Dissipative port-Hamiltonian Formulation of Maxwell Viscoelastic Fluids

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Abstract: In this paper we consider general port-Hamiltonian formulations of multi-dimensional Maxwell's viscoelastic fluids. Two different cases are considered to describe the energy fluxes in isentropic compressible and incompressible fluids. In the compressible case, the viscoelastic effects of shear and dilatational strains on the stress tensor are described individually through the corresponding constitutive equations. In the incompressible case, an approach based on the bulk modulus definition is proposed in order to obtain an appropriate characterization, from the port-Hamiltonian point of view, of the pressure and nonlinear terms in the momentum equation, associated with both dynamic pressure and vorticity of the flow.

Keywords: Port-Hamiltonian systems, Non-Newtonian Fluids, Maxwell's viscoelasticity.

1. INTRODUCTION

Port-Hamiltonian (PH) formulations are particularly well adapted for the description of energy fluxes in mutiphysical systems, highlighting useful properties for control theory, such as passivity, for stability analysis in the Lyapunov sense and for power-preserving connectivity by ports (van der Schaft and Jeltsema, 2014). This is particularly useful for energy-based control methods, such as energy-shaping (Macchelli et al., 2017), IDA-PBC (Vu et al., 2015), among others, that require models describing the energy fluxes in the system to derive a physically meaningful controller. PH formulations have been extended to distributed parameter systems in the conservative case in Le Gorrec et al. (2005) and for systems with dissipative effects in Villegas et al. (2006).

Different energy-based approaches have been proposed in the literature to describe Newtonian fluids considering several assumptions. One can cite for example (van der Schaft and Maschke, 2002) for inviscid fluids, (Matignon and Hélie, 2013) for irrotational flows, (Kotyczka, 2013) for pipeline networks, and (Altmann and Schulze, 2017) for reactive 1D fluids. Similarly, in (Mora et al., 2020), a general formulation is presented for non-reactive Newtonian compressible fluids under isentropic and non-isentropic assumptions.

Another category of fluids, that is of interest for industrial applications, are the so-called viscoelastic fluids. These fluids, when undergoing deformation, exhibit both viscous and elastic characteristics. This is the case for example of shampoo, blood, polymeric solutions, liquid crystals, and glass-forming liquids (Huo and Yong, 2016). Several approaches, like Maxwell's, Boltzmann's and Jeffrey's type models (Joseph, 1990), can be used to describe the deformation effects on the fluid stress tensor directly (Bollada and Phillips, 2012) or through structural variables (Öttinger, 2002; Mackay and Phillips, 2019; Hütter et al., 2020).

In this work we present an energy-based formulation for 3D compressible and incompressible Maxwell's viscoelastic fluids under an isentropic assumption. We consider a simple linear constitutive equation to describe the changes of the fluid stress tensor according to the deformation tensor variations. For the compressible case, the stress tensor is divided into two parts with the corresponding constitutive equations. One part describes the shear stress associated with the deformation tensor and the other describes the dilatational stress associated with fluid compressibility. For incompressible fluids, we propose an approach based on the bulk modulus definition to obtain an energy coherent characterization of the pressure (without the use of Lagrangian multipliers), focused on an appropriate description of the nonlinear terms in the velocity field.

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In what follows we denote by Ω the spatial domain of the fluid, enclosed by the boundary $\partial\Omega$. The set of spatial variables is denoted by $\mathbf{z}$ and the time by t. $\partial_t = \frac{\partial}{\partial t}$ is the temporal partial derivative, div is the divergence operator that applies to vector functions and return a scalar function, $\mathbf{div}$ denotes the divergence of a second order tensor functions and return a vector. $\mathbf{grad}$ denotes the gradient of a scalar function, returning a vector. $\mathbf{curl}$ is the rotational operator and $\overline{\mathbf{Grad}}$ is the symmetric part of the gradient of a vector function, returning a symmetric second order tensor.

2. MAXWELL'S VISCOELASTIC COMPRESSIBLE FLUIDS

The governing equations for compressible fluids are given by the following mass and momentum balances:

$$\partial_t \rho = -div\ \rho\mathbf{v} \tag{1}$$

$$\partial_t \mathbf{v} = -\left(\mathbf{v}\cdot\mathbf{grad}\ \right)\mathbf{v} - \frac{1}{\rho}\mathbf{grad}\ p - \frac{1}{\rho}\mathbf{div}\ \boldsymbol{\tau} \tag{2}$$

where $\rho = \rho(\mathbf{z},t)$, $\mathbf{v} = \mathbf{v}(\mathbf{z},t)$, $p = p(\mathbf{z},t)$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{z},t)$, with $\mathbf{z} \in \Omega$, denote the density, velocity, pressure and stress tensor of the fluid, respectively. In viscoelastic fluids, contrary to the Newtonian case, the stress $\boldsymbol{\tau}$ exhibits an explicit time dependency with respect to the gradient of $\mathbf{v}$ (Bird et al., 2015), combining the properties of viscous fluids and elastic materials (Joseph, 1990).

In order to define the constitutive equations to describe the stress dynamics, we first analyze the different parts of the stress tensor in Newtonian fluids and their effect on the fluid velocity. In Newtonian compressible fluids the viscous stress tensor is given by:

$$\boldsymbol{\tau}_N = \underbrace{-2\mu\overline{\mathbf{Grad}}\ \mathbf{v}}_{shear\ part} + \underbrace{\left(\frac{2}{3}\mu - \kappa\right)(div\ \mathbf{v})\,\mathbf{I}}_{dilatational\ part}$$

where $\mathbf{I}$ is the identity matrix of proper dimensions, μ and κ are the shear and dilatational viscosities, respectively. Noticing that $\mathbf{div}\ [(div\ \mathbf{v})\,\mathbf{I}] = \mathbf{grad}\ (div\ \mathbf{v})$, the contribution of $\boldsymbol{\tau}_N$ to the momentum equation (2) is given by:

$$-\frac{1}{\rho}\mathbf{div}\ \boldsymbol{\tau}_N = \frac{1}{\rho}\underbrace{\mathbf{div}\ \left[2\mu\overline{\mathbf{Grad}}\ \mathbf{v}\right]}_{shear\ part} + \frac{1}{\rho}\underbrace{\mathbf{grad}\ \left(\mu_B div\ \mathbf{v}\right)}_{dilatational\ part}$$

where $\mu_B = \kappa - \frac{2}{3}\mu$. In this work, in order to obtain a port-Hamiltonian formulation of Maxwell's viscoelastic fluids, we propose to describe separately the shear and dilatational (compressible part) effects of the stress tensor through the corresponding constitutive equations. We assume from now that $\mu_B \geq 0$.

In the literature one can find several formulations of the Maxwell's constitutive equation to describe the changes of the stress tensor according to the strain of the fluid, such as the upper convected, lower convected and co-rotational invariant derivatives (Joseph, 1990). In this work, we use the basic formulation of the Maxwell's constitutive equation, i.e., the stress tensor $\boldsymbol{\tau}$ is described by

$$\lambda\partial_t\boldsymbol{\tau} + \boldsymbol{\tau} = \eta\partial_t\gamma \tag{3}$$

where $\partial_t\gamma$ denotes the time variations of the fluid strain γ, η is the zero-rate viscosity, and $\lambda = \eta/G$ is the relaxation time, with G the elastic modulus of the fluid, such that, $\boldsymbol{\tau} \to \eta\partial_t\gamma$ when $\lambda \to 0$, which describes a Newtonian fluid, and $\boldsymbol{\tau} \to G\gamma$ when $\lambda \to \infty$, which describes an elastic material (Joseph, 1990).

