\ m\frac{d}{dt}(L\dot{\theta} + \dot{w}(t,L)) = -K\left[\frac{\partial w}{\partial \xi}(L,t) - \phi(L,t)\right] \\ \qquad -k_i\mathbb{1}(L\theta + w(t,L))(L\theta + w(t,L)) \\ \qquad -f_i(\theta,w)\mathbb{1}(L\theta + w(t,L))(L\dot{\theta} + \dot{w}(t,L)) \\ J_2\frac{d}{dt}(\dot{\theta} + \dot{\phi}(t,L)) = -EI\frac{\partial \phi}{\partial \xi}(t,L) \end{cases} \tag{11}$$

with boundary conditions

$$w(t,0) = 0 \quad \phi(t,0) = 0. \tag{12}$$

The energy states of the infinite dimensional system are defined by

$$\begin{aligned} \varepsilon_t &= \frac{\partial w}{\partial \xi} - \phi & p_t &= \rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta}\right) \\ \varepsilon_r &= \frac{\partial \phi}{\partial \xi} & p_r &= I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right). \end{aligned} \tag{13}$$

The equations describing the infinite dimensional system can be written as a PH system

$$\dot{x}_b = \mathscr{J} x_b = P_1\frac{\partial}{\partial \xi}(\mathscr{H}_b x_b) + P_0(\mathscr{H}_b x_b) \tag{14}$$

with $x_b = [p_t\ p_r\ \varepsilon_t\ \varepsilon_r]^T \in X_b \subset L^2([0,L], \mathbb{R}^4)$ representing the system's state. The matrices in equation (14) are defined as

$$P_1 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \quad P_0 = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{bmatrix} \quad \mathscr{H}_b = \begin{bmatrix} \rho^{-1}&0&0&0 \\ 0&I_\rho^{-1}&0&0 \\ 0&0&K&0 \\ 0&0&0&EI \end{bmatrix}.$$

The state space X_b is equipped with the L^2 inner product $\langle x_b, x_b\rangle_{X_b} = \langle x_b, \mathscr{H}_b x_b\rangle_{L^2}$, such to express the energy related to the flexible part of the system as $H_b = \frac{1}{2}\langle x_b, x_b\rangle_{X_b}$. The boundary variables are defined as [15]

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} (\mathscr{H}_b x_b)(t,0) \\ (\mathscr{H}_b x_b)(t,L) \end{bmatrix}.$$

Then, define the boundary input and output operators as

$$\begin{aligned} \mathscr{B}_1(\mathscr{H}_b x_b) &= W_2\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I_\rho^{-1}p_r(t,0) \\ \rho^{-1}p_t(t,L) \\ I_\rho^{-1}p_r(t,L) \end{bmatrix} = u_{b,1} \\ \mathscr{B}_2(\mathscr{H}_b x_b) &= W_1\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \rho^{-1}p_t(t,0) = u_{b,2} \\ \mathscr{C}_1(\mathscr{H}_b x_b) &= \tilde{W}_2\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} -EI\varepsilon_r(t,0) \\ K\varepsilon_t(t,L) \\ EI\varepsilon_r(t,L) \end{bmatrix} = y_{b,1} \\ \mathscr{C}_2(\mathscr{H}_b x_b) &= \tilde{W}_1\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = -K\varepsilon_t(t,0) = y_{b,2} \end{aligned} \tag{15}$$

where $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ and $\tilde{W} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix}$ are appropriate matrices, and are such that $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ is non-singular. The total boundary input-output operators are defined as

$$\begin{aligned}\mathscr{B}(\mathscr{H}_b x_b) &= \begin{bmatrix}\mathscr{B}_1(\mathscr{H}_b x_b)\\ \mathscr{B}_2(\mathscr{H}_b x_b)\end{bmatrix} = \begin{bmatrix}W_1\\ W_2\end{bmatrix}\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix} = u_b\\ \mathscr{C}(\mathscr{H}_b x_b) &= \begin{bmatrix}\mathscr{C}_1(\mathscr{H}_b x_b)\\ \mathscr{C}_2(\mathscr{H}_b x_b)\end{bmatrix} = \begin{bmatrix}\tilde{W}_1\\ \tilde{W}_2\end{bmatrix}\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix} = y_b\end{aligned} \tag{16}$$

