# Irreversible port-Hamiltonian modelling of 1D compressible fluids

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**Abstract:** In this paper, an irreversible port-Hamiltonian formulation for 1D compressible Newtonian fluids is presented. We separate the fluid dynamics into reversible and irreversible parts. Given the compressibility assumption, we define a state-dependent matrix that modulates the skew-symmetric operators that describe the irreversible part of the fluid dynamics. As a result we obtain an energy-based formulation that reflects appropriately the first and second of principles of Thermodynamics.

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## 1. INTRODUCTION

The energy based formulation of compressible Newtonian fluids, in particular using the port-Hamiltonian (PH) framework, have raised a major attention in the last decade. The PH formulation and control of inviscid fluids are analyzed by van der Schaft and Maschke (2002) and Macchelli et al. (2017), respectively. Kotyczka (2013) and Macchelli et al. (2017) propose dissipative PH models of viscous fluids with an irrotational flow, considering different formulations for the dissipative term. Similarly, pseudo port-Hamiltonian formulations are proposed by Altmann and Schulze (2017) for 1D reactive fluids and by Mora et al. (2020) for 3D non-isentropic fluids.

An alternative formulation for irreversible processes is the irreversible port-Hamiltonian systems (IPHS) formulation. Initially proposed in Ramírez et al. (2013) for finite-dimensional systems, this approach allows to represent through the IPHS structure both the first and second principles of Thermodynamics. Unlike dissipative port-Hamiltonian representations, where the irreversible phenomena are described through dissipative elements, neglecting the thermal domain, irreversible port-Hamiltonian representations account for the thermal domain through the entropy balance. In this case the port-Hamiltonian structure includes a non linear element related to the thermodynamic driving force that allows to describe the second law of Thermodynamics (see Ramírez et al. (2013) for details). The extension of this framework to 1D infinite-dimensional systems has been proposed initially by Ramirez and Le Gorrec (2016) for diffusion processes and generalized in Ramírez et al. (2021) for a class of distributed systems.

In this work, an irreversible port-Hamiltonian formulation of 1D compressible Newtonian fluids is presented, including the thermal domain. The fluid dynamics is split into a reversible and an irreversible part. The thermodynamic driving forces are described through locally defined pseudo-brackets, that allow us to relate the conversion of mechanical energy into heat, due to the dissipative phenomena, with the internal entropy production of the fluid, obtaining an appropriated description of the second principle of Thermodynamics.

This paper is organized as follows. In Section 3 the dynamics of inviscid and viscous compressible fluids is considered, under an isentropic assumption, describing the conservative and dissipative PH models, respectively, analyzing the internal energy characterization of the fluid for these PH formulations. In Section 4, we consider a viscous non-isentropic and non-reactive compressible fluid, deriving its irreversible port-Hamiltonian formulation. The thermodynamic effects of the heat generation by internal phenomena and heat flux are included in the model. The entropy production is described considering the thermodynamic driving forces characterized through locally defined pseudo-brackets. Finally, in Section 4 we present the conclusion and future works.

## 2. NOTATION

We consider infinite dimensional systems described on a one dimensional (1D) spatial domain, where $\zeta \in [a, b]$ and $t \in [0, +\infty)$ stand for the spatial and temporal variables. For the sake of simplicity in what follows $\partial_\zeta$ stands for the partial derivative with respect to $\zeta$, *i.e.* $\partial_\zeta x(\zeta, t) = \frac{\partial}{\partial \zeta} x(\zeta, t)$ and $\partial_t$ stands for the partial derivative with respect to $t$, *i.e.* $\partial_t x(\zeta, t) = \frac{\partial}{\partial t} x(\zeta, t)$.

The variational derivative of a functional $U(x)$ such that

$$U(x) = \int_a^b u(\zeta, x) d\zeta$$

for any smooth real vector function $x(\zeta, t)$ is defined by

$$\delta_x U(x) = \partial_x u(\zeta, x)$$

## 3. ISENTROPIC COMPRESSIBLE FLUIDS

We consider in this section isentropic compressible fluids defined on a one dimensional domain. We denote by $\rho = \rho(\zeta, t)$, $v = v(\zeta, t)$ and $p = p(\zeta, t)$ the fluid density per unit length, velocity and pressure, respectively, where $\zeta \in [a, b]$ and $t \in [0, +\infty)$ stand for the spatial and temporal variables. As the fluid is compressible, we consider the mass balance

$$D_t \rho = -\rho \partial_\zeta v,$$

where $D_t = \partial_t + v\partial_\zeta$ denotes the material derivative. Considering the isentropic assumption the variation of the specific internal energy $u$ is only due to the changes on $\rho$, i.e., $u = u(\rho)$ and the Gibbs equation reads

$$du = -pd\frac{1}{\rho} \tag{1}$$

As a consequence, using the local equilibrium assumption, from (1) $D_t u = \frac{p}{\rho^2} D_t \rho$ and the balance on $u$ reads:

$$\partial_t u = -v\partial_\zeta u - \frac{p}{\rho}\partial_\zeta v \tag{2}$$

where the thermal domain in the production of internal energy has been neglected.