We denote by $\boldsymbol{\tau}_1 \in L^2(\Omega, \mathbb{R}^{3\times 3})$ the symmetric tensor that describes shear stress effects on the fluid velocity, whose constitutive equation, from (3), is given by:

$$\boldsymbol{\tau}_1 + \lambda_1\partial_t\boldsymbol{\tau}_1 = 2\mu\overline{\mathbf{Grad}}\ \mathbf{v} \tag{4}$$

where $\lambda_1 = \mu/G_1$ with G_1 the shear elastic modulus, and where the corresponding density of potential elastic energy is given by $\frac{\lambda_1}{4\mu}\boldsymbol{\tau}_1 : \boldsymbol{\tau}_1 = \frac{\lambda_1}{4\mu}tr(\boldsymbol{\tau}_1^2)$.

Similarly, we use the scalar function $\tau_2 \in L^2(\Omega, \mathbb{R})$ to describe the dilatational stress effects trough the following constitutive equation:

$$\tau_2 + \lambda_2\partial_t\tau_2 = \mu_B div\ \mathbf{v} \tag{5}$$

where $\lambda_2 = \mu_B/G_2$ with G_2 the dilatational elastic modulus. The associated density of elastic potential energy is given by $\frac{1}{2}\frac{\lambda_2}{\mu_b}\tau_2^2$. Then, when $\lambda_1 = \lambda_2 \to 0$ we have that $\frac{1}{\rho}\mathbf{div}\ \boldsymbol{\tau}_1 + \frac{1}{\rho}\mathbf{grad}\ \tau_2$ converges to $-\frac{1}{\rho}\mathbf{div}\ \boldsymbol{\tau}_N$, obtaining the effects of pure viscous fluids on the the velocity field dynamics.

This implies that a Maxwell viscoelastic compressible fluid can be described by the following governing equations:

$$\partial_t\rho = -div\ \rho\mathbf{v} \tag{6a}$$

$$\partial_t\mathbf{v} = -\left(\mathbf{v}\cdot\mathbf{grad}\ \right)\mathbf{v} - \frac{1}{\rho}\mathbf{grad}\ p + \frac{1}{\rho}\mathbf{div}\ \boldsymbol{\tau}_1 + \frac{1}{\rho}\mathbf{grad}\ \tau_2 \tag{6b}$$

$$\partial_t\boldsymbol{\tau}_1 = -\frac{1}{\lambda_1}\boldsymbol{\tau}_1 + \frac{2\mu}{\lambda_1}\overline{\mathbf{Grad}}\ \mathbf{v} \tag{6c}$$

$$\partial_t\tau_2 = -\frac{1}{\lambda_2}\tau_2 + \frac{\mu_B}{\lambda_2}div\ \mathbf{v} \tag{6d}$$

with a total energy on the spatial domain Ω given by:

$$\mathcal{H} = \int_\Omega \left(\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho u(\rho) + \frac{\lambda_1}{4\mu}tr(\boldsymbol{\tau}_1^2) + \frac{\lambda_2}{2\mu_B}\tau_2^2\right)d\Omega \tag{7}$$

where $u(\rho)$ denotes the specific internal energy of the fluid, such that, $p = \rho^2\partial_\rho u(\rho)$. According to the port-Hamiltonian framework (van der Schaft and Jeltsema, 2014), the efforts associated with the state variables $\mathbf{x} = \left[\rho\ \mathbf{v}^\top\ \boldsymbol{\tau}_1\ \tau_2\right]^\top$ are given by:

$$\delta_\mathbf{x}\mathcal{H} = \begin{bmatrix}\delta_\rho\mathcal{H}\\ \delta_\mathbf{v}\mathcal{H}\\ \delta_{\boldsymbol{\tau}_1}\mathcal{H}\\ \delta_{\tau_2}\mathcal{H}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \hbar\\ \rho\mathbf{v}\\ \lambda_1\boldsymbol{\tau}_1/2\mu\\ \lambda_2\tau_2/\mu_B\end{bmatrix} \tag{8}$$

where $\hbar = u + p/\rho = u + \rho\partial_\rho u$ is the specific enthalpy.

Remark 1. One can notice that $\overline{\mathbf{Grad}}\ \mathbf{v}$ and the strain tensor $\boldsymbol{\gamma}$ of an elastic material are related by $\overline{\mathbf{Grad}}\ \mathbf{v} = \partial_t\boldsymbol{\gamma}$ (Bird et al., 2015, p.239). Since $div\ \mathbf{v} = tr(\overline{\mathbf{Grad}}\ \mathbf{v}) = tr\left(\partial_t\boldsymbol{\gamma}\right)$ we have that $\boldsymbol{\tau}_1 = 2G_1\boldsymbol{\gamma}$ when $\lambda_1 \to \infty$, and $\tau_2 = G_2 tr(\boldsymbol{\gamma})$ when $\lambda_2 \to \infty$. This implies that when $\{\lambda_1, \lambda_2\} \to \infty$ the viscoelastic energy density converges to $G_1 tr(\boldsymbol{\gamma}^2) + G_2 tr(\boldsymbol{\gamma})^2/2$, that is equivalent to the energy density of the Saint Venant-Kirchhoff model of elastic materials (Bodnár et al., 2014, p.103), i.e., G_1 and G_2 are equivalent to the Lamé coefficients of elastic materials.

In order to propose a port-Hamiltonian representation of the considered viscoelastic compressible fluids, we first define some skew-symmetric operators and their formal adjoints used to split the symmetric part of (6) in the product of two formally skew symmetric operators. In this sense, we introduce $\mathscr{H}_0 = L^2(\Omega, \mathbb{R})$ as the space of square integrable scalar functions with inner product $\langle e_1, e_2 \rangle_{\mathscr{H}_0} = \int_\Omega e_1 e_2 d\Omega$; $\mathscr{H}_1 = L^2(\Omega, \mathbb{R}^n)$ denotes the Hilbert space of square integrable vectors of size n satisfying that $\mathbf{f}/\rho \in \mathscr{H}_1$, $\forall \mathbf{f} \in \mathscr{H}_1$ and inner product $\langle \mathbf{f}_1, \mathbf{f}_2 \rangle_{\mathscr{H}_1} = \int_\Omega (\mathbf{f}_1 \cdot \mathbf{f}_2)\, d\Omega$, $\forall \mathbf{f}_1, \mathbf{f}_2 \in L^2(\Omega, \mathbb{R}^n)$; and $\mathscr{H}_2 = L^2(\Omega, \mathbb{R}^{n\times n}_{sym})$ denotes the Hilbert space of the square integrable symmetric tensors of size $n \times n$ with inner product $\langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \rangle_{\mathscr{H}_2} = \int_\Omega tr\left(\boldsymbol{\sigma}_1^\top \boldsymbol{\sigma}_2\right) d\Omega$, $\forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym})$.