Denote the restoring torques and forces with u_r and the hub's and end-effector's velocities with y_r

$$u_r = \begin{bmatrix} EI\frac{\partial\phi}{\partial\xi}(t,0)\\ -K\left[\frac{\partial w}{\partial\xi}(t,L)-\phi(t,L)\right]\\ -EI\frac{\partial\phi}{\partial\xi}(t,L)\end{bmatrix} \quad y_r = \begin{bmatrix}\dot\theta\\ L\dot\theta+\dot w(t,L)\\ \dot\theta+\dot\phi(t,L)\end{bmatrix}. \tag{17}$$

The states related to the finite dimensional part are defined as

$$\begin{aligned} p_1 &= J_1\dot\theta & q_1 &= \theta\\ p_2 &= m(L\dot\theta+\dot w(t,L)) & q_2 &= L\theta + w(t,L)\\ p_3 &= J_2(\dot\theta+\dot\phi(t,L)) & & \end{aligned} \tag{18}$$

and the related equations write

$$\begin{cases}\dot p = +u_r(t) + f(p,q) + gu(t)\\ \dot q = \begin{bmatrix}\frac{1}{J_1}p_1\\ \frac{1}{m}p_2\end{bmatrix}\\ y_r(t) = \begin{bmatrix}\frac{1}{J_1}p_1 & \frac{1}{m}p_2 & \frac{1}{J_2}p_3\end{bmatrix}^T\end{cases} \tag{19}$$

where $p = [p_1\ p_2\ p_3]^T$, $q = [q_1\ q_2]^T$, the matrices and the nonlinear vector are defined as

$$g = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad f(p,q) = \begin{bmatrix}0\\ -k_i\mathbb{1}(q_2)q_2 - \frac{c_i}{m}\mathbb{1}(q_2)q_2p_2\\ 0\end{bmatrix} \tag{20}$$

Use the original boundary conditions (12) together with the state variables definition (13) to derive the interconnection relation between the infinite dimensional and the finite dimensional parts of the system

$$u_{b,1} = y_r \quad u_r = -y_{b,1}. \tag{21}$$

while the remaining boundary condition of (14) is equal to zero, i.e. $u_{b,2} = 0$. We can now define the input control torque as a simple PD controller

$$u(t) = -k(\theta(t)-\theta^o) - c\dot\theta(t) \tag{22}$$

and defining the new error state $\tilde q_1 = \theta - \theta^o$, we can write the closed loop equations in the following semi-linear operator form

$$\begin{aligned}\dot x &= \begin{bmatrix} P_1\frac{\partial}{\partial\xi}(\mathscr{H}x_b) + P_0(\mathscr{H}x_b)\\ +EI\varepsilon_r(t,0) - k\tilde q_1 - \frac{c}{J_1}p_1\\ +K\varepsilon_t(t,L) - k_iq_2\\ +EI\varepsilon_r(t,L)\\ \frac{1}{J_1}p_1\\ \frac{1}{m}p_2\end{bmatrix} + \begin{bmatrix}0\\0\\ k_i\mathbb{1}(-q_2)q_2\\ -\frac{c_i}{m}\mathbb{1}(q_2)q_2p_2\\0\\0\\0\end{bmatrix}\\ &= \mathscr{A}x + f(x)\end{aligned} \tag{23}$$

where $x = [x_b\ p_1\ p_2\ p_3\ \tilde q_1\ q_2]^T \in X = L^2([0,L],\mathbb{R}^4)\times\mathbb{R}^5$ is the state of the system and the domain of the linear operator $\mathscr{A}$ is defined as

$$\begin{aligned}D(\mathscr{A}) = \ & \{x\in X | x_b \in H^1([0,L],\mathscr{R}^4),\ \mathscr{B}_2x = 0,\\ & \mathscr{B}_1x = [p_1/J_1\ \ p_2/m\ \ p_3/J_2]^T\}.\end{aligned} \tag{24}$$

The inner product in the state space is defined for $x_1, x_2 \in X$ as

$$\begin{aligned}\langle x_1,x_2\rangle_X = \ & \langle x_1,\mathscr{H}x_2\rangle_{L^2} + \tfrac{1}{J_1}p_{1,1}p_{1,2} + \tfrac{1}{m}p_{2,1}p_{2,2}\\ & + \tfrac{1}{J_2}p_{3,1}p_{3,2} + k\tilde q_{1,1}\tilde q_{1,2} + k_iq_{2,1}q_{2,2}\end{aligned} \tag{25}$$

with associated norm $||x||_X = \sqrt{\langle x,x\rangle_X}$.