### 3.1 *Inviscid fluid*

The governing equations of an isentropic and inviscid compressible fluid defined on the 1D domain $\Omega = \{\zeta \in [a, b] \subset \mathbb{R}\}$ are given by :

$$\partial_t \rho = -\partial_\zeta \rho v \tag{3a}$$

$$\partial_t v = -\partial_\zeta \left(\frac{1}{2}v^2\right) - \frac{1}{\rho}\partial_\zeta p \tag{3b}$$

The total energy is given by the sum of the kinetic and internal energy:

$$\mathcal{H} = \int_a^b \left(\frac{1}{2}\rho v^2 + \rho u(\rho)\right) d\zeta. \tag{4}$$

As shown in van der Schaft and Maschke (2002), using the Gibbs equation (1) we have

$$\frac{1}{\rho}\partial_\zeta p = \partial_\zeta\left(\frac{p}{\rho}\right) - \partial_\zeta\left(\frac{1}{\rho}\right)p \tag{5}$$

$$= \partial_\zeta\left(\frac{p}{\rho} + u\right) = \partial_\zeta \hbar \tag{6}$$

where $\hbar = u + p/\rho$ is the specific enthalpy. Furthermore

$$\partial_\rho \left(\rho u(\rho)\right) = u(\rho) + \rho\partial_\rho\left(u(\rho)\right) \tag{7}$$

$$= u(\rho) - p\partial_\rho\left(\frac{1}{\rho}\right) \tag{8}$$

$$= u(\rho) + \frac{p}{\rho} = \hbar \tag{9}$$

Then, system (3) is expressed as the following port-Hamiltonian formulation:

$$\partial_t \mathbf{x} = P_1 \partial_\zeta \delta_{\mathbf{x}}\mathcal{H}, \ \forall \zeta \in \Omega \tag{10a}$$

$$\mathbf{u} = W_B R_{ext} \begin{bmatrix} \delta_{\mathbf{x}}\mathcal{H}|_b \\ \delta_{\mathbf{x}}\mathcal{H}|_a \end{bmatrix} \tag{10b}$$

$$\mathbf{y} = W_C R_{ext} \begin{bmatrix} \delta_{\mathbf{x}}\mathcal{H}|_b \\ \delta_{\mathbf{x}}\mathcal{H}|_a \end{bmatrix} \tag{10c}$$

where $\mathbf{x} = [\rho \ v]^\top$ is the state vector, $\mathbf{u}$ and $\mathbf{y}$ are the boundary input and outputs, respectively, $P_1 = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\delta_{\mathbf{x}}\mathcal{H} = \begin{bmatrix} \frac{1}{2}v^2 + \hbar \\ \rho v \end{bmatrix}$ and $R_{ext} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ \mathbf{I} & \mathbf{I} \end{bmatrix}$, with $W_B, W_C \in \mathbb{R}^{(2,4)}$, $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ invertible (Le Gorrec et al., 2005) such that

$$\begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_B^T & W_C^T \end{bmatrix} = \Sigma, \text{ with } \Sigma = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix} \tag{11}$$

In this case the total energy balance is given by

$$\dot{\mathcal{H}} = \mathbf{u}^\top \mathbf{y} \tag{12}$$

Balance (12) means that system (10) is conservative (Duindam et al., 2009). However, governing equations (3) describe a highly idealized fluid where any internal friction occur. For a more realistic description of compressible fluids, it is necessary to consider the viscous frictions during the fluid movement.

### 3.2 *Viscous fluid*

Governing equations of viscous compressible fluids are given by

$$\partial_t \rho = -\partial_\zeta \rho v \tag{13a}$$

$$\partial_t v = -\partial_\zeta \left(\frac{1}{2}v^2\right) - \frac{1}{\rho}\partial_\zeta p - \frac{1}{\rho}\partial_\zeta \tau \tag{13b}$$

where $\tau = -\mu\partial_\zeta v$ describes the viscous stress, with $\mu$ as the fluid viscosity.

The conversion of kinetic energy into heat due to the viscous friction is described through the term $-\frac{1}{\rho}\partial_\zeta \tau$ in (13b). Due to the isentropic assumption in the fluid, form a port-Hamiltonian point of view the term $-\frac{1}{\rho}\partial_\zeta \tau = \frac{1}{\rho}\partial_\zeta\left(\mu\partial_\zeta \frac{\delta_v \mathcal{H}}{\rho}\right)$ is a dissipative term representing the heat losses (Matignon and Hélie, 2013).