Lemma 1. Let $\mathcal{D}_{\boldsymbol{\tau}_1} : \mathscr{H}_2 \to \mathscr{H}_1$ ans $\mathcal{D}^*_{\boldsymbol{\tau}_1} : \mathscr{H}_1 \to \mathscr{H}_2$ be the operators defined as $\mathcal{D}_{\boldsymbol{\tau}_1}\boldsymbol{\sigma} = \frac{1}{\rho}\mathbf{div}\ \left[\frac{2\mu}{\lambda_1}\boldsymbol{\sigma}\right]$, $\forall \boldsymbol{\sigma} \in \mathscr{H}_2$ and $\mathcal{D}^*_{\boldsymbol{\tau}_1}\mathbf{f} = -\frac{2\mu}{\lambda_1}\overline{\mathbf{Grad}}\ (\mathbf{f}/\rho)$, $\forall \mathbf{f} \in \mathscr{H}_1$, respectively. Then, $\mathcal{D}^*_{\boldsymbol{\tau}_1}$ is the formal adjoint of $\mathcal{D}_{\boldsymbol{\tau}_1}$, satisfying the following relationship:

$$\langle \mathbf{f}, \mathcal{D}_{\boldsymbol{\tau}_1}\boldsymbol{\sigma} \rangle_{\mathscr{H}_1} - \langle \boldsymbol{\sigma}, \mathcal{D}^*_{\boldsymbol{\tau}_1}\mathbf{f} \rangle_{\mathscr{H}_2} = \int_{\partial\Omega} \left(\tilde{\mathbf{f}} \cdot [\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}]\right) \partial\Omega \tag{9}$$

where $\tilde{\mathbf{f}} = \mathbf{f}/\rho$, $\tilde{\boldsymbol{\sigma}} = 2\mu\boldsymbol{\sigma}/\lambda_1$ and $\mathbf{n}$ is the unitary outward vector normal to the boundary $\partial\Omega$.

Proof. Consider $\mathbf{f} \in \mathscr{H}_1$ and $\boldsymbol{\sigma} \in \mathscr{H}_2$, Then,

$$\langle \mathbf{f}, \mathcal{D}_{\boldsymbol{\tau}_1}\boldsymbol{\sigma} \rangle_{\mathscr{H}_1} = \int_\Omega \mathbf{f} \cdot \frac{1}{\rho}\mathbf{div}\ \left[\frac{2\mu}{\lambda_1}\boldsymbol{\sigma}\right] d\Omega = \left\langle \tilde{\mathbf{f}}, \mathbf{div}\ \tilde{\boldsymbol{\sigma}} \right\rangle_{\mathscr{H}_1}$$

where $\tilde{\mathbf{f}} = \mathbf{f}/\rho$ and $\tilde{\boldsymbol{\sigma}} = 2\mu\boldsymbol{\sigma}/\lambda_1$. According to (Brugnoli et al., 2019, Theorem 4) the formal adjoint of $\mathbf{div}$ is $-\overline{\mathbf{Grad}}$, and following the Theorem 7 in Mora et al. (2020), we obtain

$$\left\langle \tilde{\mathbf{f}}, \mathbf{div}\ \tilde{\boldsymbol{\sigma}} \right\rangle_{\mathscr{H}_1} = \left\langle \tilde{\boldsymbol{\sigma}}, -\overline{\mathbf{Grad}}\ (\tilde{\mathbf{f}}) \right\rangle_{\mathscr{H}_2} + \int_{\partial\Omega} \left(\tilde{\mathbf{f}} \cdot [\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}]\right) \partial\Omega$$

where $\mathbf{n}$ is the unitary outward vector normal to the boundary $\partial\Omega$. Additionally, notice that

$$\begin{aligned}\left\langle \tilde{\boldsymbol{\sigma}}, -\overline{\mathbf{Grad}}\ (\tilde{\mathbf{f}}) \right\rangle_{\mathscr{H}_2} &= \langle 2\mu\boldsymbol{\sigma}/\lambda_1, -\overline{\mathbf{Grad}}\ (\mathbf{f}/\rho) \rangle_{\mathscr{H}_2} \\ &= \langle \boldsymbol{\sigma}, \mathcal{D}^*_{\boldsymbol{\tau}_1}\mathbf{f} \rangle_{\mathscr{H}_2}\end{aligned}$$

Then, we obtain that

$$\langle \mathbf{f}, \mathcal{D}_{\boldsymbol{\tau}_1}\boldsymbol{\sigma} \rangle_{\mathscr{H}_1} = \langle \boldsymbol{\sigma}, \mathcal{D}^*_{\boldsymbol{\tau}_1}\mathbf{f} \rangle_{\mathscr{H}_2} + \int_{\partial\Omega} \left(\tilde{\mathbf{f}} \cdot [\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}]\right) \partial\Omega$$

Rewriting this equation we obtain the relationship (9), and considering boundary conditions equal to 0, we have that $\langle \mathbf{f}, \mathcal{D}_{\boldsymbol{\tau}_1}\boldsymbol{\sigma} \rangle_{\mathscr{H}_1} = \langle \boldsymbol{\sigma}, \mathcal{D}^*_{\boldsymbol{\tau}_1}\mathbf{f} \rangle_{\mathscr{H}_2}$, i.e., $\mathcal{D}^*_{\boldsymbol{\tau}_1}$ is the formal adjoint of $\mathcal{D}_{\boldsymbol{\tau}_1}$. □

Lemma 2. Let $\mathcal{D}^*_{\tau_2} : \mathscr{H}_1 \to \mathscr{H}_0$ and $\mathcal{D}_{\tau_2} : \mathscr{H}_0 \to \mathscr{H}_1$ be the operators defined by $\mathcal{D}^*_{\tau_2}\mathbf{f} = -\frac{\mu_B}{\lambda_2} div\ [\mathbf{f}/\rho]$, $\forall \mathbf{f} \in \mathscr{H}_1$ and $\mathcal{D}_{\tau_2} e = \frac{1}{\rho}\mathbf{grad}\ (\frac{\mu_B}{\lambda_2} e)$, $\forall e \in \mathscr{H}_0$, respectively. The operator $\mathcal{D}^*_{\tau_2}$ is the formal adjoint of $\mathcal{D}_{\tau_2}$, satisfying the following relationship:

$$\langle \mathbf{f}, \mathcal{D}_{\tau_2} e \rangle_{\mathscr{H}_1} - \langle e, \mathcal{D}^*_{\tau_2}\mathbf{f} \rangle_{\mathscr{H}_0} = \int_{\partial\Omega} \tilde{e} \left(\tilde{\mathbf{f}} \cdot \mathbf{n}\right) \partial\Omega \tag{10}$$

where $\tilde{\mathbf{f}} = \mathbf{f}/\rho$, $\tilde{e} = \mu_B e/\lambda_2$ and $\mathbf{n}$ is the unitary outward vector normal to the boundary $\partial\Omega$.