B. Equilibrium position

Since we are interested in the collision between the flexible beam and the external environment, we assume $\theta^o > 0$. Therefore, we first notice that the equilibrium position for the non contact equations corresponds to a state that is in the contact region, thus it is not a possible hybrid state. Consequently we investigate the equilibrium position for the contact situation. Studying $\mathscr{A}x + f(x) = 0$ with $q_2 > 0$, in the original coordinates with $q_1 = \theta$, $q_1^o = \theta^o$, results in equations

$$\begin{cases} P_1\frac{\partial}{\partial\xi}(\mathscr{H}x_b) + P_0(\mathscr{H}x_b) = 0\\ EI\varepsilon_r(t,0) - k(q_1 - q_1^o) - \frac{c}{m}p_1 = 0\\ EI\varepsilon_r(t,L) = 0\\ -K\varepsilon_t(t,L) - k_iq_2 - \frac{c_i}{m}q_2p_2 = 0\\ \frac{1}{J_1}p_1 = 0\\ \frac{1}{m}p_2 = 0\end{cases} \tag{26}$$

with boundary conditions

$$\begin{aligned}\tfrac{1}{\rho}p_t(t,0) &= 0 & \tfrac{1}{I_\rho}p_r(t,0) &= \tfrac{p_1}{J_1}\\ \tfrac{1}{\rho}p_t(t,L) &= \tfrac{p_2}{m} & \tfrac{1}{I_\rho}p_r(t,L) &= \tfrac{p_3}{J_2}\end{aligned} \tag{27}$$

Since we are studying an equilibrium position, all the momentum are equal to zero $p_1 = p_2 = p_3 = 0$, consequently the set of equations transforms into

$$\begin{cases} P_1\frac{\partial}{\partial z}(\mathscr{H}x_b) + P_0(\mathscr{H}x_b) = 0\\ EI\varepsilon_r(t,0) = k(q_1 - q_1^o)\\ EI\varepsilon_r(t,L) = 0\\ K\varepsilon_t(t,L) = -k_iq_2\end{cases} \tag{28}$$

with boundary conditions

$$\frac{1}{\rho}p_t(t,0) = \frac{1}{I_\rho}p_r(t,0) = \frac{1}{\rho}p_t(t,L) = \frac{1}{I_\rho}p_r(t,L) = 0. \tag{29}$$

Take the first differential set of equations and write it in the extended form

$$\begin{aligned}&\tfrac{\partial}{\partial\xi}K\varepsilon_t(t,\xi) = 0\\ &\tfrac{\partial}{\partial\xi}EI\varepsilon_r(t,\xi) + K\varepsilon_t(t,\xi) = 0\\ &\tfrac{\partial}{\partial\xi}\tfrac{1}{\rho}p_t(t,\xi) - \tfrac{1}{I_\rho}p_r(t,\xi) = 0\\ &\tfrac{\partial}{\partial\xi}\tfrac{1}{I_\rho}p_r(t,\xi) = 0\end{aligned} \tag{30}$$

with the same boundary conditions as before. The last two equations of (30) with boundary conditions (29) imply

$p_t(t,\xi) = p_r(t,\xi) = 0$. From the the first equation of (30) and the boundary condition $\varepsilon_t(t,L) = -\frac{k_i}{K} q_2$ we obtain

$$\varepsilon_t(t,\xi) = -\frac{k_i}{K} q_2. \tag{31}$$

From the second equation of (30) together with the first boundary condition (28) we obtain

$$\varepsilon_r(t,\xi) = +\frac{k}{EI}(q_1 - q_1^o) + \frac{k_i}{EI} q_2 \xi. \tag{32}$$