Then, according to (Mora, 2020, Remark 5.5), system (13) can be expressed as the following dissipative port-Hamiltonian formulation:

$$\begin{bmatrix} \partial_t \mathbf{x} \\ f_r \end{bmatrix} = \begin{bmatrix} P_1 \partial_\zeta & G_1 \partial_\zeta \\ \partial_\zeta \left(G_1^\top \cdot\right) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta_{\mathbf{x}}\mathcal{H} \\ e_r \end{bmatrix} \tag{14}$$

where $f_r = \partial_\zeta v$ and $e_r = \mu f_r$ are the dissipative flow and effort, $G_1 = [0 \ 1/\rho]^\top$. The boundary inputs and outputs of (14) are given by

$$\mathbf{u} = W_B R_{ext} \begin{bmatrix} \delta_{\mathbf{x}}\mathcal{H}|_b \\ e_r|_b \\ \delta_{\mathbf{x}}\mathcal{H}|_a \\ e_r|_a \end{bmatrix} \text{ and } \mathbf{y} = W_C R_{ext} \begin{bmatrix} \delta_{\mathbf{x}}\mathcal{H}|_b \\ e_r|_b \\ \delta_{\mathbf{x}}\mathcal{H}|_a \\ e_r|_a \end{bmatrix},$$

respectively, where $R_{ext} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{Q}_1 & -\tilde{Q}_1 \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} M_Q & 0 \\ 0 & M_Q \end{bmatrix}$, where $\tilde{Q}_1 = M^T \tilde{P}_1 M$, $M_Q = \left(M^T M\right)^{-1} M^T$ with $\tilde{P}_1 =$

$\begin{bmatrix} P_1 & G_1 \\ G_1^\top & \mathbf{0} \end{bmatrix}$ and $M$ spanning the columns of $\tilde{P}_1$. $W_B, W_C \in \mathbb{R}^{(2,4)}$ are such that $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible and satisfy (11) (Villegas et al., 2006). Satisfying the energy balance

$$\dot{\mathcal{H}} = \mathbf{u}^\top \mathbf{y} - \int_a^b \mu \left(\partial_\zeta v\right)^2 d\zeta \leq \mathbf{u}^\top \mathbf{y} \tag{15}$$

where $\mu \left(\partial_\zeta v\right)^2$ is the power dissipated into heat per unit length.

*Remark 1.* In some references, *e.g.* in Hauge et al. (2007) where fluids flowing in pipes are considered, the friction is described as $-\frac{\lambda}{2D}|v|v$, where $\lambda$ is the friction coefficient. This alternative formula stands in for the term $-\frac{1}{\rho}\partial_\zeta \tau$ in the right hand side of (13b). However, as $-\frac{\lambda}{2D}|v|v$ is a zero-order dissipative term (Kotyczka, 2013) it does not contribute in the boundary conditions (Villegas et al., 2006). The inputs and outputs are in this case the same as for the inviscid port-Hamiltonian model and the power dissipated per unit length is given by $\frac{1}{2}\frac{\lambda \rho}{D}|v|v^2$.

The port-Hamiltonian formulation (14) provides a nice framework reflecting the macroscopic properties of fluids, such as passivity. Yet it does not allow to characterize precisely what occurs within the matter, and the underlying structure does not reflect the first principle of Thermodynamics. It makes it difficult to use for control design purposes, leading to what is known as the dissipation obstacle (Macchelli et al., 2015). This motivates the use of an alternative representation, the irreversible port Hamiltonian formulation, in which the entropy balance and the effect of the temperature are explicitly taken into account, that reflects both the first and second principles of Thermodynamics. This representation is formulated in the most general case of non isentropic compressing fluids.

## 4. NON-ISENTROPIC COMPRESSIBLE FLUIDS

### *4.1 Governing equations*

For non-isentropic fluids, the thermal domain is added to the dynamics of specific internal energy (2), i.e.,

$$\partial_t u = -v\partial_\zeta u - \frac{p}{\rho}\partial_\zeta v + \frac{1}{\rho}Q - \frac{1}{\rho}\partial_\zeta q \tag{16}$$

where $q = -k\partial_\zeta T$ is the heat flux, with $k$ the heat conduction coefficient. $T = T(\zeta, t)$ denotes the fluid temperature, and $Q$ the internal heat production per unit length. From (15), the internal heat production is defined as $Q = \mu\left(\partial_\zeta v\right)^2 = -\tau \partial_\zeta v$. In this case, the Gibbs equation reads:

$$du = -pd\frac{1}{\rho} + Tds \tag{17}$$

where $s = s(\zeta, t)$ is the specific entropy. Considering $\rho$ and $s$ as independent variables, from (17) we obtain the relationships

$$\begin{aligned} p =& \rho^2 \partial_\rho u, & (18)\\ T =& \partial_s u, & (19)\\ TD_t s =& D_t u - \left(p/\rho^2\right) D_t \rho & (20)\\ -\frac{1}{\rho}\partial_\zeta p =& -\partial_\zeta h + T\partial_\zeta s & (21) \end{aligned}$$