Proof. Let $e \in \mathscr{H}_0$ and $\mathbf{f} \in \mathscr{H}_1$ be a square integrable scalar and vector functions, respectively. Considering that $-\mathbf{grad}$ is the formal adjoint of the operator div, it is easy to prove that $\mathcal{D}^*_{\tau_2} = -\frac{\mu_B}{\lambda_2} div\ \left[\frac{\cdot}{\rho}\right]$ is the formal adjoint of $\mathcal{D}_{\tau_2} = \frac{1}{\rho}\mathbf{grad}\ (\frac{\mu_B}{\lambda_2}\cdot)$ following the same procedure used in Lemma 1, obtaining the relationship (10). □

Using these Lemmas we can describe the energy fluxes between the velocity field of the fluid and the stresses $\boldsymbol{\tau}_1$ and τ_2, leading to the port-Hamiltonian formulation described in the following proposition.

Proposition 1. Let $\mathbf{x} = \begin{bmatrix}\rho & \mathbf{v}^\top & \boldsymbol{\tau}_1 & \tau_2\end{bmatrix}^\top$ be the state variables of system (6). Considering the total energy $\mathcal{H}$ and efforts $\delta_\mathbf{x}\mathcal{H}$ defined in (7) and (8), respectively, the dynamics of an isentropic Maxwell's viscoelastic compressible fluid can be described by the following dissipative port-Hamiltonian system:

$$\begin{bmatrix}\partial_t\mathbf{x} \\ \mathbf{f}_{r1} \\ f_{r2}\end{bmatrix} = \mathcal{J}(\mathbf{x}) \begin{bmatrix}\delta_\mathbf{x}\mathcal{H} \\ \mathbf{e}_{r1} \\ e_{r2}\end{bmatrix} \tag{11}$$

where $\mathbf{f}_{r1} = \delta_{\boldsymbol{\tau}_1}\mathcal{H}$, $f_{r2} = \delta_{\tau_2}\mathcal{H}$, $\mathbf{e}_{r1} = \frac{2\mu}{\lambda_1^2}\mathbf{f}_{r1}$ and $e_{r2} = \frac{\mu_B}{\lambda_2^2} f_{r2}$ are the dissipative flows and efforts associated with the incompressible and compressible stresses, $\boldsymbol{\tau}_1$ and τ_2, respectively, and $\mathcal{J}(\mathbf{x})$ is a formal skew-symmetric operator defined as:

$$\mathcal{J}(\mathbf{x}) = \begin{bmatrix} 0 & -div & \mathbf{0} & 0 & \mathbf{0} & 0 \\ -\mathbf{grad} & -\frac{1}{\rho}G_{\boldsymbol{\omega}} & \mathcal{D}_{\boldsymbol{\tau}_1} & \mathcal{D}_{\tau_2} & \mathbf{0} & 0 \\ 0 & -\mathcal{D}^*_{\boldsymbol{\tau}_1} & \mathbf{0} & 0 & -\mathbf{I} & 0 \\ 0 & -\mathcal{D}^*_{\tau_2} & \mathbf{0} & 0 & \mathbf{0} & -1 \\ 0 & \mathbf{0} & \mathbf{I} & 0 & \mathbf{0} & 0 \\ 0 & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 0 \end{bmatrix} \tag{12}$$

and satisfying the balance $\dot{\mathcal{H}} \leq \langle \mathbf{f}_\partial, \mathbf{e}_\partial \rangle_{\partial\Omega}$ where the boundary port variables are given by

$$\mathbf{f}_\partial = \begin{bmatrix} -\rho\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} \\ \mathbf{v}|_{\partial\Omega} \\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} \end{bmatrix} \quad \text{and} \quad \mathbf{e}_\partial = \begin{bmatrix} \left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \hbar\right)|_{\partial\Omega} \\ [\boldsymbol{\tau}_1 \cdot \mathbf{n}]\,|_{\partial\Omega} \\ \tau_2|_{\partial\Omega} \end{bmatrix} \tag{13}$$

Proof. Consider the system (6). As shown in Mora et al. (2020), we have that $(\mathbf{v}\cdot\mathbf{grad}\ )\mathbf{v} = \mathbf{grad}\ \left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + G_{\boldsymbol{\omega}}\mathbf{v}$ where $G_{\boldsymbol{\omega}}$ is a skew-symmetric matrix called Gyroscope. Similarly, considering an isentropic assumption, the Gibbs equation is reduced to $du = -pd\frac{1}{\rho}$, leading to the relationship $\frac{1}{\rho}\mathbf{grad}\ p = \mathbf{grad}\ \hbar$. Then, using the efforts (8) the governing equations can be written as:

$$\begin{aligned}
\partial_t\rho &= -\,div\ \delta_\mathbf{v}\mathcal{H} \\
\partial_t\mathbf{v} &= -\,\mathbf{grad}\ \delta_\rho\mathcal{H} - \frac{1}{\rho}G_{\boldsymbol{\omega}}\delta_\mathbf{v}\mathcal{H} + \frac{1}{\rho}\mathbf{div}\ \left[\frac{2\mu}{\lambda_1}\delta_{\boldsymbol{\tau}_1}\mathcal{H}\right] \\
&\quad + \frac{1}{\rho}\mathbf{grad}\ \left(\frac{\mu_B}{\lambda_2}\delta_{\tau_2}\mathcal{H}\right) \\
\partial_t\boldsymbol{\tau}_1 &= -\,\frac{2\mu}{\lambda_1^2}\delta_{\boldsymbol{\tau}_1}\mathcal{H} + \frac{2\mu}{\lambda_1}\overline{\mathbf{Grad}}\ \left(\frac{\delta_\mathbf{v}\mathcal{H}}{\rho}\right) \\
\partial_t\tau_2 &= -\,\frac{\mu_B}{\lambda_2^2}\delta_{\tau_2}\mathcal{H} + \frac{\mu_B}{\lambda_2} div\ \left[\frac{\delta_\mathbf{v}\mathcal{H}}{\rho}\right]
\end{aligned}$$

Defining $\mathbf{f}_{r1} = \delta_{\boldsymbol{\tau}_1}\mathcal{H}$, $f_{r2} = \delta_{\tau_2}\mathcal{H}$ and $\mathbf{e}_{r1} = \frac{2\mu}{\lambda_1^2}\mathbf{f}_{r1}$, $e_{r2} = \frac{\mu_B}{\lambda_2^2} f_{r2}$ as the corresponding dissipative flows and efforts, and using the operator $\mathcal{J}(\mathbf{x})$ defined in (12), we obtain the port-Hamiltonian formulation (11).