Using $\varepsilon_r(L,t) = 0$ we obtain

$$L k_i q_2 = k(q_1^o - q_1). \tag{33}$$

Now, from the state variable definition (13) compute

$$\begin{aligned} w(t,\xi) = &\ + \int_0^\xi \varepsilon(t,z) + \int_0^\xi \varepsilon_r(t,\alpha) d\alpha dz \\ = &\ -\int_0^\xi \frac{k_i q_2}{K} + \int_0^\alpha \frac{k(q_1 - q_1^o)}{EI} + \frac{k_i q_2 \alpha}{EI} d\alpha d\xi \\ = &\ -\frac{k_i q_2 \xi}{K} - \frac{k(q_1^o - q_1)}{2EI}\xi^2 + \frac{k_i q_2}{6EI}\xi^3 \end{aligned} \tag{34}$$

and because of (33), we can write

$$w(t,\xi) = -\frac{k_i q_2}{K}\xi - \frac{k_i q_2 L}{2EI}\xi^2 + \frac{k_i q_2}{6EI}\xi^3 \tag{35}$$

that computed at the $\xi = L$ boundary gives

$$w(t,L) = -\frac{k_i q_2}{k} L - \frac{k_i q_2}{3EI} L^3. \tag{36}$$

Using the q_1 and q_2 definitions of (18), we know that

$$q_2 = L q_1 + w(t,L) \tag{37}$$

then, substitute the $w(t,L)$ definition in the former equation to obtain

$$q_2 = A q_1, \quad A = \frac{L}{1 + k_i\left(\frac{L}{K} + \frac{L^3}{3EI}\right)}. \tag{38}$$

Define $B = \frac{k}{k_i L A + k}$ and substitute the former equation in (33) to get

$$q_1^* = B q_1^o, \quad q_2^* = A B q_1^o. \tag{39}$$

Then we are able to find all the equilibrium positions for all the states as a function of the control parameter q_1^o

$$\varepsilon_t^*(\xi) = -\frac{k_i A}{K} B q_1^o \quad \varepsilon_r^*(\xi) = \frac{k_i A}{EI} B(\xi - L) q_1^o \tag{40}$$

We can now define a new set of shifted variables with respect to the founded equilibrium states:

$$\begin{array}{ll} \varepsilon_t'(t,\xi) = \varepsilon_t(t,\xi) - \varepsilon_t^*(\xi) & \varepsilon_r'(t,\xi) = \varepsilon_r(t,\xi) - \varepsilon_r^*(\xi) \\ p_t'(t,\xi) = p_t(t,\xi) & p_r'(t,\xi) = p_r(t,\xi) \\ q_1'(t) = q_1(t) - q_1^* & q_2'(t) = q_2(t) - q_2^* \end{array} \tag{41}$$

and $p_1'(t) = p_1(t),\ p_2'(t) = p_2(t),\ p_3'(t) = p_3(t)$. The equations in the new variables become

$$\begin{cases} \dot p_t'(t,\xi) = \frac{\partial}{\partial\xi} K(\varepsilon_t'(t,\xi) + \varepsilon_t^*(\xi)) \\ \dot p_r'(t,\xi) = \frac{\partial}{\partial\xi} EI(\varepsilon_r'(t,\xi) + \varepsilon_r^*(\xi)) + K(\varepsilon_t'(t,\xi) + \varepsilon_t^*(\xi)) \\ \dot\varepsilon_t'(t,\xi) = \frac{\partial}{\partial\xi}\frac{1}{\rho} p_t'(t,\xi) - \frac{1}{I_\rho} p_r'(t,\xi) \\ \dot\varepsilon_r'(t,\xi) = \frac{\partial}{\partial\xi}\frac{1}{I_\rho} p_r'(t,\xi) \\ \dot p_1'(t) = +EI(\varepsilon_r'(t,0) + \varepsilon_r^*(0)) - k(q_1'(t) + q_1^* - q_1^o) - \frac{c}{m} p_1'(t) \\ \dot p_2'(t) = -EI(\varepsilon_r'(t,L) + \varepsilon_r^*(L)) \\ \dot p_3'(t) = -K(\varepsilon_t'(t,L) + \varepsilon_t^*(L)) - k_i \mathbb{1}(q_2'(t) + q_2^*)(q_2'(t) + q_2^*) \\ \quad -\frac{c_i}{m}\mathbb{1}(q_2'(t) + q_2^*)(q_2'(t) + q_2^*) p_3'(t). \end{cases} \tag{42}$$