Tacking into account the relations (18) to (21) in (13) we exhibit the following governing equations:

$$\partial_t \rho = -\partial_\zeta \rho v \tag{22a}$$

$$\partial_t v = -\partial_\zeta \left(\frac{1}{2}v^2 + h\right) + T\partial_\zeta s - \frac{1}{\rho}\partial_\zeta \tau \tag{22b}$$

$$\partial_t s = -v\partial_\zeta s - \frac{\tau}{\rho T}\partial_\zeta v - \frac{1}{\rho T}\partial_\zeta q \tag{22c}$$

In what follows we denote by $\mathbf{x} = [\rho \ v]^\top$ the state variables that describe the fluid dynamics and by $s$ the state variable used to describe the thermal domain. The total energy $\mathcal{H} = \mathcal{H}(\mathbf{x}, s)$ of the fluid is defined as:

$$\mathcal{H} = \int_a^b \frac{1}{2}\rho v^2 + \rho u(\rho, s) d\zeta \tag{23}$$

and the co-energy variables are given by:

$$\begin{bmatrix} \delta_\rho \mathcal{H} \\ \delta_v \mathcal{H} \\ \delta_s \mathcal{H} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}v^2 + h \\ \rho v \\ \rho T \end{bmatrix} \tag{24}$$

### *4.2 IPHS formulation of compressible fluids*

The IPHS formulation consists in choosing the energy and entropy density variables as state variables and in expressing the dynamics with respect to the total energy (including the internal energy). In order to derive the irreversible port-Hamiltonian formulation of the system (22), it is necessary to identify the reversible and irreversible parts of the system dynamics, $\mathcal{W}_r$ and $\mathcal{W}_i$, respectively, such that $\partial_t \mathbf{x} = \mathcal{W}_r + \mathcal{W}_i$. In the considered case, the irreversible part of the governing equations is given by the terms associated with entropy creation by heat flux and conversion of kinetic energy into heat by viscous friction, i.e.,

$$\mathcal{W}_r = \begin{bmatrix} -\partial_\zeta \left(\delta_v \mathcal{H}\right) \\ \frac{\delta_s \mathcal{H}}{\rho}\partial_\zeta s - \partial_\zeta\left(\delta_\rho \mathcal{H}\right) \\ -\frac{\delta_v \mathcal{H}}{\rho}\partial_\zeta s \end{bmatrix}, \mathcal{W}_i = \begin{bmatrix} 0 \\ -\frac{1}{\rho}\partial_\zeta \tau \\ -\frac{\tau}{\rho T}\frac{\delta_v \mathcal{H}}{\rho} - \frac{1}{\rho T}\partial_\zeta q \end{bmatrix}$$

The reversible part $\mathcal{W}_r$ can be described by a skew-symmetric operator $\mathcal{J}_r$, i.e.,

$$\mathcal{W}_r = \underbrace{\begin{bmatrix} P_1 \partial_\zeta & L(\mathbf{x}, s) \\ -L^\top(\mathbf{x}, s) & \mathbf{0} \end{bmatrix}}_{\mathcal{J}_r} \begin{bmatrix} \delta_\mathbf{x} \mathcal{H} \\ \delta_s \mathcal{H} \end{bmatrix} \tag{25}$$

with $P_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, and $L(\mathbf{x}, s) = \begin{bmatrix} 0 & \frac{\partial_\zeta s}{\rho} \end{bmatrix}^\top$.

Regarding the irreversible part $\mathcal{W}_i$, we shall use the following instrumental Lemma.

*Lemma 1.* Let $H^1_n\left((a,b), \mathbb{R}^n\right)$ be the Sobolev space of differentiable vector functions of size $n \in \mathbb{N}$ on the interval $(a, b)$. Consider $\alpha(\mathbf{x}) \in L^2\left((a,b), \mathbb{R}\right)$ and $F(\mathbf{x}) \in L^2\left((a,b), \mathbb{R}^N\right)$ as two state-dependent continuous functions, such that $\alpha(\mathbf{x})\mathbf{f} \in H^1_n, \forall \mathbf{f} \in H^1_N$ and $F(\mathbf{x})e \in H^1_n, \forall e \in H^1_1$. Then, the operator $\mathcal{D}^* : L^2\left((a,b), \mathbb{R}\right) \to L^2\left((a,b), \mathbb{R}^n\right)$ defined as $\mathcal{D}^* e = -\alpha(\mathbf{x})\partial_\zeta\left(F(\mathbf{x})e\right)$ is the formal adjoint of $\mathcal{D} : L^2\left((a,b), \mathbb{R}^N\right) \to L^2\left((a,b), \mathbb{R}\right)$ defined as $\mathcal{D}\mathbf{f} = F^\top(\mathbf{x})\partial_\zeta\left(\alpha(\mathbf{x})\mathbf{f}\right)$.