On the other hand, the rate of change of the total energy is given by $\dot{\mathcal{H}} = \langle \delta_{\mathbf{x}}\mathcal{H}, \partial_t \mathbf{x}\rangle_\Omega$. From (11) and the definition of $\mathcal{J}(\mathbf{x})$ we have

$$\begin{aligned}\langle \delta_{\mathbf{x}}\mathcal{H}, \partial_t \mathbf{x}\rangle_\Omega =& -\langle \delta_\rho \mathcal{H}, div\ \delta_{\mathbf{v}}\mathcal{H}\rangle_{\mathcal{H}_0} - \langle \delta_{\mathbf{v}}\mathcal{H}, \mathbf{grad}\ \delta_\rho\mathcal{H}\rangle_{\mathcal{H}_1}\\ &+ \langle \delta_{\mathbf{v}}\mathcal{H}, \mathcal{D}_{\boldsymbol{\tau}_1}\delta_{\boldsymbol{\tau}_1}\mathcal{H}\rangle_{\mathcal{H}_1} - \left\langle \delta_{\boldsymbol{\tau}_1}\mathcal{H}, \mathcal{D}^*_{\boldsymbol{\tau}_1}\delta_{\mathbf{v}}\mathcal{H}\right\rangle_{\mathcal{H}_2}\\ &+ \langle \delta_{\mathbf{v}}\mathcal{H}, \mathcal{D}_{\tau_2}\delta_{\tau_2}\mathcal{H}\rangle_{\mathcal{H}_1} - \left\langle \delta_{\tau_2}\mathcal{H}, \mathcal{D}^*_{\tau_2}\delta_{\mathbf{v}}\mathcal{H}\right\rangle_{\mathcal{H}_0}\\ &- \langle \delta_{\boldsymbol{\tau}_1}\mathcal{H}, \mathbf{e}_{r1}\rangle_{\mathcal{H}_2} - \langle \delta_{\tau_2}\mathcal{H}, e_{r2}\rangle_{\mathcal{H}_0}\\ &- \left\langle \delta_{\mathbf{v}}\mathcal{H}, \frac{1}{\rho}G_{\boldsymbol{\omega}}\delta_{\mathbf{v}}\mathcal{H}\right\rangle_{\mathcal{H}_1}\end{aligned}$$

Given the skew-symmetrical property of the Gyroscope matrix $G_{\boldsymbol{\omega}}$ we obtain that $\left\langle \delta_{\mathbf{v}}\mathcal{H}, \frac{1}{\rho}G_{\boldsymbol{\omega}}\delta_{\mathbf{v}}\mathcal{H}\right\rangle_{\mathcal{H}_1} = 0$, and using Lemmas 1 and 2, we have

$$\begin{aligned}\langle \delta_{\mathbf{x}}\mathcal{H}, \partial_t \mathbf{x}\rangle_\Omega =& -\int_{\partial\Omega} \delta_\rho\mathcal{H}\left(\delta_{\mathbf{v}}\mathcal{H}\cdot\mathbf{n}\right)\partial\Omega\\ &+ \int_{\partial\Omega}\left(\frac{\delta_{\mathbf{v}}\mathcal{H}}{\rho}\cdot\left[\frac{2\mu}{\lambda_1}\delta_{\boldsymbol{\tau}_1}\mathcal{H}\cdot\mathbf{n}\right]\right)\partial\Omega\\ &+ \int_{\partial\Omega}\frac{\mu_B}{\lambda_2}\delta_{\tau_2}\mathcal{H}\left(\frac{\delta_{\mathbf{v}}\mathcal{H}}{\rho}\cdot\mathbf{n}\right)\partial\Omega\\ &- \langle \delta_{\boldsymbol{\tau}_1}\mathcal{H}, \alpha_1\delta_{\boldsymbol{\tau}_1}\mathcal{H}\rangle_{\mathcal{H}_2} - \langle \delta_{\tau_2}\mathcal{H}, \alpha_2\delta_{\tau_2}\mathcal{H}\rangle_{\mathcal{H}_0}\end{aligned}$$

where $\alpha_1 = 2\mu/\lambda_1^2 > 0$ and $\alpha_2 = \mu_B/\lambda_2^2 > 0$. Using (8) and defining the boundary flows $\mathbf{f}_\partial$ and efforts $\mathbf{e}_\partial$ as shown in (13), we have $\langle \mathbf{f}_\partial, \mathbf{e}_\partial\rangle_{\partial\Omega} = -\int_{\partial\Omega}\delta_\rho\mathcal{H}\left(\delta_{\mathbf{v}}\mathcal{H}\cdot\mathbf{n}\right)\partial\Omega + \int_{\partial\Omega}\left(\frac{\delta_{\mathbf{v}}\mathcal{H}}{\rho}\cdot\left[\frac{2\mu}{\lambda_1}\delta_{\boldsymbol{\tau}_1}\mathcal{H}\cdot\mathbf{n}\right] + \frac{\mu_B}{\lambda_2}\delta_{\tau_2}\mathcal{H}\left(\frac{\delta_{\mathbf{v}}\mathcal{H}}{\rho}\cdot\mathbf{n}\right)\right)\partial\Omega$. Then, the rate of change of the total energy is given by

$$\begin{aligned}\dot{\mathcal{H}} =& -\langle \delta_{\boldsymbol{\tau}_1}\mathcal{H}, \alpha_1\delta_{\boldsymbol{\tau}_1}\mathcal{H}\rangle_{\mathcal{H}_2} - \langle \delta_{\tau_2}\mathcal{H}, \alpha_2\delta_{\tau_2}\mathcal{H}\rangle_{\mathcal{H}_0} + \langle \mathbf{f}_\partial, \mathbf{e}_\partial\rangle_{\partial\Omega}\\ &\le \langle \mathbf{f}_\partial, \mathbf{e}_\partial\rangle_{\partial\Omega}\end{aligned}$$

□

3. MAXWELL'S VISCOELASTIC INCOMPRESSIBLE FLUIDS

Consider the governing equations of Maxwell's viscoelastic incompressible fluids:

$$div\ \mathbf{v} = 0 \tag{14a}$$

$$\rho_0\partial_t\mathbf{v} = -\rho_0\left(\mathbf{v}\cdot\mathbf{grad}\ \right)\mathbf{v} - \mathbf{grad}\ p + \mathbf{div}\ \boldsymbol{\tau} \tag{14b}$$

$$\partial_t\boldsymbol{\tau} = -\frac{1}{\lambda}\boldsymbol{\tau} + \frac{2\mu}{\lambda}\overline{\mathbf{Grad}}\ \mathbf{v} \tag{14c}$$

where $\boldsymbol{\tau}$ denotes the stress tensor and ρ_0 the reference density of the fluid. The stored energy is given by $\mathcal{H} = \int_\Omega\left(\frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v} + \frac{\lambda}{4\mu}tr(\boldsymbol{\tau}^2)\right)d\Omega$, where $\frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v}$ and $\frac{\lambda}{4\mu}tr(\boldsymbol{\tau}^2)d\Omega$ denote the kinetic and potential (elastic) energy densities, respectively.

From the port-Hamiltonian point view, it is necessary to find an appropriate description of the pressure in (14). According to John (2016), there are several methods to obtain the pressure for incompressible fluids in computational models. One can cite the so-called pseudo-compressibility methods, as for example the Pressure Stabilization Petrov-Galerkin method, $-div\ \mathbf{v} + \xi div\ (\mathbf{grad}\ p) = 0$, the penalty method, $-div\ \mathbf{v} - \xi p = 0$, and the artificial compressibility method, $-div\ \mathbf{v} - \xi\partial_t p = 0$, where ξ is a tuning parameter. In this work we focus on an approach based on the artificial compressibility method. In this sense, the continuity equation (14a) is an approximation of the general mass balance $\partial_t\rho + div\ \rho\mathbf{v} = 0$, with $\rho = \hat{\rho} + \rho_0$, when $|\hat{\rho}| \ll \rho_0$ (Gresho and Sani, 1998).