Using the equilibrium definitions (40), the first two equations can be rewritten in the classical form

$$\begin{cases} \dot p_t'(t,\xi) = \frac{\partial}{\partial\xi} K\varepsilon_t'(t,\xi) \\ \dot p_r'(t,\xi) = \frac{\partial}{\partial\xi} EI\varepsilon_r'(t,\xi) + K\varepsilon_t'(t,\xi) \end{cases} \tag{43}$$

allowing the port-Hamiltonian representation for the infinite dimensional part of the system. Then, defining $x' = [\varepsilon_t'\ \varepsilon_r'\ p_t'\ p_r'\ p_1'\ p_2'\ p_3'\ q_1'\ q_2']^T$ we can write the linear operators equations in case of contact or non-contact scenario

Non contact operator

$$\dot x' = \begin{bmatrix} P_1 \frac{\partial}{\partial\xi}(\mathscr{H} x_b') + P_0(\mathscr{H} x_b') \\ EI\varepsilon_r'(t,0) - k q_1' - \frac{c}{J_1} p_1' \\ -K\varepsilon_t'(t,L) + A k_i B q_1^o \\ -EI\varepsilon_r'(t,L) \\ \frac{1}{J_1} p_1' \\ \frac{1}{m} p_2' \end{bmatrix} = \mathscr{A}_1 x' \tag{44}$$

Contact operator

$$\dot x' = \begin{bmatrix} P_1 \frac{\partial}{\partial\xi}(\mathscr{H} x_b') + P_0(\mathscr{H} x_b') \\ EI\varepsilon_r'(t,0) - k q_1' - \frac{c}{J_1} p_1' \\ -K\varepsilon_t'(t,L) - k_i q_2' - \frac{c_i}{m}(q_2' + q_2^*) p_2' \\ -EI\varepsilon_r'(t,L) \\ \frac{1}{J_1} p_1' \\ \frac{1}{m} p_2' \end{bmatrix} = \mathscr{A}_2 x' \tag{45}$$

with domains defined as

$$\begin{aligned} D(\mathscr{A}_1) = &\quad D(\mathscr{A}_2) = \{x' \in X | x_b' \in H^1([0,L],\mathbb{R}^4),\ \mathscr{B}_2 x' = 0, \\ &\quad \mathscr{B}_1 x' = [p_1'/J_1\ \ p_2'/m\ \ p_3'/J_2]^T\}, \end{aligned} \tag{46}$$

switching sets

$$\begin{aligned} S_{nc,c} = &\quad \{x' \in X |\ q_2' = -q_2^*, p_2 > 0\} \\ S_{c,nc} = &\quad \{x' \in X |\ q_2' = -q_2^*, p_2 < 0\}, \end{aligned} \tag{47}$$

and non-contact and contact region defined, respectively, as

$$\begin{aligned} \Omega_{nc} = \{x' \in X | q_2 < -q_2^*\} \\ \Omega_c = \{x' \in X | q_2 \geq -q_2^*\}. \end{aligned} \tag{48}$$

C. Stability Analysis

We now use Theorem 2.2 to study the stability of the switched system defined by operators (44)-(45) and state space partition (48), in case the control law sets the equilibrium position in the contact region, *i.e.* $q_1^o > 0$.

Proposition 3.1: The solutions of the switched system (44)-(48) are bounded for every initial condition $x_0' \in X$.