**Proof.** Let $\mathscr{H}_0 = L^2\left((a,b),\mathbb{R}\right)$ and $\mathscr{H}_1 = L^2\left((a,b),\mathbb{R}^n\right)$ be the Hilbert spaces of continuous scalar and vector functions on the interval $(a,b)$, respectively. Considering that $\alpha(\mathbf{x})\mathbf{f}$ and $F(\mathbf{x})e$ are differentiable, the inner product $\langle e, \mathcal{D}\mathbf{f}\rangle_{\mathscr{H}_0}$ reads:

$$\begin{aligned}\langle e, \mathcal{D}\mathbf{f}\rangle_{\mathscr{H}_0} &= \int_a^b eF^\top(\mathbf{x})\partial_\zeta\left(\alpha(\mathbf{x})\mathbf{f}\right)d\zeta \\ &= -\int_a^b \mathbf{f}^\top\alpha(\mathbf{x})\partial_\zeta\left(F(\mathbf{x})e\right)d\zeta \\ &\quad + \left[\alpha(\mathbf{x})\mathbf{f}\right]^\top\left[F(\mathbf{x})e\right]\Big|_a^b \\ &= \langle \mathcal{D}^*e, \mathbf{f}\rangle_{\mathscr{H}_1} + \left[\alpha(\mathbf{x})\mathbf{f}\right]^\top\left[F(\mathbf{x})e\right]\Big|_a^b \qquad (26)\end{aligned}$$

Considering boundary conditions equal to 0, from (26) we have that $\langle e, \mathcal{D}\mathbf{f}\rangle_{\mathscr{H}_0} = \langle \mathcal{D}^*e, \mathbf{f}\rangle_{\mathscr{H}_1}$, i.e., $\mathcal{D}^*$ is the formal adjoint of $\mathcal{D}$. ■

Defining $G_1 = [0\ 1/\rho]^\top$, $g_s = 1/\rho$, and considering the relationship $\frac{1}{\rho T}\partial_\zeta q = \frac{q}{\rho T^2}\partial_\zeta T + \frac{1}{\rho}\partial_\zeta\frac{q}{T}$, $\mathscr{W}_i$ is expressed as:

$$\mathscr{W}_i = \begin{bmatrix} -G_1\partial_\zeta\left[\tau\right] \\ -\frac{\tau}{\rho T}\partial_\zeta G_1^\top\delta_{\mathbf{x}}\mathcal{H} - \frac{q}{\rho T^2}\partial_\zeta\left(g_s\delta_{\mathscr{s}}\mathcal{H}\right) - g_s\partial_\zeta\frac{q}{T}\end{bmatrix}$$

In order to describe the IPHS, the following pseudo (locally defined) brackets will be used to define the thermodynamic driving forces of the system

$$\{\mathscr{E}|\mathscr{G}|\mathscr{F}\} = \begin{bmatrix}\delta_{\mathbf{x}}\mathscr{E} \\ \delta_{\mathscr{s}}\mathscr{E}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathscr{G} \\ -\mathscr{G}^* & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta_{\mathbf{x}}\mathscr{F} \\ \delta_{\mathscr{s}}\mathscr{F}\end{bmatrix} \qquad (27)$$

$$\{\mathscr{E}|g_s(\mathbf{x})|\mathscr{F}\} = \delta_{\mathscr{s}}\mathscr{E}\partial_\zeta\left(g_s(\mathbf{x})\delta_{\mathscr{s}}\mathscr{F}\right) \qquad (28)$$

for some smooth function $\mathscr{E}$ and $\mathscr{F}$, operator $\mathscr{G}$ and its corresponding formal adjoint $\mathscr{G}^*$, and real continuous scalar function $g_s(\mathbf{x})$.

Considering that the total entropy of system (22), and the corresponding variational derivative $\delta_{\mathbf{x}}\mathcal{S}$ and $\delta_{\mathscr{s}}\mathcal{S}$, are given by

$$\mathcal{S} = \int_a^b \rho\mathscr{s}d\zeta,\ \delta_{\mathbf{x}}\mathcal{S} = \begin{bmatrix}\mathscr{s} \\ 0\end{bmatrix} \text{ and } \delta_{\mathscr{s}}\mathcal{S} = \rho \qquad (29)$$

respectively, we have that

$$\{\mathcal{S}|G_1\partial_\zeta|\mathcal{H}\} = \rho\partial_\zeta v \text{ and } \{\mathcal{S}|g_s(\mathbf{x})|\mathcal{H}\} = \rho\partial_\zeta T$$