Considering the Bulk modulus definition $\beta_S = \rho_0\left(\frac{\partial p}{\partial\rho}\right)_S$ (Massey and Smith, 2012; Murdock, 1993), we have

$$\hat{p} = p - p_0 = \frac{\beta_S}{\rho_0}\left(\rho - \rho_0\right) \tag{15}$$

where p_0 is the reference pressure at ρ_0. From (15) we have that $\partial_t\hat{p} = \frac{\beta_S}{\rho_0}\partial_t\rho$ and $\frac{\rho}{\rho_0} = \alpha(\hat{p})$ with $\alpha(\hat{p}) = \frac{\hat{p}}{\beta_S} + 1$. Then, the mass balance can be rewritten as

$$\partial_t\hat{p} = -\beta_S div\ \alpha(\hat{p})\mathbf{v} \tag{16}$$

Assuming $|\hat{\rho}| \ll \rho$, from (15), we have that $\frac{|\hat{p}|}{\beta_S} \ll 1$, i.e., $\alpha(\hat{p}) \approx 1$, obtaining the following approximation: $\alpha(\hat{p})\mathbf{v} \approx \mathbf{v}$, i.e., (16) is equivalent to the artificial compressibility method described by John (2016). Similarly, (16) can be rewritten as $\partial_t\hat{p} + \mathbf{v}\cdot\mathbf{grad}\ p = -\beta_S\alpha(\hat{p})div\ \mathbf{v}$, as discussed in Bollada and Phillips (2012). Neglecting the therm $\frac{\hat{p}}{\beta_S}$ in $\alpha(\hat{p})$ we obtain the pressure description used in the weak compressible model proposed by Edwards and Beris (1990).

Defining $\boldsymbol{\pi} = \rho_0\mathbf{v}$ as the fluid momentum, considering that $\mathbf{grad}\ p = \mathbf{grad}\ \hat{p}$ and substituting the continuity equation (14a) by (16), we obtain the following governing equations:

$$\partial_t\hat{p} = -\beta_S div\ \alpha(\hat{p})\frac{\boldsymbol{\pi}}{\rho_0} \tag{17a}$$

$$\partial_t\boldsymbol{\pi} = -\left(\boldsymbol{\pi}\cdot\mathbf{grad}\ \right)\frac{\boldsymbol{\pi}}{\rho_0} - \mathbf{grad}\ \hat{p} + \mathbf{div}\ \boldsymbol{\tau} \tag{17b}$$

$$\partial_t\boldsymbol{\tau} = -\frac{1}{\lambda}\boldsymbol{\tau} + \frac{2\mu}{\lambda}\overline{\mathbf{Grad}}\ \frac{\boldsymbol{\pi}}{\rho_0} \tag{17c}$$

Noticing that (17a) is equivalent to the pressure propagation of sound waves in fluid media (Landau and Lifshitz, 1987, Ch.8), and according to Trenchant et al. (2018, 2015) the density of potential energy associated with the the pressure propagation of sound waves can be expressed as $\frac{1}{2}\hat{p}^2/\beta_S$. Similarly, including the small perturbations of the fluid density, the kinetic energy density can be expressed as $\frac{1}{2}\left(\hat{\rho} + \rho_0\right)\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}\alpha(\hat{p})\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{\rho_0}$. Then, the total energy stored by the system (17), in a domain Ω, is given by:

$$\mathcal{H} = \frac{1}{2}\int_\Omega\left(\alpha(\hat{p})\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{\rho_0} + \frac{\hat{p}^2}{\beta_S} + \frac{\lambda}{2\mu}tr(\boldsymbol{\tau}^2)\right)d\Omega \tag{18}$$

such that

$$\begin{bmatrix}\delta_{\hat{p}}\mathcal{H}\\ \delta_{\boldsymbol{\pi}}\mathcal{H}\\ \delta_{\boldsymbol{\tau}}\mathcal{H}\end{bmatrix} = \begin{bmatrix}\dfrac{1}{\beta_S}\left(\dfrac{1}{2}\dfrac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{\rho_0} + \hat{p}\right)\\ \alpha(\hat{p})\dfrac{\boldsymbol{\pi}}{\rho_0}\\ \dfrac{\lambda}{2\mu}\boldsymbol{\tau}\end{bmatrix} \tag{19}$$

Lemma 3. Let $\mathcal{D}_\beta : \mathcal{H}_1 \to \mathcal{H}_0$ and $\mathcal{D}^*_\beta : \mathcal{H}_0 \to \mathcal{H}_1$ be the operators defined as $\mathcal{D}_\beta\mathbf{f} = -\beta_S div\ \mathbf{f}$, $\forall\mathbf{f} \in \mathcal{H}_1$ and $\mathcal{D}^*_\beta e = \mathbf{grad}\ \left(\beta_S e\right)$, $\forall e \in \mathcal{H}_0$, respectively. The operator $\mathcal{D}^*_\beta$ is the formal adjoint of $\mathcal{D}_\beta$, satisfying the relationship

$$\langle e, \mathcal{D}_\beta\mathbf{f}\rangle_{\mathcal{H}_0} - \left\langle \mathcal{D}^*_\beta e, \mathbf{f}\right\rangle_{\mathcal{H}_1} = -\int_{\partial\Omega}\beta_S e\left(\mathbf{f}\cdot\mathbf{n}\right)\partial\Omega \tag{20}$$

Proof. Consider $e \in \mathscr{H}_0$ and $\mathbf{f} \in \mathscr{H}_1$, then, the inner product $\langle e, \mathcal{D}_\beta \mathbf{f}\rangle_{\mathscr{H}_0}$ is given by:

$$\begin{aligned}\langle e, \mathcal{D}_\beta \mathbf{f}\rangle_{\mathscr{H}_0} &= \int_\Omega -e\beta_S div\ \mathbf{f} d\Omega \\ &= \int_\Omega \mathbf{f}\cdot \mathbf{grad}\ (\beta_S e)\,\beta_S d\Omega - \int_\Omega div\ (\beta_S e\mathbf{f})\, d\Omega\end{aligned}$$

Using the Gauss Divergence Theorem (Bird et al., 2015, Appendix A) we obtain:

$$\langle e, \mathcal{D}_\beta \mathbf{f}\rangle_{\mathscr{H}_0} = \langle \mathcal{D}_\beta^* e, \mathbf{f}\rangle_{\mathscr{H}_1} - \int_{\partial\Omega} \beta_S e\,(\mathbf{f}\cdot\mathbf{n})\,\partial\Omega$$

Rearranging this result we obtain the relationship (20), and considering boundary conditions equal to 0 we have that $\langle e, \mathcal{D}_\beta \mathbf{f}\rangle_{\mathscr{H}_0} = \left\langle \mathcal{D}_\beta^* e, \mathbf{f}\right\rangle_{\mathscr{H}_1}$, i.e., $\mathcal{D}_\beta^*$ is the formal adjoint of $\mathcal{D}_\beta$. □