Sketch of the proof: Let's consider the following Lyapunov functions for the noncontact and contact operators

$$\begin{aligned} V_{nc} = &\ \frac{1}{2}\int_0^L \left(K(\varepsilon_t' + \varepsilon_t^*)^2 + EI(\varepsilon_r' + \varepsilon_r^*)^2 + \frac{1}{\rho} p_t'^2 + \frac{1}{I_\rho} p_r'^2\right) d\xi \\ &+ \frac{1}{2J_1} p_1'^2 + \frac{1}{2J_2} p_2'^2 + \frac{1}{2m} p_3'^2 + \frac{1}{2} k(q_1' - (1-B) q_1^o)^2, \end{aligned} \tag{49}$$

$$\begin{aligned} V_c = &\ \frac{1}{2}\langle x_b', \mathscr{H} x_b'\rangle_{L^2} + \frac{1}{2J_1} p_1'^2 + \frac{1}{2J_2} p_2'^2 + \frac{1}{2m} p_3'^2 \\ &+ \frac{1}{2} k q_1'^2 + \frac{1}{2} k_i q_2'^2. \end{aligned} \tag{50}$$

We can see that both functions are positive definite in X, and in particular $V_{nc} > 0$ in Ω_{nc} and $V_c \geq 0$ in Ω_c. It is possible to show that both Lyapunov functions are non-increasing in the respective region of the state space

$$\begin{aligned}\dot{V}_{nc}(x') &= dV_{nc}(x')\mathscr{A}_{nc}x' = -\tfrac{c}{J_1^2}p_1'^2 \quad \forall x' \in \Omega_{nc}\\ \dot{V}_c(x') &= dV_c(x')\mathscr{A}_c x' = -\tfrac{c}{J_1^2}p_1'^2 - \tfrac{c_i}{m^2}(q_2'+q_2^*)p_2'^2 \;\; \forall x' \in \Omega_c\end{aligned} \tag{51}$$

and that they are non increasing in $\mathscr{E}(S(x_0)|_{nc})$ and $\mathscr{E}(S(x_0)|_c)$, respectively. By means of Theorem 2.2, we can conclude that the solutions of system (44)-(48) are bounded for every initial condition $x_0' \in X$. The detailed proof will be given in the journal version of this paper. □

In the next section we will show, through the use of numerical simulation, the behaviour along solution of the selected Lyapunov functions. In particular we will see that they are non-increasing in the time periods in which they are active, and that they both meet the "switching in" condition.

IV. NUMERICAL SIMULATIONS

To perform the numerical simulations, a finite dimensional discretization of the infinite dimensional system has been considered. In particular, it has been used the distretization procedure described in [16], that allows to spatially approximate the resulting linear PDEs with a linear PH systems of dimensions depending on the number of discretizing elements (in the shown simulations, the flexible beam has been divided in 150 elements). A modification of the algorithm proposed in [16] was necessary to be able to discretize the model with clamped-clamped boundary conditions as obtained in (15). Simulations were made in the Simulink® environment using the "ode23t" time integration algorithm, and the set of parameters used for simulation are listed in Table I.

TABLE I
SIMULATION PARAMETERS

Name	Variable	Value
Beam's Length	L	$1\ m$
Beam's Width	L_w	$0.1\ m$
Beam's Thickness	L_t	$0.02\ m$
Density	ρ	$8000\ \frac{kg}{m^3}$
Young's modulus	E	$2\times10^9\ \frac{N}{m^2}$
Bulk's modulus	K	$6.85\times10^8\ \frac{N}{m^2}$
Hub's inertia	J_1	$1\ kg\cdot m^2$
Load's mass	m	$1\ kg$
Load's inertia	J_2	$1\ kg\cdot m^2$

The controller parameters are set as $k=10$, $c=3$ and $\theta^o=1$, while the impact's model parameters are set equal to $k_i=1000$ and $c_i=30$. In accordance with section III-B, it is possible to compute the equilibrium configuration of the system as: $q_1^*=0.0424\ rad$, $q_2^*=0.0096\ m$, $\varepsilon_t^*(\xi)=-1.3981\times10^{-8}$ and $\varepsilon_r(\xi)=0.0985(\xi-L)$.
To perform numerical simulations, the beam's states as well as the finite dimensional momentum states are initialized to zero initial conditions $x_b(0,\xi)=0$,