Then, defining

$$\alpha_1(\mathbf{x}) = \gamma_1\{\mathcal{S}|G_1\partial_\zeta|\mathcal{H}\} = -\tau/\rho T \qquad (30)$$

$$\alpha_s(\mathbf{x}) = \gamma_s\{\mathcal{S}|g_s(\mathbf{x})|\mathcal{H}\} = -q/\rho T^2 \qquad (31)$$

where $\gamma_1 = \mu/\rho^2 T > 0$ and $\gamma_s = k/\left(\rho T\right)^2 > 0$ the irreversible part of the non-isentropic compressible fluid is rewritten as:

$$\mathscr{W}_i = \underbrace{\begin{bmatrix}\mathbf{0} & \mathcal{D}_\tau \\ -\mathcal{D}_\tau^* & \mathcal{D}_T - \mathcal{D}_T^*\end{bmatrix}}_{\mathscr{J}_i}\begin{bmatrix}\delta_{\mathbf{x}}\mathcal{H} \\ \delta_{\mathscr{s}}\mathcal{H}\end{bmatrix} \qquad (32)$$

where, according to Lemma 1, $\mathcal{D}_\tau^*(.) = -\alpha_1(\mathbf{x})\partial_\zeta\left(G_1^\top.\right)$ is the formal adjoint of $\mathcal{D}_\tau(.) = G_1\partial_\zeta\left(\alpha_1(\mathbf{x}).\right)$, and $\mathcal{D}_T^*(.) = -\alpha_s(\mathbf{x})\partial_\zeta\left(g_s(\mathbf{x}).\right)$ is the formal adjoint of $\mathcal{D}_T(.) = g_s(\mathbf{x})\partial_\zeta\left(\alpha_s(\mathbf{x}).\right)$, i.e., the operator $\mathscr{J}_i$ in (32) is formally skew-symmetric.

Note that the operator $\mathcal{D}_\tau^*$ is associated with the entropy production per unit mass due to the conversion of kinetic energy into heat by viscous friction, i.e.,

$$-\mathcal{D}_\tau^*\delta_{\mathbf{x}}\mathcal{H} = \alpha_1(\mathbf{x})\partial_\zeta\left(G_1^\top\delta_{\mathbf{x}}\mathcal{H}\right) = \frac{\mu}{\rho T}\left(\partial_\zeta v\right)^2 \geq 0$$

Similarly, the operator $\mathcal{D}_T^*$ describes the entropy production per unit mass due to the heat flux, i.e.,

$$-\mathcal{D}_T^*\delta_{\mathscr{s}}\mathcal{H} = \alpha_s(\mathbf{x})\partial_\zeta\left(g_s(\mathbf{x})\delta_{\mathscr{s}}\mathcal{H}\right) = \frac{k}{\rho T^2}\left(\partial_\zeta T\right)^2 \geq 0$$

Then, the dynamics of a non-isentropic and non-reactive compressible fluid can be formulated through the irreversible port-Hamiltonian formulation given in Proposition 1.

*Proposition 1.* Let $\mathcal{H}$ be the total energy defined in (23). Using the skew-symmetric operators $\mathscr{J}_r$, and $\mathscr{J}_i$ defined in (25) and (32), respectively, the governing equations (22) of a non-reactive compressible fluid can be formulated as the following irreversible port-Hamiltonian system

$$\begin{bmatrix}\partial_t\mathbf{x} \\ \partial_t\mathscr{s}\end{bmatrix} = \underbrace{\begin{bmatrix}P_1\partial_\zeta & L(\mathbf{x}) + \mathcal{D}_\tau \\ -L^\top(\mathbf{x}) - \mathcal{D}_\tau^* & \mathcal{D}_T - \mathcal{D}_T^*\end{bmatrix}}_{\mathscr{J}_{iphs}}\begin{bmatrix}\delta_{\mathbf{x}}\mathcal{H} \\ \delta_{\mathscr{s}}\mathcal{H}\end{bmatrix} \qquad (33)$$

with boundary inputs and outputs given by

$$\mathbf{u}(t) = \begin{bmatrix}\left(\frac{1}{2}\rho v^2 + \rho\mathscr{h} + \tau\right)(b) \\ -\mathscr{f}_s(b) \\ \left(\frac{1}{2}\rho v^2 + \rho\mathscr{h} + \tau\right)(a) \\ -\mathscr{f}_s(a)\end{bmatrix}, \mathbf{y}(t) = \begin{bmatrix}-v(b) \\ T(b) \\ v(a) \\ -T(a)\end{bmatrix} \qquad (34)$$

where $\mathbf{x} = [\rho\ v]^\top$, and $\mathscr{f}_s = q/T$ is the entropy flux. This IPHS formulation allows to represent both the first and second principles of Thermodynamics, the conservation of energy

$$\dot{\mathcal{H}} = \mathbf{u}^\top\mathbf{y} \qquad (35)$$

$$(36)$$

and the irreversible entropy creation

$$\dot{\mathcal{S}} = \int_a^b \sigma_s d\zeta - \left(\mathscr{s}\rho v + \mathscr{f}_s\right)|_a^b \geq u_s y_s \qquad (37)$$

where $\sigma_s = \frac{\mu}{T}\left(\partial_\zeta v\right)^2 + \frac{\kappa}{T^2}\left(\partial_\zeta T\right)^2 \geq 0$ and $u_s y_s = -\left(\mathscr{s}\rho v + \mathscr{f}_s\right)|_a^b$.