Lemma 4. Let $\mathcal{D}_{\boldsymbol{\tau}} : \mathscr{H}_2 \to \mathscr{H}_1$ and $\mathcal{D}_{\boldsymbol{\tau}}^* : \mathscr{H}_1 \to \mathscr{H}_2$ be the operators defined as $\mathcal{D}_{\boldsymbol{\tau}}\boldsymbol{\sigma} = \mathbf{div}\ \left[\frac{2\mu}{\lambda}\boldsymbol{\sigma}\right]$, $\forall \boldsymbol{\sigma} \in \mathscr{H}_2$ and $\mathcal{D}_{\boldsymbol{\tau}}^*\mathbf{f} = -\frac{2\mu}{\lambda}\overline{\mathbf{Grad}}\ \mathbf{f}$, $\forall \mathbf{f} \in \mathscr{H}_1$, respectively. The operator $\mathcal{D}_{\boldsymbol{\tau}}^*$ is the formal adjoint of $\mathcal{D}_{\boldsymbol{\tau}}$, satisfying the relationship

$$\langle \mathbf{f}, \mathcal{D}_{\boldsymbol{\tau}}\boldsymbol{\sigma}\rangle_{\mathscr{H}_1} - \langle \mathcal{D}_{\boldsymbol{\tau}}^*\mathbf{f}, \boldsymbol{\sigma}\rangle_{\mathscr{H}_2} = \int_{\partial\Omega} \mathbf{f}\cdot\left[\frac{2\mu}{\lambda}\boldsymbol{\sigma}\cdot\mathbf{n}\right]\partial\Omega \quad (21)$$

Proof. Considering that $\overline{\mathbf{Grad}}$ is formal adjoint of $-\mathbf{div}$ (Brugnoli et al., 2019, Theorem 4), it is easy to verify that $\mathcal{D}_{\boldsymbol{\tau}}^*$ is the formal adjoint of $\mathcal{D}_{\boldsymbol{\tau}}$ and the relationship (21), following the same procedure of Lemma 3. □

Proposition 2. Let $\mathbf{x} = \begin{bmatrix}\hat{p} & \boldsymbol{\pi}^\top & \boldsymbol{\tau}\end{bmatrix}^\top$ be the state variables of system (17). Considering the total energy (18) and efforts (19), the dynamics of an incompressible Maxwell's viscoelastic fluid can be expressed as the following dissipative port-Hamiltonian system

$$\begin{bmatrix}\partial_t \mathbf{x}\\ \mathbf{f}_r\end{bmatrix} = \mathcal{J}(\mathbf{x})\begin{bmatrix}\delta_{\mathbf{x}}\mathcal{H}\\ \mathbf{e}_r\end{bmatrix} \quad (22)$$

where $\mathbf{f}_r = \delta_{\boldsymbol{\tau}}\mathcal{H}$ and $\mathbf{e}_r = \frac{2\mu}{\lambda^2}\mathbf{f}_r$ are the dissipative flow and effort tensors, and the skew-symmetric operator $\mathcal{J}(\mathbf{x})$ is defined as

$$\mathcal{J}(\mathbf{x}) = \begin{bmatrix} 0 & \mathcal{D}_\beta & \mathbf{0} & \mathbf{0}\\ -\mathcal{D}_\beta^* & -\dfrac{1}{\alpha(\hat{p})}G_{\boldsymbol{\pi}} & \mathcal{D}_{\boldsymbol{\tau}} & \mathbf{0}\\ 0 & -\mathcal{D}_{\boldsymbol{\tau}}^* & \mathbf{0} & -\mathbf{I}\\ 0 & \mathbf{0} & \mathbf{I} & \mathbf{0}\end{bmatrix} \quad (23)$$

satisfying the following energy balance $\dot{\mathcal{H}} \leq \langle \mathbf{e}_\partial, \mathbf{f}_\partial\rangle_{\partial\Omega}$ where

$$\mathbf{f}_\partial = \begin{bmatrix}-\alpha(\hat{p})\,(\mathbf{v}\cdot\mathbf{n})\,|_{\partial\Omega}\\ \alpha(\hat{p})\mathbf{v}|_\Omega\end{bmatrix} \quad \mathbf{e}_\partial = \begin{bmatrix}\left.\left(\dfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\hat{p}\right)\right|_{\partial\Omega}\\ (\boldsymbol{\tau}\cdot\mathbf{n})\,|_{\partial\Omega}\end{bmatrix} \quad (24)$$

are the boundary flows and efforts, respectively.

Proof. Consider the dynamic system (17). Using the differential relation described in (Bird et al., 2015, eq. (A.4-23)), the term $(\boldsymbol{\pi}\cdot\mathbf{grad}\ )\frac{\boldsymbol{\pi}}{\rho_0}$ can be rewritten as

$$(\boldsymbol{\pi}\cdot\mathbf{grad}\ )\frac{\boldsymbol{\pi}}{\rho_0} = \mathbf{grad}\ \left(\frac{1}{2}\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{\rho_0}\right) + [\mathbf{curl}\ \boldsymbol{\pi}]\times\frac{\boldsymbol{\pi}}{\rho_0}$$

where the term $[\mathbf{curl}\ \boldsymbol{\pi}]\times\frac{\boldsymbol{\pi}}{\rho_0}$ describes the energy flux between the velocity field components due to the fluid vorticity. Using the Gyroscope matrix $G_{\boldsymbol{\omega}}$ defined by Mora et al. (2020) we obtain that $[\mathbf{curl}\ \boldsymbol{\pi}]\times\frac{\boldsymbol{\pi}}{\rho_0} = G_{\boldsymbol{\pi}}\frac{\boldsymbol{\pi}}{\rho_0}$ where $G_{\boldsymbol{\pi}} = \rho_0 G_{\boldsymbol{\omega}}$ is a skew-symmetric matrix dependent on the vorticity vector $\boldsymbol{\omega} = \mathbf{curl}\ \mathbf{v}$.

On the other hand, assuming $\frac{|\hat{p}|}{\beta_S} \ll 1$, i.e., $\alpha(\hat{p}) \approx 1$, we have that $\overline{\mathbf{Grad}}\ \frac{\boldsymbol{\pi}}{\rho_0} \approx \overline{\mathbf{Grad}}\ \delta_{\boldsymbol{\pi}}\mathcal{H}$. Then, the governing equations can be expressed as:

$$\begin{aligned}\partial_t\hat{p} &= -\ \beta_S div\ \delta_{\boldsymbol{\pi}}\mathcal{H}\\ \partial_t\boldsymbol{\pi} &= -\ \mathbf{grad}\ \delta_{\hat{p}}\mathcal{H} - \frac{G_{\boldsymbol{\pi}}}{\alpha(\hat{p})}\delta_{\boldsymbol{\pi}}\mathcal{H} + \mathbf{div}\ \left[\frac{2\mu}{\lambda}\delta_{\boldsymbol{\tau}}\mathcal{H}\right]\\ \partial_t\boldsymbol{\tau} &= -\ \frac{2\mu}{\lambda^2}\delta_{\boldsymbol{\tau}}\mathcal{H} + \frac{2\mu}{\lambda}\overline{\mathbf{Grad}}\ \delta_{\boldsymbol{\pi}}\mathcal{H}\end{aligned}$$

Defining $\mathbf{f}_r = \delta_{\boldsymbol{\tau}}\mathcal{H}$ and $\mathbf{e}_r = \frac{2\mu}{\lambda^2}\mathbf{f}_r$ as the dissipative flow and effort tensors, and using the skew-symmetric operator $\mathcal{J}(\mathbf{x})$ defined in (23) we obtain the port-Hamiltonian formulation (22).