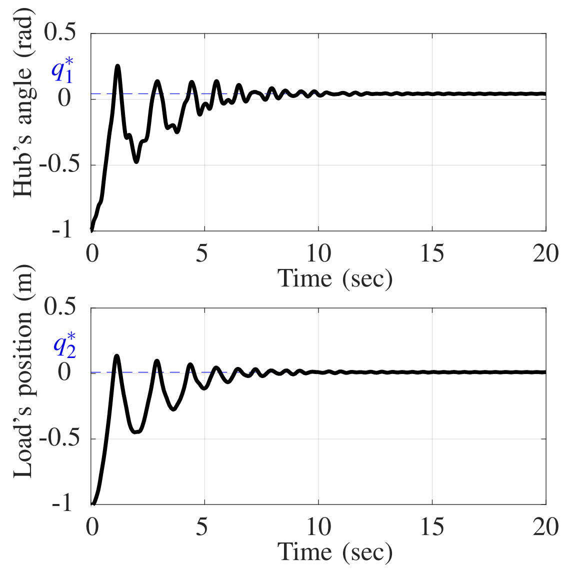

Fig. 2. Hub's angle evolution along time $q_1(t)$ and Load's position evolution along time $q_2(t)$

$p_1(0)=p_2(0)=p_3(0)$. The initial hub's angle has been initialized to $\theta(0)=q_1(0)=-1\ rad$, accordingly with the load's initial position $q_2(0)=L\theta(0)+w(0,L)=-1\ m$.

Fig. 2 shows the evolution in time of the hub's angle and of the load position. It is important to note that the contact occurs when $q_2(t)\geq 0$, and in fact when it dynamically reaches this value, the q_2 variable is rejected back because of the spring force of the impact model. It is possible to appreciate that both angles asymptotically stabilize to the computed equilibrium positions. Fig. **??** shows the Lyapunov functions (49)-(50) behaviour along solutions in the entire simulation time interval without distinguish between the active or non active time intervals, while Fig. **??** shows their behaviour during the respective activation time intervals. It is possible to appreciate that both the selected Lyapunov functions are non-increasing in their activation phases, and that the "Switching in" conditions are met.

V. CONCLUSIONS

In this preliminary work a general framework for switched infinite dimensional linear systems, together with a theorem concerning Lagrange stability has been presented. The proposed result makes use of multiple Lyapunov functions, each one associated to one of the operators defining the system. The theorem assures Lagrange stability if the Lyapunov functions have non-increasing time derivative in the subspace on which they are active and they respect the so called "switching-in" condition. Then, the modelling procedure together with the equilibrium computation of a rotating flexible beam in impact scenario has been detailed using the port-Hamiltonian framework. The obtained free motion and contact scenario operators have been written such to be cast in the defined framework for switched infinite dimensional systems. Next, Lyapunov functions for

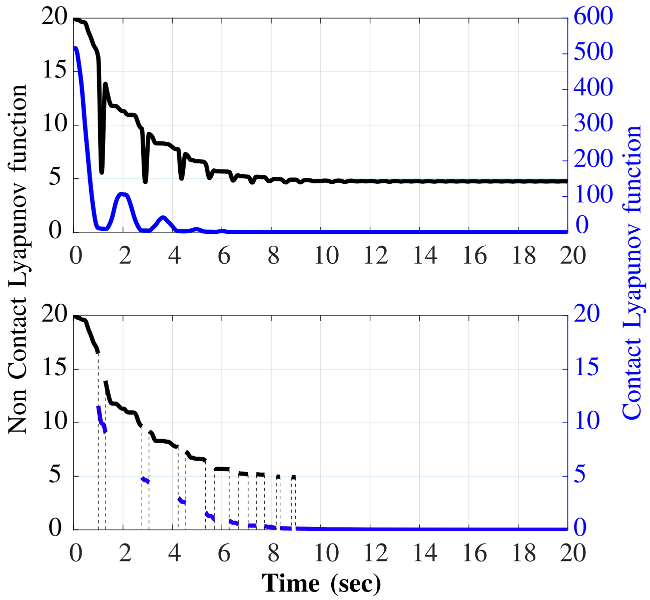

Fig. 3. Behaviour of the noncontact and contact Lyapunov functions along the solution.

the free and the contact case fulfilling the assumptions of the presented theorem have been proposed. Finally, with the help of numerical simulations, their non increasing behaviour in the respective active region together with the "switching-in" condition fulfilment have been shown. The complete proof of the proposed theorems will be given in the journal version of this paper.

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