**Proof.** Using (25) and (32) the PDE (33) is obtained. On the other hand, the energy balance is given by $\dot{\mathcal{H}} = \int_a^b\left(\delta_{\mathbf{x}}\mathcal{H}^\top\partial_t\mathbf{x} + \delta_{\mathscr{s}}\mathcal{H}^\top\partial_t\mathscr{s}\right)d\zeta$. Then, using (33) the energy balance can be expressed as:

$$\begin{aligned}\dot{\mathcal{H}} &= \int_a^b\left[\delta_{\mathbf{x}}\mathcal{H}\right]^\top P_1\partial_\zeta\delta_{\mathbf{x}}\mathcal{H}d\zeta \\ &\quad + \int_a^b\left[\delta_{\mathbf{x}}\mathcal{H}\right]^\top\left[\alpha(\mathbf{x})L(\mathscr{s})\delta_{\mathscr{s}}\mathcal{H} + \mathcal{D}_\tau\delta_{\mathscr{s}}\mathcal{H}\right]d\zeta \\ &\quad + \int_a^b\delta_{\mathscr{s}}\mathcal{H}\left(-\alpha(\mathbf{x})L^\top(\mathscr{s})\delta_{\mathbf{x}}\mathcal{H} - \mathcal{D}_\tau^*\delta_{\mathbf{x}}\mathcal{H}\right)d\zeta \\ &\quad + \int_a^b\delta_{\mathscr{s}}\mathcal{H}\left(\mathcal{D}_T\delta_{\mathscr{s}}\mathcal{H} - \mathcal{D}_T^*\delta_{\mathscr{s}}\mathcal{H}\right)d\zeta \\ &= \int_a^b\left[\delta_{\mathbf{x}}\mathcal{H}\right]^\top P_1\partial_\zeta\delta_{\mathbf{x}}\mathcal{H}d\zeta \\ &\quad + \langle\delta_{\mathbf{x}}\mathcal{H}, \mathcal{D}_\tau\delta_{\mathscr{s}}\mathcal{H}\rangle_{\mathcal{H}_1} - \langle\delta_{\mathscr{s}}\mathcal{H}, \mathcal{D}_\tau^*\delta_{\mathbf{x}}\mathcal{H}\rangle_{\mathcal{H}_0} \\ &\quad + \langle\delta_{\mathscr{s}}\mathcal{H}, \mathcal{D}_T\delta_{\mathscr{s}}\mathcal{H}\rangle_{\mathcal{H}_0} - \langle\delta_{\mathscr{s}}\mathcal{H}, \mathcal{D}_T^*\delta_{\mathscr{s}}\mathcal{H}\rangle_{\mathcal{H}_0}\end{aligned}$$

From (26), and considering that $[\delta_{\mathbf{x}}\mathcal{H}]^\top P_1\partial_\zeta\delta_{\mathbf{x}}\mathcal{H} = -\delta_\rho\mathcal{H}\partial_\zeta\delta_v\mathcal{H} - \delta_v\mathcal{H}\partial_\zeta\mathcal{H} = -\partial_\zeta\left(\delta_\rho\mathcal{H}\delta_v\mathcal{H}\right)$ we obtain

$$\begin{aligned}\dot{\mathcal{H}} =& -\left(\delta_\rho\mathcal{H}\delta_v\mathcal{H}\right)\Big|_a^b + \left[\alpha(\mathbf{x})\delta_{\mathbf{x}}\mathcal{H}\right]^\top G_1\mathbf{R}_1(\mathbf{x})\delta_{\mathscr{s}}\mathcal{H}\Big|_a^b \\ &+ \left(\alpha(\mathbf{x})\delta_{\mathscr{s}}\mathcal{H}\right) r_1(\mathbf{x})\delta_{\mathscr{s}}\mathcal{H}\Big|_a^b \\ =& -\left(\delta_\rho\mathcal{H}\delta_v\mathcal{H}\right)\Big|_a^b - \frac{\delta_v\mathcal{H}}{\rho}\tau\Big|_a^b - \frac{\delta_{\mathscr{s}}\mathcal{H}}{\rho}\mathscr{f}_s\Big|_a^b\end{aligned}$$

Regrouping terms we obtain

$$\begin{aligned}\dot{\mathcal{H}} =& -\left(\frac{\delta_v\mathcal{H}}{\rho}\left(\rho\delta_\rho\mathcal{H}+\tau\right)\right)\Big|_a^b - \frac{\delta_{\mathscr{s}}\mathcal{H}}{\rho}\mathscr{f}_s\Big|_a^b \\ =& -\left(v\left(\frac{1}{2}\rho v^2 + \rho\mathscr{h} + \tau\right)\right)\Big|_a^b - \left(T\mathscr{f}_s\right)\Big|_a^b\end{aligned}$$

and defining the boundary inputs and outputs as shown in (34), the relationship (35) is obtained.