Finally, the rate of change of the total energy (18) is given by $\dot{\mathcal{H}} = \langle \delta_{\mathbf{x}}\mathcal{H}, \mathcal{J}(\mathbf{x})\delta_{\mathbf{x}}\mathcal{H}\rangle_\Omega$. Using (23) we obtain

$$\begin{aligned}\dot{\mathcal{H}} &= \langle \delta_{\hat{p}}\mathcal{H}, \mathcal{D}_\beta\delta_{\boldsymbol{\pi}}\mathcal{H}\rangle_{\mathscr{H}_0} - \langle \mathcal{D}_\beta^*\delta_{\hat{p}}\mathcal{H}, \delta_{\boldsymbol{\pi}}\mathcal{H}\rangle_{\mathscr{H}_1}\\ &\quad + \langle \delta_{\boldsymbol{\pi}}\mathcal{H}, \mathcal{D}_{\boldsymbol{\tau}}\delta_{\boldsymbol{\tau}}\mathcal{H}\rangle_{\mathscr{H}_1} - \langle \mathcal{D}_{\boldsymbol{\tau}}^*\delta_{\boldsymbol{\pi}}\mathcal{H}, \delta_{\boldsymbol{\tau}}\mathcal{H}\rangle_{\mathscr{H}_2}\\ &\quad - \left\langle \delta_{\boldsymbol{\pi}}\mathcal{H}, \frac{G_{\boldsymbol{\pi}}}{\alpha(\hat{p})}\delta_{\boldsymbol{\tau}}\mathcal{H}\right\rangle_{\mathscr{H}_1} - \langle \delta_{\boldsymbol{\tau}}\mathcal{H}, \mathbf{e_r}\rangle_{\mathscr{H}_2}\end{aligned}$$

We have that $\left\langle \delta_{\boldsymbol{\pi}}\mathcal{H}, \frac{G_{\boldsymbol{\pi}}}{\alpha(\hat{p})}\delta_{\boldsymbol{\tau}}\mathcal{H}\right\rangle_{\mathscr{H}_1} = 0$ given the skew-symmetry of $G_{\boldsymbol{\pi}}$. Then, using Lemmas 3 and 4, we obtain

$$\begin{aligned}\dot{\mathcal{H}} = &-\int_{\partial\Omega}\left(\beta_S\delta_{\hat{p}}\mathcal{H}\,(\delta_{\boldsymbol{\pi}}\mathcal{H}\cdot\mathbf{n}) - \delta_{\boldsymbol{\pi}}\mathcal{H}\cdot\left[\frac{2\mu}{\lambda}\delta_{\boldsymbol{\tau}}\mathcal{H}\cdot\mathbf{n}\right]\right)\partial\Omega\\ &- \left\langle \delta_{\boldsymbol{\tau}}\mathcal{H}, \frac{2\mu}{\lambda^2}\delta_{\boldsymbol{\tau}}\mathcal{H}\right\rangle_{\mathscr{H}_2}\end{aligned}$$

Defining the boundary flows and efforts as

$$\mathbf{f}_\partial = \begin{bmatrix}-\,(\delta_{\boldsymbol{\pi}}\mathcal{H}\cdot\mathbf{n})\,|_{\partial\Omega}\\ \delta_{\boldsymbol{\pi}}\mathcal{H}|_{\partial\Omega}\end{bmatrix} = \begin{bmatrix}-\alpha(\hat{p})\,(\mathbf{v}\cdot\mathbf{n})\,|_{\partial\Omega}\\ \alpha(\hat{p})\mathbf{v}|_\Omega\end{bmatrix} \text{ and}$$

$$\mathbf{e}_\partial = \begin{bmatrix}\beta_S\delta_{\hat{p}}\mathcal{H}|_{\partial\Omega}\\ \left.\left[\dfrac{2\mu}{\lambda}\delta_{\boldsymbol{\tau}}\mathcal{H}\cdot\mathbf{n}\right]\right|_{\partial\Omega}\end{bmatrix} = \begin{bmatrix}\left.\left(\dfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\hat{p}\right)\right|_{\partial\Omega}\\ (\boldsymbol{\tau}\cdot\mathbf{n})\,|_{\partial\Omega}\end{bmatrix},$$

respectively, and considering that $\langle \delta_{\boldsymbol{\tau}}\mathcal{H}, \frac{2\mu}{\lambda^2}\delta_{\boldsymbol{\tau}}\mathcal{H}\rangle_{\mathscr{H}_2} \geq 0$, the time derivative of $\mathcal{H}$ can be rewritten as

$$\begin{aligned}\dot{\mathcal{H}} &= \langle \mathbf{e}_\partial, \mathbf{f}_\partial\rangle_{\partial\Omega} - \left\langle \delta_{\boldsymbol{\tau}}\mathcal{H}, \frac{2\mu}{\lambda^2}\delta_{\boldsymbol{\tau}}\mathcal{H}\right\rangle_{\mathscr{H}_2}\\ &\leq \langle \mathbf{e}_\partial, \mathbf{f}_\partial\rangle_{\partial\Omega}\end{aligned}$$

□

4. CONCLUSION

In this work port-Hamiltonian formulations of compressible and incompressible Maxwell's viscoelastic fluids have been presented. For the compressible case, we divide the stress tensor to consider the shear and dilatational (compressibility) effects separately, proposing a constitutive equation to describe viscoelasticity associated with each of these effects. Unlike other models in the literature, in this formulation, terms G_1 and G_2 on relaxation time λ_1 and λ_2, respectively, are related with the Lamé parameters of elastic materials, as shown in Remark 1. For the incompressible case, in order to obtain an energetically coherent description of the pressure, without the use of Lagrangian multipliers, a weak-compressible approach based on the bulk modulus definition has been proposed. This approach allows us to obtain an appropriate characterization of the nonlinear terms in the velocity field associated with the dynamic pressure, $\mathbf{grad}\ \left(\frac{1}{2}\frac{\boldsymbol{\pi}\cdot\boldsymbol{\pi}}{\rho_0}\right)$, and the fluid vorticity, $[\mathbf{curl}\ \boldsymbol{\pi}] \times \frac{\boldsymbol{\pi}}{\rho_0}$. It is important to notice that we have used a simple Maxwell's constitutive equation to describe the stress tensors dynamics. In case of descriptions with invariant derivatives, such as upper and lower converted formulations, the additional terms in the stress tensor constitutive equation increase the complexity of the skew-symmetric operator that describes the energy flux in port-Hamiltonian systems. As a future work, we consider addressing this issue in order to obtain a port-Hamiltonian formulation of invariant derivative Maxwell's models of viscoelastic fluids.

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