Regarding the entropy balance, notice that

$$\mathcal{S} = \int_a^b \rho\mathscr{s}d\zeta,\ \delta_{\mathbf{x}}\mathcal{S} = \begin{bmatrix}\mathscr{s}\\0\end{bmatrix} \text{ and } \delta_{\mathscr{s}}\mathcal{S} = \rho \tag{38}$$

Then, the entropy rate of change is given by:

$$\begin{aligned}\dot{\mathcal{S}} =& \int_a^b \left[\delta_{\mathbf{x}}\mathcal{S}\right]^\top \partial_t\mathbf{x}d\zeta + \int_a^b \left[\delta_{\mathscr{s}}\mathcal{S}\right]^\top \partial_t\mathscr{s}d\zeta \\ =& \int_a^b \left[\delta_{\mathbf{x}}\mathcal{S}\right]^\top P_1\partial_\zeta\delta_{\mathbf{x}}\mathcal{H}d\zeta \\ &+ \int_a^b \left[\delta_{\mathbf{x}}\mathcal{S}\right]^\top \left[\alpha(\mathbf{x})L(\mathscr{s})\delta_{\mathscr{s}}\mathcal{H} + \mathcal{D}_\tau\delta_{\mathscr{s}}\mathcal{H}\right]d\zeta \\ &- \int_a^b \delta_{\mathscr{s}}\mathcal{H}\left[\alpha(\mathbf{x})L(\mathscr{s})\delta_{\mathbf{x}}\mathcal{H} + \mathcal{D}_\tau^*\delta_{\mathbf{x}}\mathcal{H}\right]d\zeta \\ &+ \int_a^b \delta_{\mathscr{s}}\mathcal{H}\left(\mathcal{D}_T\delta_{\mathscr{s}}\mathcal{H} - \mathcal{D}_T^*\delta_{\mathscr{s}}\mathcal{H}\right)d\zeta\end{aligned}$$

Considering that $[\delta_{\mathbf{x}}\mathcal{S}]^\top\left[L(\mathbf{x})\delta_{\mathscr{s}}\mathcal{H} + \mathcal{D}_\tau\delta_{\mathscr{s}}\mathcal{H}\right] = 0$ and $[\delta_{\mathbf{x}}\mathcal{S}]^\top P_1\partial_\zeta\delta_{\mathbf{x}}\mathcal{H} = -\mathscr{s}\partial_\zeta\delta_v\mathcal{H}$, we have that

$$\begin{aligned}\dot{\mathcal{S}} =& -\int_a^b \left(\mathscr{s}\partial_\zeta\delta_v\mathcal{H} + \delta_v\mathcal{H}\partial_\zeta\mathscr{s} + \partial_\zeta\mathscr{f}_s\right)d\zeta \\ &+ \int_a^b \left(\frac{\mu}{T}\left(\partial_\zeta\frac{\delta_v\mathcal{H}}{\rho}\right)^2 + \frac{\kappa}{T^2}\left(\partial_\zeta\frac{\delta_{\mathscr{s}}\mathcal{H}}{\rho}\right)^2\right)d\zeta \\ =& \int_a^b \sigma_s d\zeta - \int_a^b \partial_\zeta\left(s\delta_v\mathcal{H} + \mathscr{f}_s\right)d\zeta \\ =& \int_a^b \sigma_s d\zeta - \left(s\rho v + \mathscr{f}_s\right)\Big|_a^b\end{aligned}$$

where $\sigma_s = \frac{\mu}{T}\left(\partial_\zeta v\right)^2 + \frac{\kappa}{T^2}\left(\partial_\zeta T\right)^2 \geq 0$. ■

Note that the irreversible port-Hamiltonian formulation of compressible Newtonian fluids described in Proposition 1 is conservative, i.e., if the system is isolated, with boundary conditions (B.C.) equal to 0, the energy stored by the fluid is constant in time, $\dot{\mathcal{H}} = 0$. Similarly, with B.C.=0 the entropy balance (37) is reduced to the inequality $\dot{\mathcal{S}} \geq 0$, i.e., system (33) provides an appropriated description of the first and second laws of Thermodynamics.

On the other hand, operator $\mathcal{J}_{iphs}$ in (33) is skew-symmetric. However, scalar functions $\alpha_1(\mathbf{x})$ and $\alpha_s(\mathbf{x})$ depend explicitly of the entropy effort $\delta_{\mathscr{s}}\mathcal{H} = \rho T$, as shown in (30) and (31). This implies that $\mathcal{J}_{iphs}$ does not satisfy the identity of Jacobi and, as a consequence, a Dirac structure is not generated.

## 5. CONCLUSION

In this paper an irreversible port-Hamiltonian formulation of compressible Newtonian fluids has been proposed. The thermodynamic driving forces associated with the entropy production due to viscous friction and heat flux are described through locally defined pseudo-brackets. The model obtained is conservative with a non-decreasing entropy under boundary conditions equal to 0, satisfying the fist and second laws of Thermodynamics. As future work we shall first consider the irreversible port-Hamiltonian formulation of reactive processes and systems defined on $n$-dimensional domains. We will then generalize the control by interconnection method to the control of irreversible thermodynamic processes using IPHS formulations.

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